Ramsey theory of the universal homogeneous triangle-free graph

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Ramsey Theory in Logic, Combinatorics and Complexity

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Ramsey Theory, Small and Big

Property	**Examples**
$\mathcal{K}$ has Ramsey Property $\forall A \le B \in \mathcal{K}\ \forall k$, $\mathbf{K} \to (B)^A_k$	finite: linear orders, Boolean algebras finite ordered: graphs, hypergraphs, graphs omitting k-cliques
$\mathcal{K}$ has Small Ramsey Degrees $\forall A\ \exists t_{\mathcal{K}}(A)\ \forall B\ \forall k$, $\mathbf{K} \to (B)^A_{k,t_{\mathcal{K}}(A)}$	finite: graphs, hypergraphs graphs omitting k-cliques hypergraphs omitting irreducibles
$\mathcal{K}$ has Big Ramsey Degrees $\forall A\ \exists T_{\mathcal{K}}(A)\ \forall k$, $\mathbf{K} \to (\mathbf{K})^A_{k,T_{\mathcal{K}}(A)}$	the rationals, Rado graph, dense local order $\mathbf{S}(2)$, tournament $\mathbf{S}(3)$ $\mathbb{Q}_n$, $n \ge 2$.

$\mathcal{K}$ a Fraïssé class.

$\mathbf{K}$ = Fraïssé limit of $\mathcal{K}$, called a Fraïssé structure.

Missing Pieces: Forbidden Configurations

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The Problem: Lack of tools for representing such Fraïssé structures and lack of a viable Ramsey theory for such (non-existent) representations.

We address this lack of representations and techniques, starting with my submitted paper, *The Ramsey theory of the universal homogeneous triangle-free graph*, 48 pp.

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- Homogeneous hypergraphs with forbidden configurations (in progress)
- Homogeneous bowtie-free graph (in progress, with Hubička)

Connections with Topological Dynamics

Ramsey Theory	corr.	Topological Dynamics
Ramsey Property $\mathcal{K}$ an order class with RP	KPT $\longleftrightarrow$	Extreme Amenability G is EA
Small Ramsey Degrees precpct expansion $\mathcal{K}^*$ has RP	KPT/NVT $\longleftrightarrow$	Computation of UMF of G X^* is UMF of G
Big Ramsey Degrees $\mathbf{K}$ admits a big R-structure	Zucker $\longleftrightarrow$	Universal Completion Flow Big Ramsey Flow = UCF of G

$\mathcal{K}$ a Fraïssé class
$\mathbf{K} = \mathrm{Flim}(\mathcal{K})$
$G = \mathrm{Aut}(\mathbf{K})$

A Brief (incomplete) Intro to Graph Colorings

Example: Ordered graph A embeds into ordered graph B.

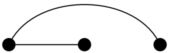


Figure: Ordered Graph A

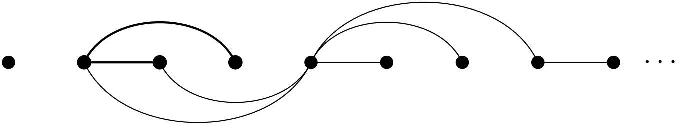
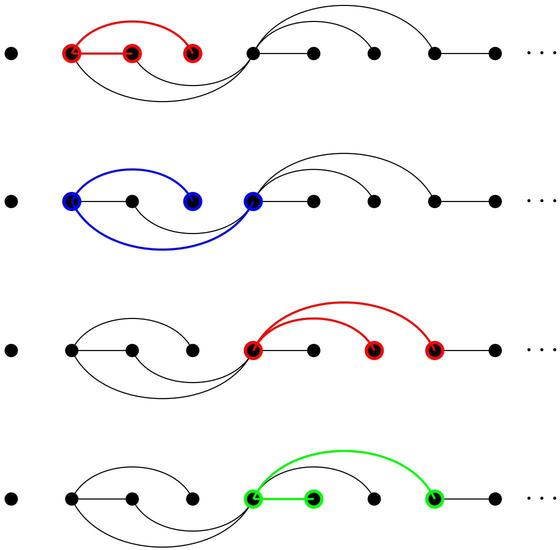


Figure: Ordered Graph B

Some copies of A in B



Thm. (Nešetřil/Rödl 1977 and Abramson/Harrington 1978)
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It follows that the class of finite graphs has small Ramsey degrees.

Case Study: Rado Graph

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• Actual degrees were found structurally in (LSV 2006) and computed in (J. Larson 2008).

Other Structures known to have big Ramsey degrees

• the natural numbers (Ramsey 1929)

• the rationals (Galvin, Laver, Devlin 1979)

• the Rado graph and similar binary relational structures (Sauer 2006)

• the countable ultrametric Urysohn space (Nguyen Van Thé 2008)

• the dense local order, circular tournament, $\mathbb{Q}_n$ (Laflamme, NVT, Sauer 2010).

