Ramsey complete sequences

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Joint with Jacob Fox

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Complete sequences

Definition

A sequence $A = (a_i)_{i=1}^{\infty}$ of natural numbers is said to be *complete* if every sufficiently large positive integer can be written as a sum of distinct elements of A and *entirely complete* if this holds for every positive integer.

We write $\Sigma(A)$ for the set of all natural numbers representable as the sum of distinct elements of A.

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- $(2^i)_{i=0}^{\infty}$ - entirely complete
- $(2^i)_{i=1}^{\infty}$ - not complete
- $(i^k)_{i=1}^{\infty}$ - complete
- $\{p^i q^j : i, j \geq 0\}$, $(p, q) = 1$ - complete

A criterion for completeness

Lemma

$A = (a_i)_{i=1}^{\infty}$ is entirely complete iff $a_1 = 1$ and, for all $k \geq 2$,

$$\sum_{i=1}^{k-1} a_i \geq a_k - 1.$$

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$\Longrightarrow$ If $a_k > 1 + \sum_{i=1}^{k-1} a_i$, then $\sum_{i=1}^{k-1} a_i + 1$ not represented.

$\Longleftarrow$ Suppose $\Sigma((a_i)_{i=1}^{k-1})$ contains $[\sum_{i=1}^{k-1} a_i]$. Then

$$\Sigma((a_i)_{i=1}^{k}) \supseteq [\sum_{i=1}^{k-1} a_i] \cup (a_k + [\sum_{i=1}^{k-1} a_i]) \supseteq [\sum_{i=1}^{k} a_i].$$

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A sequence $A = (a_i)_{i=1}^{\infty}$ of natural numbers is said to be *r-Ramsey complete* if whenever A is partitioned into $A_1, \ldots, A_r$, every sufficiently large positive integer is in $\cup_{i=1}^{r} \Sigma(A_i)$ and *entirely r-Ramsey complete* if this holds for every positive integer.

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If a_i are all distinct, can't guarantee the same colour everywhere. To see this, consider the 2-colouring χ where

$$\chi(i) = \begin{cases} 0 & \text{if } i \text{ is in } (2^{2^j} - 1, 2^{2^{j+1}}] \text{ with } j \text{ even,} \\ 1 & \text{if } i \text{ is in } (2^{2^j} - 1, 2^{2^{j+1}}] \text{ with } j \text{ odd.} \end{cases}$$

Theorem (Burr–Erdős, 1985)

There exists a constant C and an entirely 2-Ramsey complete sequence A such that, for all n,

$$|A \cap [n]| \leq C \log^3 n.$$

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Two problems of Burr and Erdős, reiterated by Erdős

- Improve these bounds. $100
- Extend to r-colour case. $250

New results

Theorem (C.–Fox)

For every integer $r \geq 2$, there exist $C = C(r)$ and an entirely Ramsey complete sequence A with

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Solves both problems at once.

Lower bound

In their paper, Burr and Erdős state that their proof for the lower bound is 'quite complicated' and because of the gap between their upper and lower bounds, they could not 'justify the effort' of reproducing their proof. Consequently, they only proved that

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There exists $c > 0$ such that there is no 2-Ramsey complete sequence with $|A \cap [n, 2n)| \leq c \log n$ for all large n.

We give a full proof of the lower bound, but I will only try to explain the result of Burr and Erdős here.

Lower bound

Again consider the 2-colouring χ where

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The rough idea then is that if there are only $c \log n$ numbers in each interval $[n, 2n)$, then sums of numbers from the interval $[1, 2^{2^j} - 1]$ can never make it as far as the interval $[2^{2^j + j}, 2^{2^j + j + 1}]$, while there are too few numbers above 2^{2^j} to cover the same interval.

Upper bound

Main Lemma

There is ϵ_0 such that the following holds for all $0 < \epsilon < \epsilon_0$: for all x sufficiently large, there is a set A of $\epsilon^{-3} \log x$ elements from $[x, 2x)$ such that, for any $A' \subseteq A$ with $|A'| \geq \epsilon |A|$, $\Sigma(A')$ contains $[y, 4y)$ for $y = 30\epsilon^{-3/2} x \log x$.

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Now, for some $\epsilon \leq 1/2$, choose such a set from $[2^j, 2^{j+1})$ for all j. Within each such set, half the elements are red or half are blue, meaning that we can cover the set $[y_j, 4y_j)$ with $y_j = 30\epsilon^{-3/2} j 2^j$ monochromatically. Since these intervals cover all sufficiently large positive integers, this proves the main result for $r = 2$.

