Ramsey's Theorem in the Weihrauch Lattice

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Ramsey's Theorem

Theorem (Ramsey 1930)

Every coloring $c : [\mathbb{N}]^n \to k$ admits an infinite homogeneous set $M \subseteq \mathbb{N}$.

- Here $[M]^n$ denotes the set of n–element subsets of $M \subseteq \mathbb{N}$.
- We identify k with $\{0, 1, ..., k-1\}$ for all $k \in \mathbb{N}$.
- A set $M \subseteq \mathbb{N}$ is called homogeneous for the coloring c, if there is some $i \in k$ such that $c(A) = i$ for all $A \in [M]^n$.
- By $\mathcal{C}^n_k$ we denote the set of colorings $c : [\mathbb{N}]^n \to k$.
- $c : [\mathbb{N}]^n \to k$ is called stable if $\lim_{i\to\infty} c(A \cup \{i\})$ exists for all $A \in [\mathbb{N}]^{n-1}$.
- We also consider the case $k = \mathbb{N}$, which corresponds to an unspecified but finite number of colors.

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- By $\mathcal{C}_k^n$ we denote the set of colorings $c : [\mathbb{N}]^n \to k$.
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Specker (1969) proved that there are computable colorings of pairs without computable homogenous sets.

Jockusch (1972) showed the following now classical results:

- There is a computable $c : [\mathbb{N}]^n \to 2$ for each $n \geq 2$ without an infinite homogeneous set $M \subseteq \mathbb{N}$ that is computable in $\emptyset^{(n-1)}$.
- For every computable coloring $c : [\mathbb{N}]^n \to 2$ with $n \geq 1$ there exists an infinite homogeneous set $M \subseteq \mathbb{N}$ with $M' \leq_{\mathrm{T}} \emptyset^{(n)}$.

Hence, the instancewise complexity of Ramsey's theorem RT_k^n is $\boldsymbol{\Sigma}_{n+1}^0$ in the arithmetical hierarchy.

General Question on the Complexity

How complicated is Ramsey's theorem RT^n_k seen as a mathematical problem?

- How do computability properties of homogeneous sets do depend on computability properties of colorings?
- In this sense it has been studied for a long time, starting with Specker (1969), Jockusch (1972), Seetapun (1995), Cholak, Jockusch and Slaman (2001) and many others.
- How can Ramsey's theorem be classified in reverse mathematics (from a proof theoretic perspective)?
- How can Ramsey's theorem be classified in descriptive set theory?
- How can Ramsey's theorem be classified in the Weihrauch lattice?

The Weihrauch lattice refines the Borel hierarchy and can be seen as a uniform computability theoretic version of reverse mathem.

Mathematical Problems

- We consider partial multi-valued functions $f :\subseteq X \rightrightarrows Y$ as mathematical problems.
- We assume that the underlying spaces X and X are represented spaces, hence notions of computability and continuity are well-defined.
- Every theorem of the form

$$(\forall x \in X)(\exists y \in Y)(x \in D \Longrightarrow P(x,y))$$

can be identified with $F :\subseteq X \rightrightarrows Y$ with $\mathrm{dom}(F) := D$ and $F(x) := \{y \in Y : P(x,y)\}$.
- **Example**: Ramsey's Theorem is the mathematical problem $\mathrm{RT}_k^n : \mathcal{C}_k^n \rightrightarrows 2^{\mathbb{N}}$ with

$$\mathrm{RT}_k^n(c) := \{M \subseteq \mathbb{N} : M \text{ is an infinite homogenous set for } c\}.$$

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Examples of Mathematical Problems

- Weak Kőnig's Lemma is the mathematical problem

$$\mathrm{WKL} :\subseteq \mathrm{Tr} \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T]$$

  with $\mathrm{dom}(\mathrm{WWKL}) := \{T \in \mathrm{Tr} : T \text{ infinite}\}$.
- Bolzano Weierstraß Theorem is the mathematical problem

$$\mathrm{BWT}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X, (x_n) \mapsto \{x \in X : x \text{ is a cluster point of } (x_n)\}$$

  where $\mathrm{dom}(\mathrm{BWT}_X)$ contains only sequences (x_n) with a relatively compact range.
- $\lim_X :\subseteq X^{\mathbb{N}} \to X$ is called the limit problem of the space X.
- The cohesiveness problems $\mathrm{COH} : (2^{\mathbb{N}})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ is defined such that $\mathrm{COH}((R_i)_{i\in\mathbb{N}})$ is the set of all infinite sets A such that

$$A \cap R_i \text{ is finite or } A \cap (\mathbb{N} \setminus R_i) \text{ is finite}$$

  for each i, i.e., $A \subseteq^* R_i$ or $A \subseteq^* \mathbb{N} \setminus R_i$ for each i.