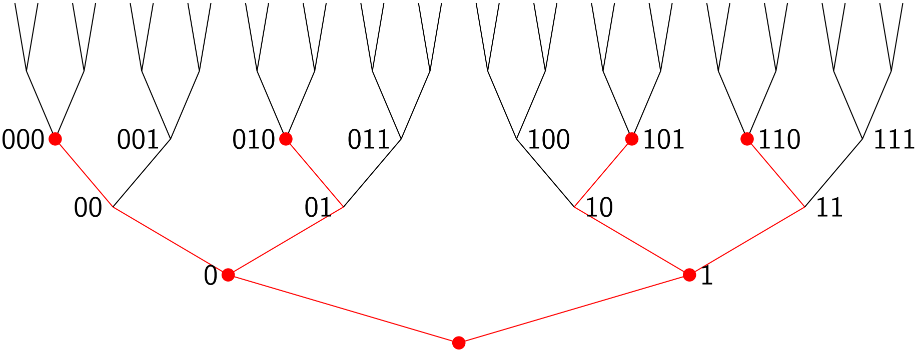
The crux of all but two of these proofs is a theorem of Milliken (or variant).

(The Urysohn space result uses Ramsey's Theorem.)

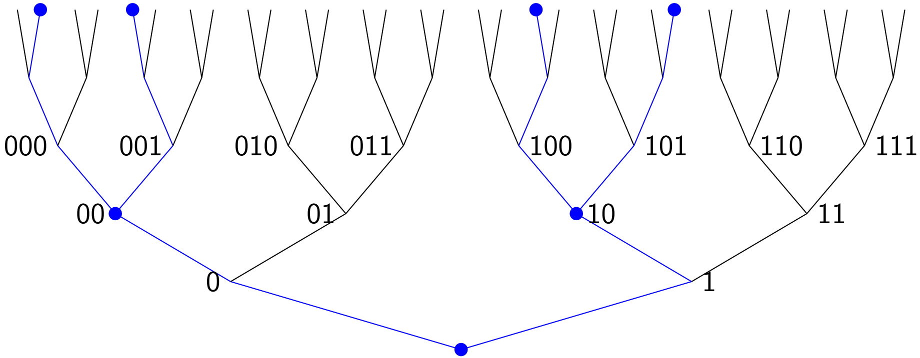
Strong Trees and Milliken's Theorem

A tree $T \subseteq 2^{<\omega}$ is a strong tree iff it is (strongly) isomorphic either to $2^{<\omega}$ or to $2^{\leq k}$ for some finite k.

Strong Subtree $\cong 2^{\leq 2}$, Ex. 1



Strong Subtree $\cong 2^{\leq 2}$, Ex. 2



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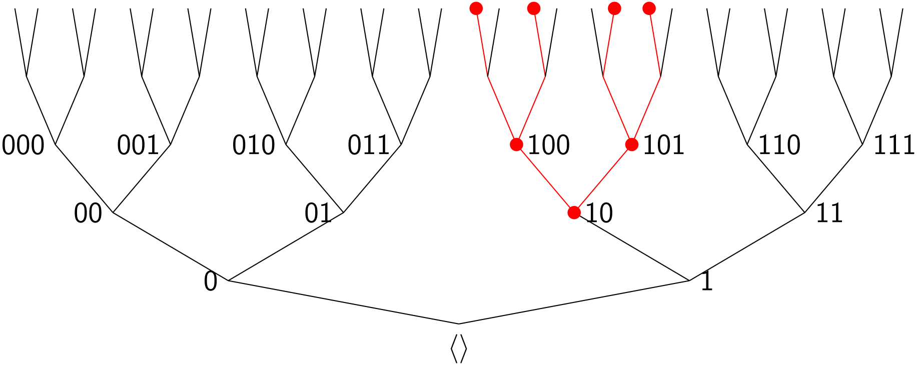

A Ramsey Theorem for Strong Trees

Thm. (Milliken 1979) Let $k \geq 0$, $l \geq 2$, and a coloring of all the subtrees of $2^{<\omega}$ which are isomorphic to $2^{\leq k}$ into l colors. Then there is an infinite strong subtree $S \subseteq 2^{<\omega}$ such that all copies of $2^{\leq k}$ in S have the same color.

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Thm. (Halpern-Läuchli 1966) Let $d \geq 1$, $l \geq 2$, and $T_i = 2^{<\omega}$ for $i < d$. Given a coloring of the product of level sets of the T_i into l colors,

$$f : \bigcup_{n<\omega} \prod_{i<d} T_i(n) \to l,$$

there are infinite strong trees $S_i \leq T_i$ and an infinite sets of levels $M \subseteq \omega$ where the splitting in S_i occurs, such that f is constant on $\bigcup_{m\in M} \prod_{i<d} S_i(m)$.

