Combining extensions of the Hales-Jewett Theorem with Ramsey Theory in other structures

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and

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Theorem (Hales-Jewett). *If S_0 is finitely colored, then there exists $w \in S_1$ such that $\{w(a) : a \in \mathbb{A}\}$ is monochromatic.*

Basic information

I. Compact right topological semigroups. Let $(X, \cdot, \mathcal{T})$ be a compact Hausdorff right topological semigroup. That is, $(X, \cdot)$ is a semigroup, $(X, \mathcal{T})$ is a compact Hausdorff space, and for each $x \in X$, ρ_x is continuous, where $\rho_x(y) = y \cdot x$. As customary, we write xy for $x \cdot y$.

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(4) For $C \subseteq S$, $\overline{C} = \{p \in \beta S : C \in p\}$. The set $\{\overline{C} : C \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

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(11) If T is an ideal of S, then βT is an ideal of βS.

(12) If X is a compact Hausdorff right topological semigroup and $\varphi : S \to X$ is a homomorphism such that $\varphi[S] \subseteq \Lambda(X)$, then the continuous extension $\widetilde{\varphi} : \beta S \to X$ is a homomorphism. If φ is injective, so is $\widetilde{\varphi}$. If φ is surjective, so is $\widetilde{\varphi}$.

Theorem (Hales-Jewett). *Let D be a piecewise syndetic subset of S_0. There is some $w \in S_1$ such that $\{w(a) : a \in \mathbb{A}\} \subseteq D$. In particular, if S_0 is finitely colored, there is some $w \in S_1$ such that $\{w(a) : a \in \mathbb{A}\}$ is monochromatic.*

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Proof. (Blass) Since D is piecewise syndetic, pick $u \in S_0$ such that $u^{-1}D$ is central in S_0. Pick $p \in E\big(K(\beta S_0)\big)$ such that $u^{-1}D \in p$.

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Now $S_0 \cup S_1 = T_1$, so $\beta S_0 \cup \beta S_1 = \beta T_1$. Pick an idempotent $q \in K(\beta T_1)$ such that $q \leq p$.

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Since $u^{-1}D \in p$ we have by the continuity of $\widetilde{h}_a$ that $h_a^{-1}[u^{-1}D] \in q$ for each $a \in \mathbb{A}$. Pick

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Note that the above proof actually shows that $\{w \in S_1 : \{w(a) : a \in \mathbb{A}\} \subseteq D\}$ is piecewise syndetic in S_1. And if D is central in S_0, then $\{w \in S_1 : \{w(a) : a \in \mathbb{A}\} \subseteq D\}$ is central in S_1.

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Proof. To see that $\widetilde{\mu}(pq) = \widetilde{\mu}(q)$, that is $\widetilde{\mu} \circ \rho_q(p) = \widetilde{\mu}(q)$, it suffices that $\widetilde{\mu} \circ \rho_q$ is identically equal to $\widetilde{\mu}(q)$ on S_0, so let $u \in S_0$. Then $\widetilde{\mu} \circ \rho_q(u) = \widetilde{\mu}(uq) = \widetilde{\mu} \circ \lambda_u(q)$, so it suffices that $\widetilde{\mu} \circ \lambda_u$ and $\widetilde{\mu}$ agree on S_1. This is true because, for $w \in S_1$, $\mu(uw) = \mu(w)$.

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One shows in a similar fashion that $\widetilde{\mu}(qp) = \widetilde{\mu}(q)$, by showing that $\widetilde{\mu} \circ \rho_p$ and $\widetilde{\mu}$ agree on S_1. □

Theorem. *Let D be a piecewise syndetic subset of S_0 and let B be an IP-set in $(\mathbb{N},+)$. There exists $w \in S_1$ such that $\{w(a) : a \in \mathbb{A}\} \subseteq D$ and $\mu(w) \in B$. Thus, if S_0 is finitely colored and $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$ (think a thin sequence like $x_n = 2^{n!}$), there exist $w \in S_1$ and $F \in \mathcal{P}_f(\mathbb{N})$ such that $\{w(a) : a \in \mathbb{A}\}$ is monochromatic and $\mu(w) = \sum_{n \in F} x_n$.*

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Note that we cannot conclude from the proof that $\{w \in S_1 : \{w(a) : a \in \mathbb{A}\} \subseteq D \text{ and } \mu(w) \in B\}$ is piecewise syndetic in S_1 because q is not known to be minimal in βS_1. In fact, unless B is central in $\mathbb{N}$, it won't be.