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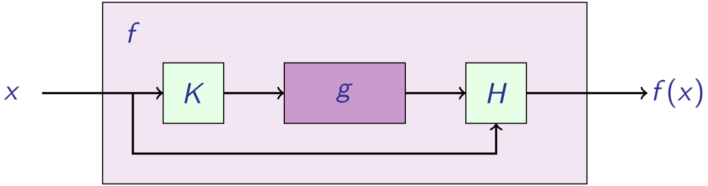
Weihrauch Reducibility

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be two mathematical problems.



- f is called Weihrauch reducible to g, in symbols $f \leq_{\mathrm{W}} g$, if there are computable $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $H\langle \mathrm{id}, GK\rangle$ realizes f whenever $G :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ realizes g.

For the "realization" we use representations $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ of the underlying objects.

Example: In order to prove $\mathsf{COH} \leq_{\mathrm{W}} \mathsf{RT}_2^2$, one would have to utilize RT_2^2 in order to compute COH.

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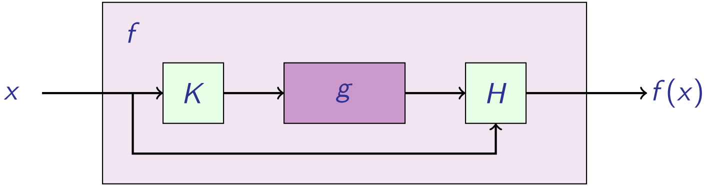

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Algebraic Operations in the Weihrauch Lattice

Let f, g be two mathematical problems. We consider:

- $f \times g$: both problems are available in parallel (Product)
- $f \sqcup g$: both problems are available, but for each instance one has to choose which one is used (Coproduct)
- $f \sqcap g$: given an instance of f and g, only one of the solutions will be provided (Sum)
- $f * g$: f and g can be used consecutively (Comp. Product)
- $g \rightarrow f$: this is the simplest problem h such that f can be reduced to $g * h$ (Implication)
- f^*: f can be used any given finite number of times in parallel (Star)
- $\widehat{f}$: f can be used countably many times in parallel (Parallelization)
- f': f can be used on the limit of the input (Jump)

Ramsey's Theorem in the Weihrauch Lattice

Theorem (Cholak, Jockusch, Slaman 2009)

RT^n_k *is equivalent to* $\mathsf{SRT}^n_k \wedge \mathsf{COH}$ *over* RCA_0 *for all* $n, k \geq 2$.

Corollary

$\mathsf{SRT}^n_k \sqcup \mathsf{COH} \leq_{\mathrm{W}} \mathsf{RT}^n_k \leq_{\mathrm{W}} \mathsf{SRT}^n_k * \mathsf{COH}$ *for all* $n, k \geq 2$.

Theorem (Dzhafarov, Goh, Hirschfeldt, Patey, Pauly 2018)

$\mathsf{RT}^2_2 \,|_{\mathrm{W}}\, \mathsf{SRT}^2_2 \times \mathsf{COH}$.

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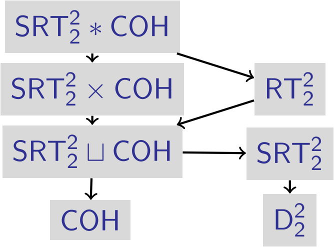

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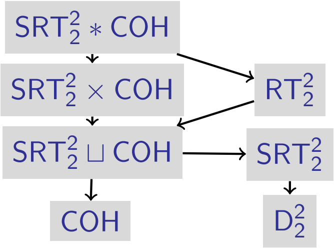