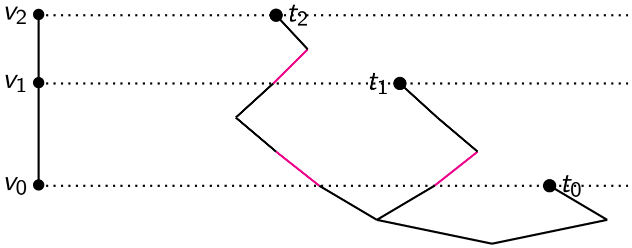
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

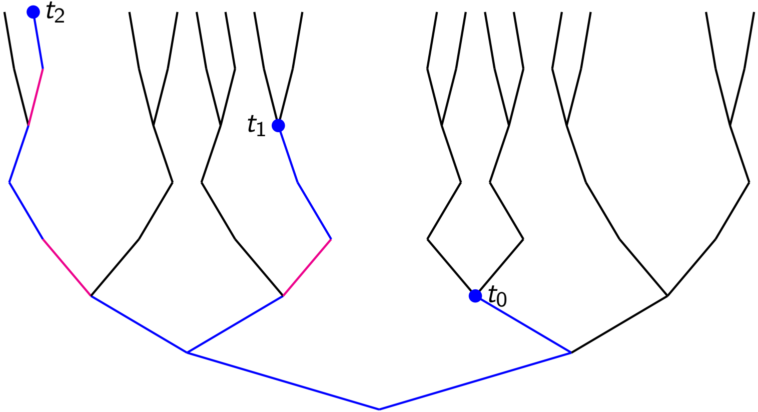
A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

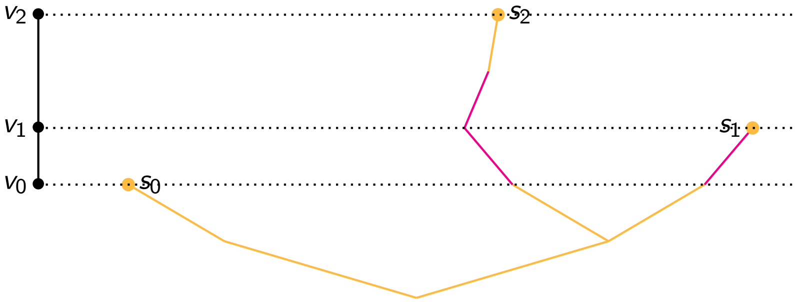
The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



A Strong Tree Envelope



A Different Antichain Coding a Path of Length 2



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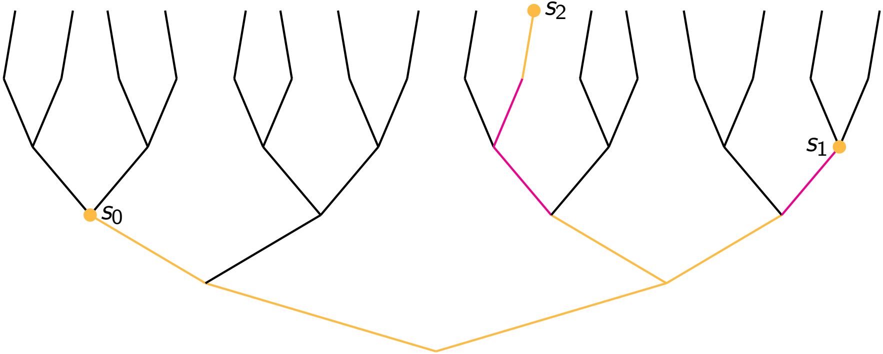

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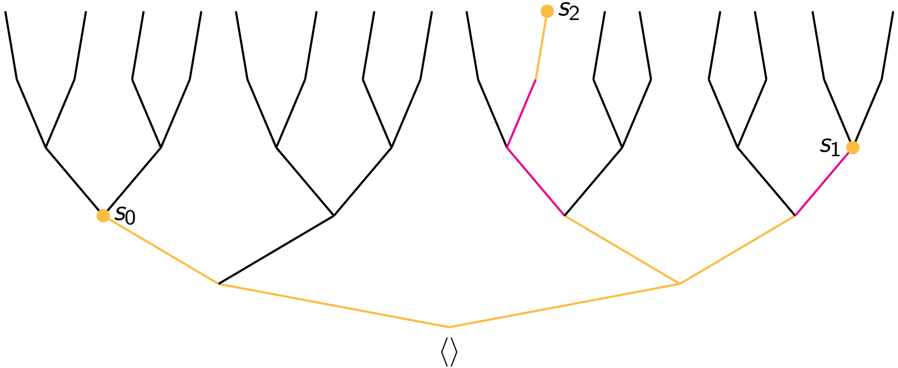

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6. Show that each type persists in each subgraph which is random to obtain exact numbers.

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"So far, the lack of tools to represent ultrahomogeneous structures is the major obstacle towards a better understanding of their infinite partition properties." (Nguyen Van Thé, Habilitation 2013)

Main Theorem: $\mathcal{H}_3$ has Finite Big Ramsey Degrees

Theorem. (D.) For each finite triangle-free graph A, there is a positive integer $T_{\mathcal{K}_3}(\mathrm{A})$ such that for any coloring of all copies of A in $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H} \leq \mathcal{H}_3$, again universal triangle-free, such that all copies of A in $\mathcal{H}$ take no more than $T_{\mathcal{K}_3}(\mathrm{A})$ colors.