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Clearly a similar proof works for any r.

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Doesn't work! Roughly half of the elements in a random set will be even and this subset cannot possibly cover the required interval (since the sum of even numbers is even).

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Second idea

Choose A randomly from the set of elements of $[x, 2x)$ with no small prime factors.

This avoids the previous issue and can be made work.

The Main Lemma

Say that a set of positive integers T is (p, x)-full if there is an interval $[z, z + x)$ such that

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Steps of the proof

1. Show that every $A' \subseteq A$ with $|A'| \geq \epsilon|A|$ contains ℓ sets $A_1, \ldots, A_\ell$, each (p, x)-full, where $p = \epsilon^{3/2}$ and $\ell = 10/p$.

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Construct the required sets greedily, adding elements from A' in order of index. At step i, there is a set A_j, called the *active* set, which is currently not full, but all sets $A_{j'}$ with $j' < j$ are full, and we consider whether or not to add a_i to A_j.

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Lev's lemma

Suppose that $\ell, q \geq 1$ and $n \geq 3$ are integers with $\ell \geq 2\lceil (q-1)/(n-2) \rceil$. If $S_1, \ldots, S_\ell$ are integer sets each having at least n elements, each a subset of an interval of at most $q+1$ integers, and none of which is a subset of an arithmetic progression of common difference greater than one, then $S_1 + \cdots + S_\ell$ contains an interval of at least $\ell(n-1)+1$ integers.

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More roughly, the sum of $10/\delta$ intervals of density δ contains an interval (provided they were not all arithmetic progressions with the same difference).

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If $y \leq 2x$ and $\Sigma((a_i)_{i=1}^k)$ contains an interval of length $2x$, say $[w, w+2x)$, then

$$\Sigma((a_i)_{i=1}^k \cup y) \supseteq [w, w+2x+y).$$

Therefore, with more choices for y, we can cover longer and longer intervals.

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Theorem (Graham, 1964)

Note that every polynomial $P : \mathbb{N} \to \mathbb{R}$ can be written in the form

$$P(x) = \sum_{i=0}^{k} \alpha_i \binom{x}{i}.$$

Then $\{P(m)\}_{m \geq 1}$ is complete if and only if

1. $\alpha_k > 0$,
2. $\alpha_i = p_i/q_i$ for integers p_i and q_i with $(p_i, q_i) = 1$,
3. $\gcd(p_0, p_1, \ldots, p_k) = 1$.

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Theorem (C.–Fox)

Suppose $\{P(m)\}_{m \geq 1}$ is a complete polynomial sequence. Then there is $C = C(P, r)$ and $A \subset \{P(m)\}_{m \geq 1}$ with

$$|A \cap [n]| \leq C \log^2 n$$

for all n such that A is r-Ramsey complete.

A density result

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However, ϵ-complete sequences do exist for all $\epsilon > 0$. For instance, a result of Szemerédi and Vu shows that any subsequence A of the primes with $|A \cap [n]| \geq C(\epsilon)\sqrt{n}$ is ϵ-complete.

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Problem

How sparse can an ϵ-complete sequence be?

A density result

Theorem (C.–Fox)

Let $F = (f_i)_{i \geq 1}$ be any sequence of positive integers for which $f_n = \sum_{i \leq \epsilon n} f_i$ for all sufficiently large n. Then every ϵ-complete sequence $A = (a_i)_{i \geq 1}$ must satisfy $a_i = O(f_i)$ and there is an ϵ-complete sequence with $a_i = \Theta(f_i)$.

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Corollary (C.–Fox)

There exists an ϵ-complete sequence A with

$$|A \cap [n]| \leq 2^{\sqrt{(2 \log_2(1/\epsilon) + o(1)) \log_2 n}}$$

and this is essentially best possible.

Complete but not Ramsey complete

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Then everything which is a sum of elements in colour 0 can be written with only 0s and 1s in base p, while everything in colour 1 (and hence everything which is a sum of elements in colour 1) is divisible by q. Together, these cannot hope to cover everything.

Complete but not Ramsey complete

Open problem

If p, q and r are pairwise coprime, then the sequence

$$\{p^i q^j r^k : i, j, k \geq 0\}$$

is complete but, by a similar argument to above, not 3-Ramsey complete. Is it 2-Ramsey complete?

The Ramsey–Waring problem

Open problem

Given natural numbers $r, k \geq 2$, does there exist $C = C(r, k)$ such that, for every r-colouring of the k^{th} powers, every natural number can be written as the sum of at most C k^{th} powers of the same colour?

Thank you for listening!