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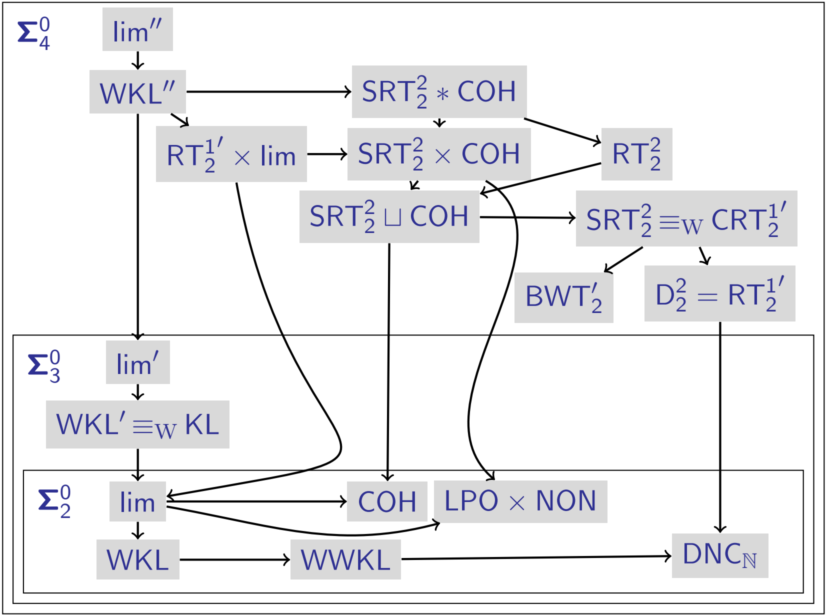

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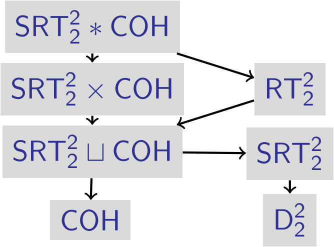

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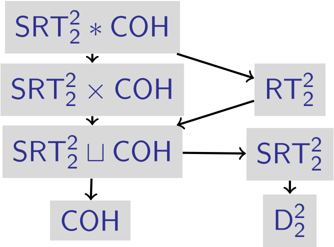

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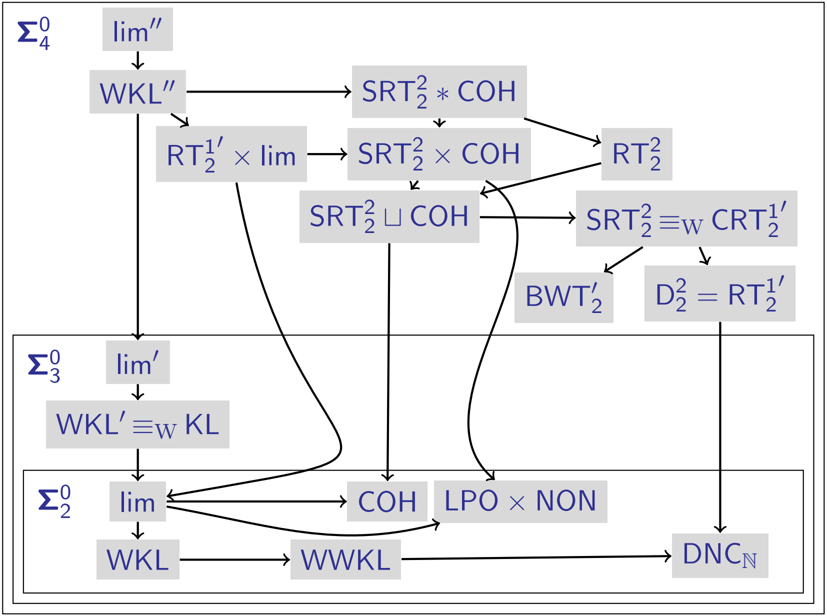
Σ^0_4
$\lim''$
WKL''
$\mathsf{SRT}^2_2 * \mathsf{COH}$
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WWKL
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A Separation Technique for Jumps

Theorem

$\mathsf{RT}_2^{1\prime} \mid_{\mathrm{W}} \mathsf{BWT}_2'$.

$\mathsf{RT}_2^{1\prime} \not\leq_{\mathrm{W}} \mathsf{BWT}_2'$ follows since $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}_2^{1\prime}$, but $\mathsf{C}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{BWT}_2'$. For $\mathsf{BWT}_2' \not\leq_{\mathrm{W}} \mathsf{RT}_2^{1\prime}$ we have used the following theorem (note that $\mathsf{BWT}_2 \equiv_{\mathrm{W}} \mathsf{RT}_2^1$).