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2016 Prague RT DocCourse - Nešetřil, Rödl, Hubička and company very kind about glitch I found. A few days later, I found the fix.

Structure of Proof: Three Main Parts

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II Prove a Ramsey Theorem for strictly similar finite antichains.

III Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_3$.

Part I: Strong Coding Trees

Idea: Want correct analogue of strong trees for setting of $\mathcal{H}_3$.

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Problem: How to make sure triangles are never encoded but branching is as thick as possible?

First Approach: Strong Triangle-Free Trees

• Use a unary predicate for distinguishing certain nodes to code vertices of a given graph (called coding nodes).

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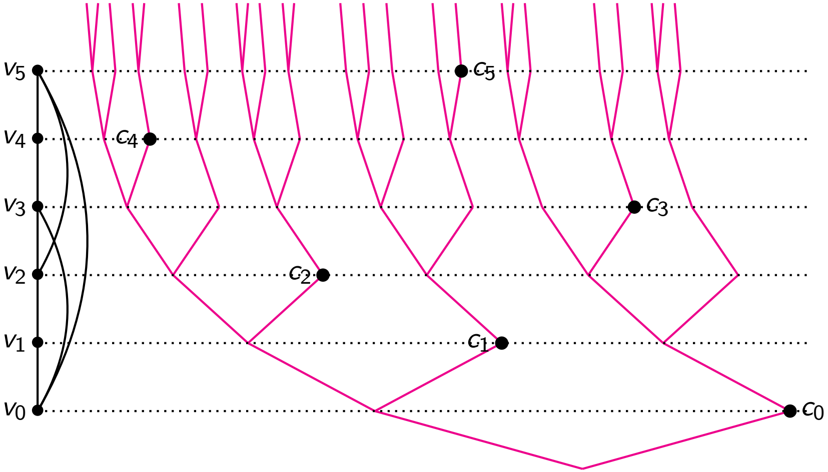
• Make a Branching Criterion so that a node s splits iff all its extensions will never code a triangle with coding nodes at or below the level of s.

Strong triangle-free tree $\mathbb{S}$



Almost sufficient

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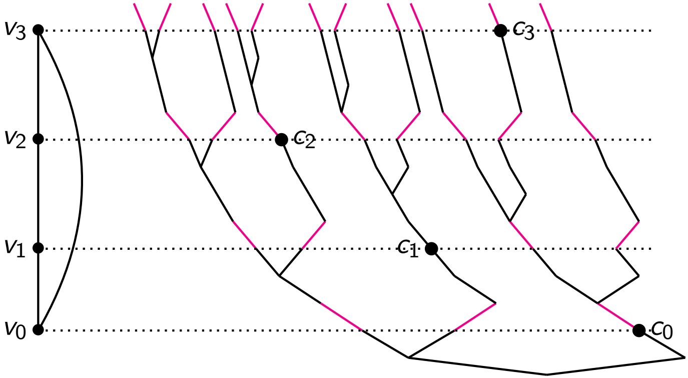
except for vertex colorings: there is a bad coloring of coding nodes.

Refined Approach: Strong coding tree $\mathbb{T}$



Skew the levels of interest.

The Space of Strong Coding Trees: $\mathcal{T}_3$

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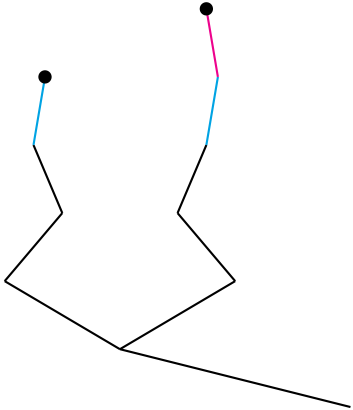
The space $\mathcal{T}_3$ of strong coding trees is very near a topological Ramsey space.

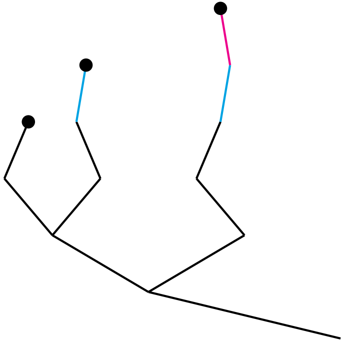
A subtree of $\mathbb{T}$ in which P1C fails

It has parallel 1's not witnessed by a coding node (P1C fails).



A subtree of $\mathbb{T}$ in which P1C holds

Its parallel 1's are witnessed by a coding node.



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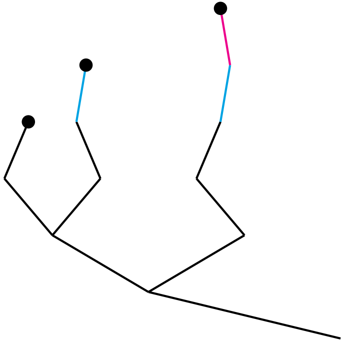

This gives the basic idea of P1C, though more subtleties are involved.