The following corollary is known to be a consequence of the Central Sets Theorem.

Corollary. *Let* $k \in \mathbb{N}$, *let* B *be an IP-set in* $\mathbb{N}$, *and let* $\mathbb{N}$ *be finitely colored. There exist* $b \in \mathbb{N}$ *and* $d \in B$ *such that* $\{b, b+d, b+2d, \ldots, b+kd\}$ *is monochromatic.*

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Proof. Assume that φ is a finite coloring of $\mathbb{N}$. Let $\mathbb{A} = \{0, 1, \ldots, k\}$. For $u = l_1 l_2 \cdots l_m$ where each $l_i \in \mathbb{A}$, let $\tau(u) = \sum_{i=1}^{m} l_i$. Let $\psi = \varphi \circ \tau$. Then ψ is a finite coloring of S_0.

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The *extensions* of the Hales-Jewett Theorem from the title involve multiple variables.

Notation. Let $n \in \mathbb{N}$. Then T_n is the free semigroup over $\mathbb{A} \cup \{v_1, v_2, \ldots, v_n\}$ where $v_1, v_2, \ldots, v_n$ are distinct *variables* that are not members of $\mathbb{A}$ and $S_n = \{w \in T_n :$ for each $i \in \{1, 2, \ldots, n\}$, v_i occurs in $w\}$.

The following corollary is known to be a consequence of the Central Sets Theorem.

Corollary. *Let* $k \in \mathbb{N}$, *let* B *be an IP-set in* $\mathbb{N}$, *and let* $\mathbb{N}$ *be finitely colored. There exist* $b \in \mathbb{N}$ *and* $d \in B$ *such that* $\{b, b+d, b+2d, \ldots, b+kd\}$ *is monochromatic.*

Proof. Assume that φ is a finite coloring of $\mathbb{N}$. Let $\mathbb{A} = \{0, 1, \ldots, k\}$. For $u = l_1 l_2 \cdots l_m$ where each $l_i \in \mathbb{A}$, let $\tau(u) = \sum_{i=1}^{m} l_i$. Let $\psi = \varphi \circ \tau$. Then ψ is a finite coloring of S_0.

Pick $w \in S_1$ such that ψ is constant on $\{w(a) : a \in \mathbb{A}\}$ and $\mu(w) \in B$. Let $b = \tau\big(w(0)\big)$ and let $d = \mu(w)$. Then for $a \in \mathbb{A}$, $b + ad = \tau\big(w(a)\big)$ so $\varphi(b + ad) = \varphi\Big(\tau\big(w(a)\big)\Big) = \psi\big(w(a)\big)$. $\square$

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If $\vec{x} \in \mathbb{A}^n$ and $w \in S_n$, then $w(\vec{x})$ is the result of replacing each occurrence of v_i by x_i for $i \in \{1, 2, \ldots, n\}$. Given $i \in \{1, 2, \ldots, n\}$, and $w \in S_n$, $\mu_i(w)$ is the number of occurrences of v_i in w.

The full generality of the following theorem has M as an $m \times m$ upper triangular matrix with positive diagonal entries and entries below the diagonal less than or equal to 0 and $w \in S_n$ for $n \geq m$. I present it with $m = n = 2$.

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$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \textit{ is of the form } \begin{pmatrix} + & 0 \\ \leq 0 & + \end{pmatrix} \textit{ or } \begin{pmatrix} 0 & + \\ + & \leq 0 \end{pmatrix}.$$

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If S_0 is finitely colored and B_1 and B_2 are IP-sets in $\mathbb{N}$, then there exists $w \in S_2$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^2\}$ is monochromatic and $M \begin{pmatrix} \tau_1(w) \\ \tau_2(w) \end{pmatrix} \in B_1 \times B_2$.

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For example, assume $M = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, $\tau_1 = \mu_2$, $\tau_2 = \mu_1 - \mu_2$, $B_1 = FS(\langle 4^n \rangle_{n=1}^{\infty})$, and $B_2 = FS(\langle 7^n \rangle_{n=1}^{\infty})$. If S_0 is finitely colored, then there exists $w \in S_2$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^2\}$ is monochromatic and

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That is, there exist F and H in $\mathcal{P}_f(\mathbb{N})$ such that

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