Theorem (Baire's grand theorem)

Let X, Y be metric spaces, X a Baire space and Y separable. Then the restriction $f|_U$ of every $\mathbf{\Sigma}_2^0$–measurable function $f : X \to Y$ to any non-empty open subset $U \subseteq X$ has a point of continuity.

Corollary

$\mathsf{CRT}_2^{1\prime} \not\leq_{\mathrm{W}} \mathsf{RT}_2^{1\prime}$.

Corollary (Dzhafarov 2016)

$\mathsf{SRT}_2^2 \not\leq_{\mathrm{W}} \mathsf{D}_2^2$.

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$\mathsf{RT}^{1\prime}_2 \not\leq_{\mathrm{W}} \mathsf{BWT}'_2$ follows since $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}^{1\prime}_2$, but $\mathsf{C}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{BWT}'_2$. For $\mathsf{BWT}'_2 \not\leq_{\mathrm{W}} \mathsf{RT}^{1\prime}_2$ we have used the following theorem (note that $\mathsf{BWT}_2 \equiv_{\mathrm{W}} \mathsf{RT}^1_2$).

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The Cohesiveness Problem

Theorem (B., Hendtlass and Kreuzer 2017)

$\mathsf{COH} \equiv_W (\lim \to \mathsf{WKL}')$ *and* $\mathsf{WKL}' \equiv_W \lim * \mathsf{COH}$.

Theorem

$\mathsf{SRT}_k^{n+1} \equiv_W \mathsf{CRT}_k^{n\prime}$.

Altogether, we have

- $\mathsf{RT}_k^n \leq_W \mathsf{SRT}_k^n * \mathsf{COH}$ and
- $\mathsf{SRT}_k^{n+1} \leq_W \mathsf{RT}_k^n * \lim$, which implies
- $\mathsf{RT}_k^{n+1} \leq_W \mathsf{RT}_k^n * \lim * \mathsf{COH} \equiv_W \mathsf{RT}_k^n * \mathsf{WKL}'$.
- $\mathsf{WKL}^{(n)} * \mathsf{WKL}^{(k)} \equiv_W \mathsf{WKL}^{(n+k-1)}$ is also known.

Corollary

$\widehat{\mathsf{RT}_k^n} \equiv_W \mathsf{WKL}^{(n)}$ *for all* $n \geq 1, k \geq 2$.

This degree is known to be $\mathbf{\Sigma}^0_{n+2}$–measurable, but not $\mathbf{\Sigma}^0_{n+1}$–measurable.

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Corollary

$\widehat{\mathsf{RT}_k^n} \equiv_{\mathrm{W}} \mathsf{WKL}^{(n)}$ *for all* $n \geq 1, k \geq 2$.

This degree is known to be $\mathbf{\Sigma}^0_{n+2}$–measurable, but not $\mathbf{\Sigma}^0_{n+1}$–measurable.

The Cohesiveness Problem

Theorem (B., Hendtlass and Kreuzer 2017)

$\mathrm{COH} \equiv_{\mathrm{W}} (\lim \to \mathrm{WKL}')$ *and* $\mathrm{WKL}' \equiv_{\mathrm{W}} \lim * \mathrm{COH}$.

Theorem

$\mathrm{SRT}_k^{n+1} \equiv_{\mathrm{W}} \mathrm{CRT}_k^{n\prime}$.

Altogether, we have

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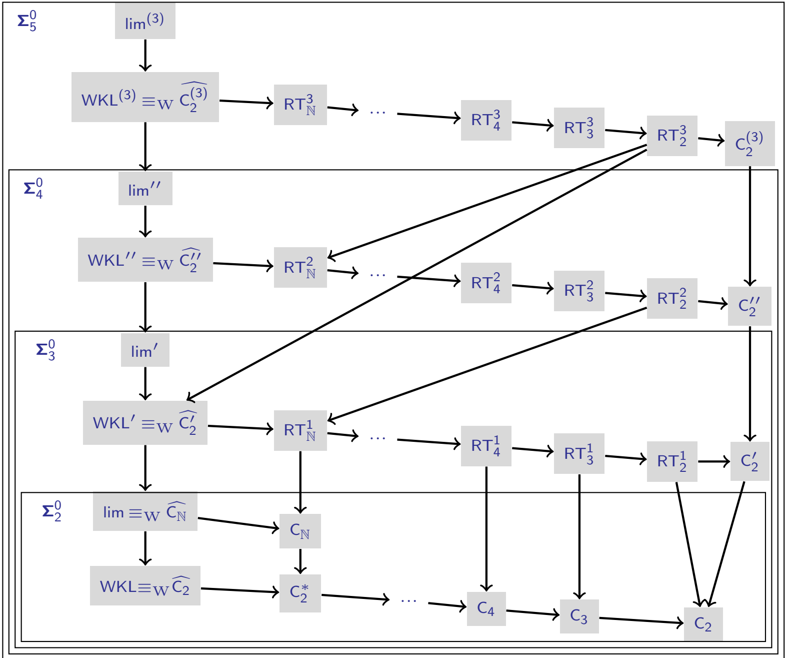
Ramsey's Theorem in the Weihrauch Lattice