Part II: A Ramsey Theorem for Strictly Similar Finite Antichains.

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Idea: Strict similarity takes into account tree isomorphism and placements of coding nodes and new sets of parallel 1's.

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Idea: Strict similarity takes into account tree isomorphism and placements of coding nodes and new sets of parallel 1's.

It persists upon taking subtrees in $\mathcal{T}_3$.

Ramsey Theorem for Strong Coding Trees

Theorem. (D.) Let A be a finite subtree of a strong coding tree T, and let c be a coloring of all strictly similar copies of A in T.

Then there is a strong coding tree $S \leq T$ in which all strictly similar copies of A in S have the same color.

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Strict similarity is a strong version of isomorphism, and forms an equivalence relation.

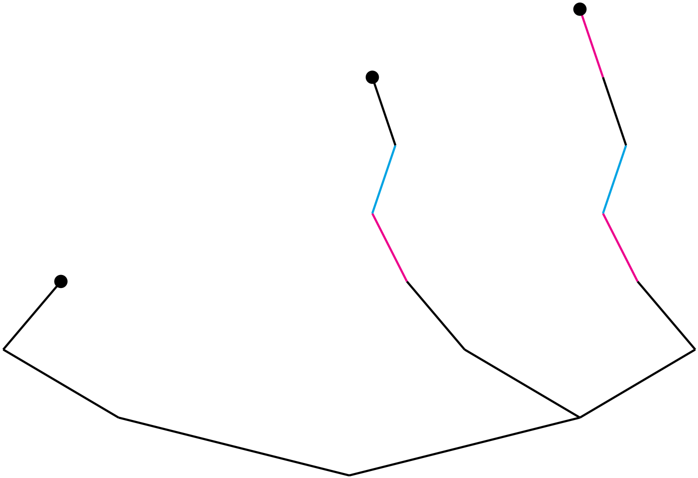
Some Examples of Strict Similarity Types

Let G be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding G.

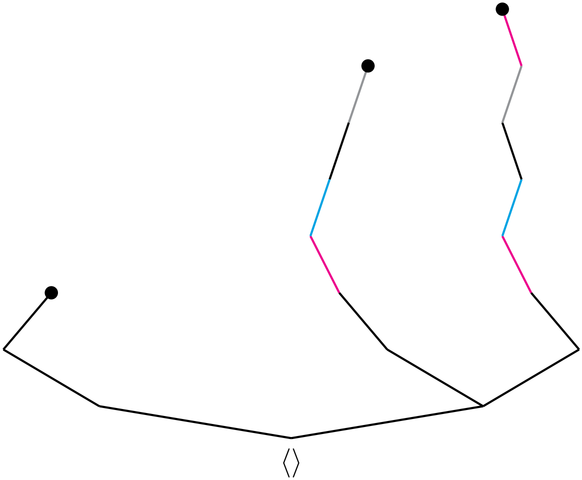
G a graph with three vertices and no edges

A tree A coding G - not P1C but still a valid strict similarity type



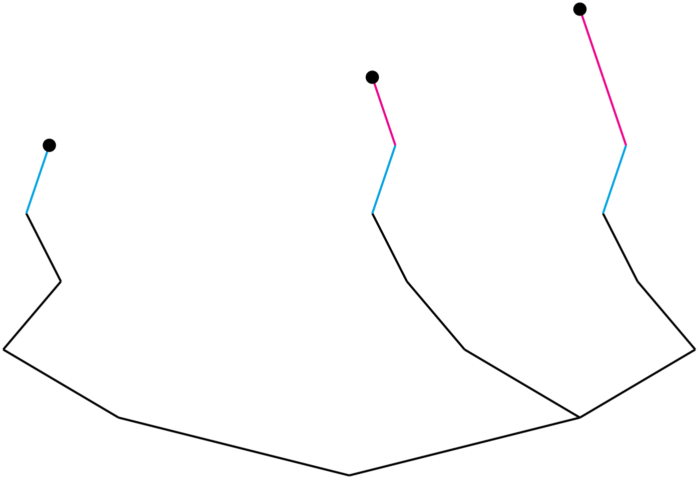
G a graph with three vertices and no edges

B codes G and is strictly similar to A.



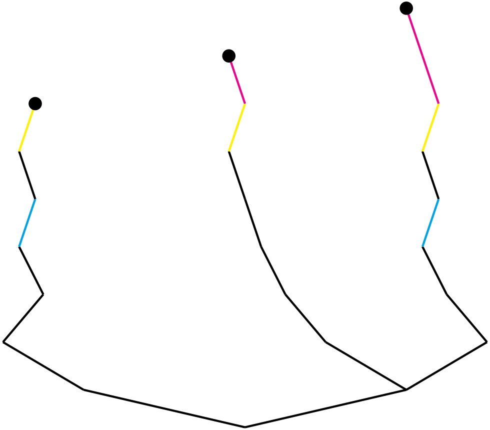
The tree C codes G

C is not strictly similar to A.