Products and Parallellization of Ramsey

Theorem

$\mathrm{RT}^n_{\mathbb{N}} \times \mathrm{RT}^{n+1}_k \leq_{\mathrm{sW}} \mathrm{RT}^{n+1}_{k+1}$ *for all* $n, k \geq 1$.

Proof. (Idea.) Given a coloring $c_1 : [\mathbb{N}]^n \to \mathbb{N}$ with finite range and a coloring $c_2 : [\mathbb{N}]^{n+1} \to k$ we construct a coloring $c^+ : [\mathbb{N}]^{n+1} \to k+1$ as follows:

$$c^+(A) := \begin{cases} c_2(A) & \text{if } A \text{ is homogeneous for } c_1 \\ k & \text{otherwise} \end{cases}$$

for all $A \in [\mathbb{N}]^{n+1}$. Then $\mathrm{RT}^{n+1}_2(c^+) \subseteq \mathrm{RT}^n_{\mathbb{N}}(c_1) \cap \mathrm{RT}^{n+1}_k(c_2)$ and hence the desired reduction follows. □

Corollary

$(\mathrm{RT}^n_k)^* \leq_{\mathrm{W}} \mathrm{RT}^n_{\mathbb{N}} \leq_{\mathrm{W}} \mathrm{RT}^{n+1}_2$ *for all* $n, k \geq 1$.

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Parallelization of Ramsey

Theorem

$\widehat{\mathrm{RT}^n_k} \leq_{\mathrm{sW}} \mathrm{RT}^{n+2}_2$ *for all* $n, k \geq 1$.

Proof. (Idea.) Given a sequence $(c_i)_i$ of colorings $c_i : [\mathbb{N}]^n \to k$, we compute a sequence $(d_m)_m$ of colorings $d_m \in \mathcal{C}^n_{k^m}$ that capture the products $(\mathrm{RT}^n_k)^m$ and a sequence $(d^+_m)_m$ of colorings $d^+_m : [\mathbb{N}]^{n+1} \to 2$ by

$$d^+_m(A) := \begin{cases} 0 & \text{if } A \text{ is homogeneous for } d_m \\ 1 & \text{otherwise} \end{cases}$$

for all $A \in [\mathbb{N}]^{n+1}$. Now, in a final step we compute a coloring $c : [\mathbb{N}]^{n+2} \to 2$ with

$$c(\{m\} \cup A) := d^+_m(A)$$

for all $A \in [\mathbb{N}]^{n+1}$ and $m < \min(A)$. Given an infinite homogeneous set $M \in \mathrm{RT}^{n+2}_2(c)$ we determine a sequence $(M_i)_i$ as follows: for each fixed $i \in \mathbb{N}$ we first search for a number $m > i$ in M and then we let $M_i := \{x \in M : x > m\}$. □

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$\widehat{\mathsf{RT}^n_k} \leq_{\mathrm{sW}} \mathsf{RT}^{n+2}_2$ *for all* $n, k \geq 1$.

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Weak Kőnig's Lemma and Ramsey's Theorem

Theorem

$\widehat{\mathrm{RT}^n_k} \leq_{\mathrm{sW}} \mathrm{RT}^{n+2}_2$ *for all* $n, k \geq 1$.

Corollary

$\mathrm{WKL}' \leq_{\mathrm{W}} \mathrm{RT}^3_2$ *and* $\mathrm{WKL}^{(n)} \leq_{\mathrm{W}} \mathrm{SRT}^{n+2}_2$ *for* $n \geq 2$.

The first statement was also proved independently by Hirschfeldt and Jockusch (2016).

Corollary

$\mathrm{PA} \not\leq_{\mathrm{W}} \mathrm{SRT}^2_2 * \mathrm{COH}$ *and in particular* $\mathrm{WKL} \not\leq_{\mathrm{W}} \mathrm{RT}^2_2$.

This follows from results of Liu (2012). We note that $\mathrm{PA} <_{\mathrm{W}} \mathrm{WKL}$.

Proposition (B., Hendtlass and Kreuzer 2017)

$\mathrm{PA} \equiv (\mathrm{C}'_{\mathbb{N}} \to \mathrm{WKL})$.

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Question

Is there a simple proof of $\mathrm{WKL} \not\leq_{\mathrm{W}} \mathrm{RT}_2^2$*?*

To prove that there is no uniform reduction should potentially be much simpler to prove than the non-uniform result.

The Squashing Theorem

Theorem (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016)

Let $f, g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ and let f be finitely tolerant and total. Then $g \times f \leq_\mathrm{W} f \implies \widehat{g} \leq_\mathrm{W} f$.

- $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ is called finitely tolerant if there is a computable $T :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that for all $p, q \in \mathrm{dom}(f)$, $r \in \mathbb{N}^\mathbb{N}$, $k \in \mathbb{N}$:
  - $(\forall n \geq k)\ p(n) = q(n)$ and
  - $r \in f(q) \implies T\langle r, k\rangle \in f(p)$.
- $\mathsf{RT}^n_k, \mathsf{RT}^n_\mathbb{N}$ are finitely tolerant.

A similar version of the squashing theorem also holds for $\leq_\mathrm{sW}$ and Dorais, Dzhafarov, Hirst, Mileti and Shafer (2016) proved $\mathsf{RT}^n_k <_\mathrm{sW} \mathsf{RT}^n_{k+1}$ for all $n, k \geq 1$.

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Color Separation

Theorem (Rakotoniaina, Hirschfeldt & Jockusch, Patey 2015)

$\mathrm{RT}^n_k <_{\mathrm{W}} \mathrm{RT}^n_{k+1}$ *for all* $n, k \geq 1$.

Proof. (B. and Rakotoniaina 2015)

- $\mathrm{RT}^n_2 \times \mathrm{RT}^{n+1}_k \leq_{\mathrm{W}} \mathrm{RT}^{n+1}_{k+1}$ by the Product Theorem.
- $\mathrm{RT}^n_2 \times \mathrm{RT}^{n+1}_k \leq_{\mathrm{W}} \mathrm{RT}^{n+1}_k$ implies $\widehat{\mathrm{RT}^n_2} \leq_{\mathrm{W}} \mathrm{RT}^{n+1}_k$ by the Squashing Theorem which leads to a contradiction: $\lim^{(n-1)} \leq_{\mathrm{W}} \mathrm{WKL}^{(n)} \equiv_{\mathrm{W}} \widehat{\mathrm{RT}^n_2} \leq_{\mathrm{W}} \mathrm{RT}^{n+1}_k$
- $\mathrm{RT}^n_2 \times \mathrm{RT}^{n+1}_k \not\leq_{\mathrm{W}} \mathrm{RT}^{n+1}_k$ for all $n, k \geq 1$ follows.
- $\mathrm{RT}^{n+1}_k <_{\mathrm{W}} \mathrm{RT}^{n+1}_{k+1}$ for all $n, k \geq 1$ follows. $\square$

Products and Colors

Can colors make up for products?

- $\mathrm{RT}^n_k \times \mathrm{RT}^n_l \leq_{\mathrm{W}} \mathrm{RT}^n_{kl}$ is easy to see, hence
- $\prod_{i=1}^m \mathrm{RT}^n_{k_i} \leq_{\mathrm{W}} \mathrm{RT}^n_{\prod_{i=1}^m k_i}$ follows.
- Dzhafarov, Goh, Hirschfeldt, Patey, Pauly (2018) proved that the upper bound is optimal in the case of $n = 1$.

Corollary

$\bigsqcup_{i=1}^{\infty} (\mathrm{RT}^n_k)^i \leq_{\mathrm{W}} \bigsqcup_{j=1}^{\infty} \mathrm{RT}^n_j$ for $n \geq 1$, $k \geq 2$.

Here $\equiv_{\mathrm{W}}$ holds at least in the case of $n = 1$.

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