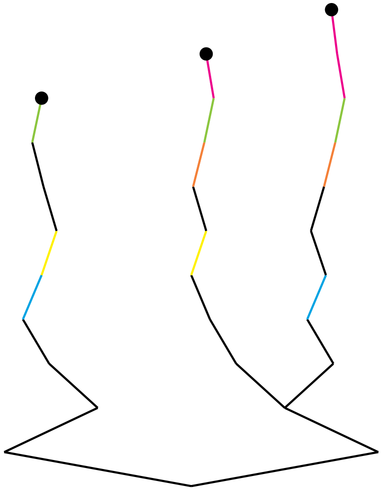


The tree D codes G

D is not strictly similar to either A or C.

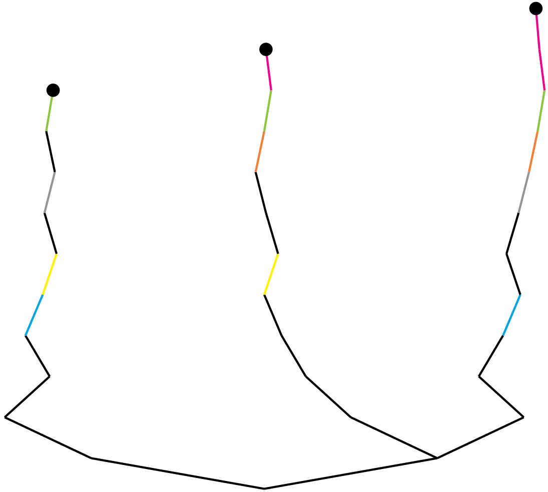


The tree E codes G and is not strictly similar to A - D



E is incremental. More on that later.

The tree F codes G and is strictly similar to E



F is also incremental.

Part III: Apply the Ramsey Theorem to Strictly Similarity Types of Antichains to obtain the Main Theorem.

Bounds for $T_{\mathcal{K}_3}(G)$

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6. Then f has no more colors on the copies of G in $\mathcal{H}'$ than the number of (incremental) strict similarity types of antichains coding G.

Reducing the Upper Bounds

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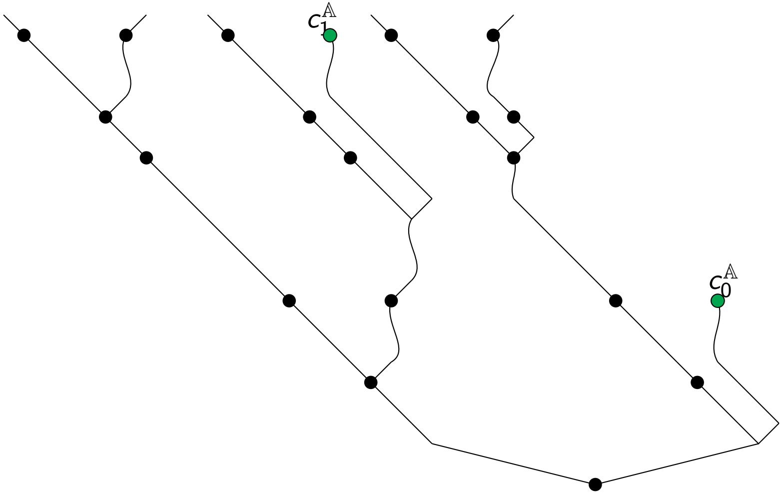
We can take S in the previous slide to be an incremental strong coding tree.

An antichain $\mathbb{A}$ of coding nodes of S coding $\mathcal{H}_3$



The tree minus the antichain of $c_n^{\mathbb{A}}$'s is isomorphic to $\mathbb{T}$.

Part II Expanded: Ideas behind the proof of the Ramsey Theorem for Strictly Similar Finite Trees

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(c) New notion of envelope.
  – Turns an antichain into a tree satisfying Strict P1C.

(a) Halpern-Läuchli-style Theorem

Thm. (D.) Given a strong coding tree T and

1. B a finite, valid strong coding subtree of T;
2. A a finite subtree of B with $\max(A) \subseteq \max(B)$; and
3. X a level set extending A into T with $A \cup X$ satisfying the P1C and valid in T.

Color all end-extensions Y of A in T for which $A \cup Y$ is strictly similar to $A \cup X$ into finitely many colors.

Then there is a strong coding tree $S \leq T$ end-extending B such that all level sets Y in S with $A \cup Y$ strictly similar to $A \cup X$ have the same color.

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Remark. The proof uses three different forcings and Harrington-style ideas. The forcings are best thought of as conducting unbounded searches for finite objects in ZFC.

(b) Ramsey Theorem for Finite Trees satisfying the SP1C

Thm. (D.) Let T be a strong coding tree, and let A be a finite valid subtree of T satisfying the Strict P1C. Suppose all the strictly similar copies of A in T are colored in finitely many colors.

Then there is a strong coding subtree $S \leq T$ such that all strictly similar copies of A in S have the same color.

A tree A satisfies the strict P1C if each new set of parallel 1's is witnessed by a coding node before anything else happens (other occurrences of new parallel 1's, splits, or coding nodes).

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Envelopes add some neutral coding nodes to a finite tree to make it satisfy the strict Parallel 1's Criterion.

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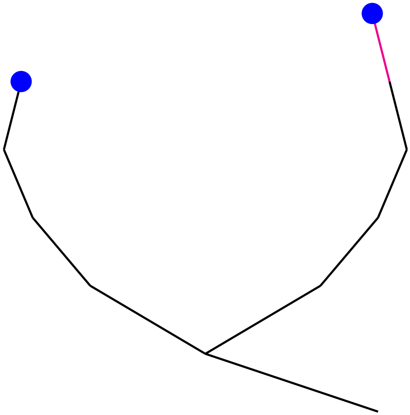
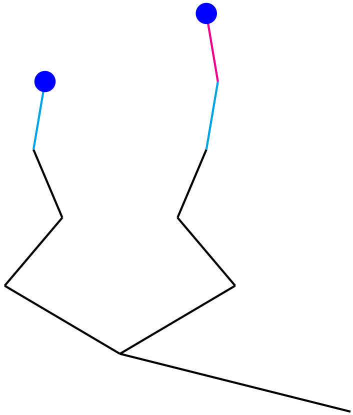
We now give some examples of envelopes.

H codes a non-edge



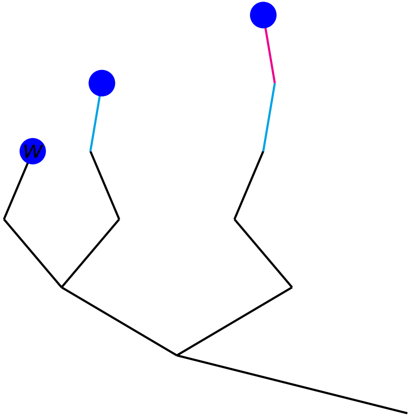
This satisfies the Strict Parallel 1's Criterion, so H is its own envelope.

I codes a non-edge



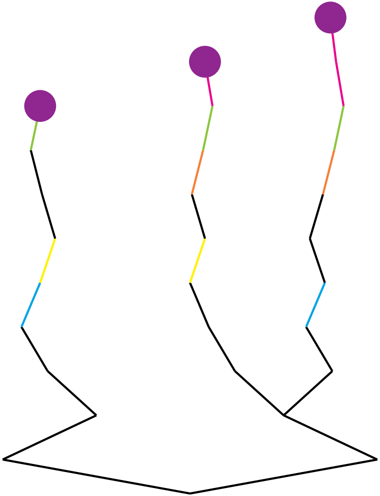
I does not satisfy the Parallel 1's Criterion.

An Envelope $\mathbf{E}(I)$

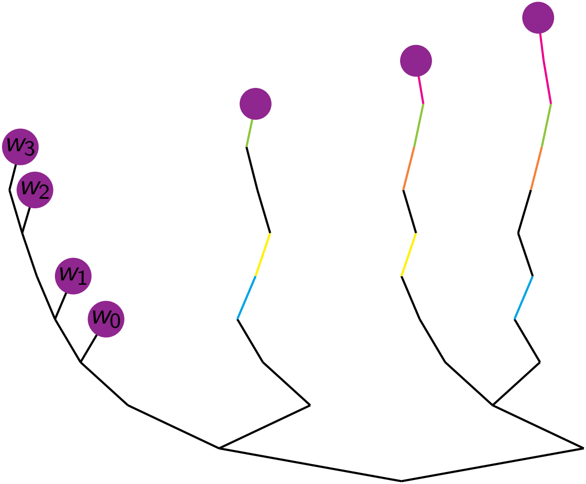


The witnessing coding node w is added to make an envelope.

The incremental tree E from before

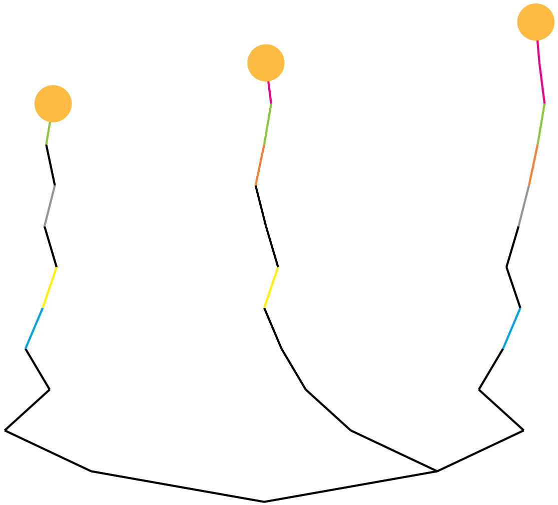


An envelope $\mathbf{E}(E)$

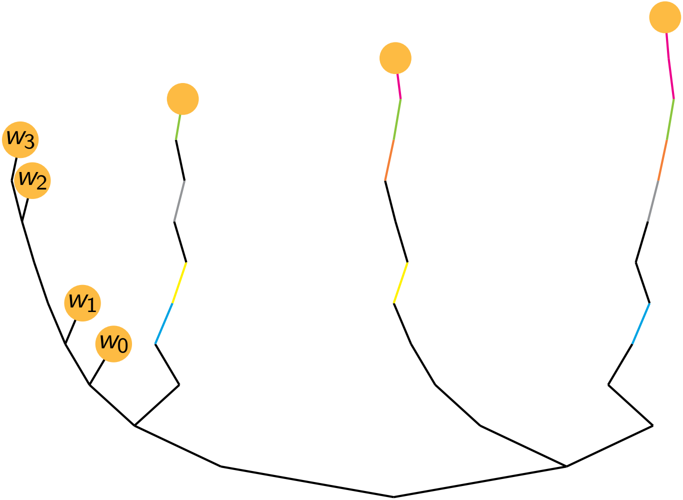


The witnessing coding nodes $w_1, \ldots, w_3$ make an envelope of E.

The tree F from before is strictly similar to E



$\mathbf{E}(F)$ is strictly similar to $\mathbf{E}(E)$



The witnessing coding nodes $w_0, \ldots, w_3$ make an envelope of F.

The Ramsey Thm for Strictly Similar Antichains follows

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3. Apply the Ramsey Theorem for Trees with the SP1C for g' on $\mathbb{T}$ to obtain $T \leq \mathbb{T}$ in which all copies of $E(A)$ have the same color.

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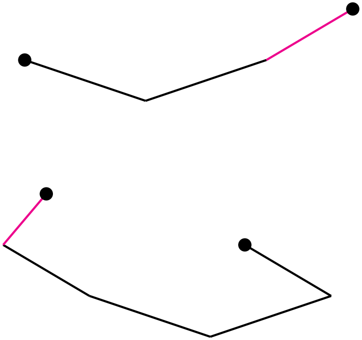
5. Then each copy of A in S has an envelop in T, by adding in some nodes from W.
6. Thus, each copy of A in S has the same color.

Proving the lower bounds in general for big Ramsey degrees of $\mathcal{H}_3$ is a work in progress.

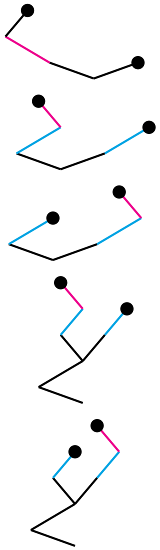
Big Ramsey degrees for edges and non-edges have been computed.

Edges have big Ramsey degree 2 in $\mathcal{H}_3$



These are their own envelopes.
$T_{\mathcal{H}_3}(Edge) = 2$ was obtained in (Sauer 1998) by different methods.

Non-edges have 5 Strict Similarity Types (D.)



Remarks

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To adapt these methods to other structures $\mathcal{S}$ with forbidden configurations, one needs to find the correct Branching Criteria, Extension Criteria guaranteeing a finite subtree can be extended inside a tree coding $\mathcal{S}$, and Ramsey theorems for relevant structures.

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II(a) HL - Case (i): level set X contains a splitting node

List the immediate successors of $\max(A)$ as $s_0, \dots, s_d$, where s_d denotes the node which the splitting node in X extends.

Let $T_i = \{t \in T : t \supseteq s_i\}$, for each $i \leq d$.

Fix κ large enough so that $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$ holds.

Such a κ is guaranteed in ZFC by a theorem of Erdős and Rado.

The forcing for Case (i)

$\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \restriction l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

(iii) ran(p) has no pre-determined new parallel 1's in T.

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$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \restriction l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

(iii) $\mathrm{ran}(p)$ has no pre-determined new parallel 1's in T.

$q \leq p$ if and only if $\vec{\delta}_q \supseteq \vec{\delta}_p$, $l_q \geq l_p$, and

(i) $q(d) \supset p(d)$, and $q(i, \delta) \supset p(i, \delta)$ for each $\delta \in \vec{\delta}_p$ and $i < d$; and

(ii) The set $\{q(i, \delta) : (i, \delta) \in d \times \vec{\delta}_p\} \cup \{q(d)\}$ has no new sets of parallel 1's above $\{p(i, \delta) : (i, \delta) \in d \times \vec{\delta}_p\} \cup \{p(d)\}$.

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(3) The assumption that $A \cup X$ satisfies the Parallel 1's Criterion is necessary.

The rest of II

II(a) Case (ii): level set X contains a coding node.

This case is more complex and requires preliminary forcings to obtain cone-homogeneity, an induction proof to construct a cone-homogeneous strong coding tree, and another forcing to obtain the Halpern-Läuchli style theorem.

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II(b): Ramsey Theorem for Finite Trees with Strict P1C.

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II(b): Ramsey Theorem for Finite Trees with Strict P1C.

This is obtained by induction using II(a).

II(c) was elaborated on already.