On existence of Ramsey expansions

Jan Hubička

Department of Applied Mathematics
Charles University
Prague

Joint work with David Evans, Matěj Konečný and Jaroslav Nešetřil

Ramsey Theory in Logic, Combinatorics and Complexity 2018

Ramsey theorem for finite relational structures

Relational structures are graphs, digraphs, posets, . . .
Structures may also have functions (operations) in addition to relations (boolean algebras, groups,. . .)

Ramsey theorem for finite relational structures

Relational structures are graphs, digraphs, posets, . . .
Structures may also have functions (operations) in addition to relations (boolean algebras, groups,. . .)
Ordered structures are having relation $\leq$ which is a linear order on vertices.

Ramsey theorem for finite relational structures

Relational structures are graphs, digraphs, posets, . . .
Structures may also have functions (operations) in addition to relations (boolean algebras, groups,. . .)
Ordered structures are having relation $\leq$ which is a linear order on vertices.

$\overrightarrow{Rel}(L)$ is a class of all finite ordered relational structures in given language L.

Theorem (Nešetřil-Rödl, 1977; Abramson-Harrington, 1978)

$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Rel}(L)} \exists_{\mathbf{C}\in\overrightarrow{Rel}(L)} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Ramsey theorem for finite relational structures

Relational structures are graphs, digraphs, posets, . . .
Structures may also have functions (operations) in addition to relations (boolean algebras, groups,. . .)
Ordered structures are having relation $\leq$ which is a linear order on vertices.

$\overrightarrow{Rel}(L)$ is a class of all finite ordered relational structures in given language L.

Theorem (Nešetřil-Rödl, 1977; Abramson-Harrington, 1978)

$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Rel}(L)} \exists_{\mathbf{C}\in\overrightarrow{Rel}(L)} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all substructures of **B** isomorphic to **A**.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$: For every 2-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ is monochromatic.

Ramsey theorem for finite relational structures

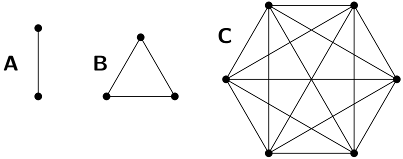
Relational structures are graphs, digraphs, posets, ...
Structures may also have functions (operations) in addition to relations (boolean algebras, groups,...)
Ordered structures are having relation $\leq$ which is a linear order on vertices.
$\overrightarrow{Rel}(L)$ is a class of all finite ordered relational structures in given language L.

Theorem (Nešetřil-Rödl, 1977; Abramson-Harrington, 1978)

$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Rel}(L)}\exists_{\mathbf{C}\in\overrightarrow{Rel}(L)} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all substructures of $\mathbf{B}$ isomorphic to $\mathbf{A}$.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$: For every 2-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ is monochromatic.



Ramsey theorem for finite relational structures

Relational structures are graphs, digraphs, posets, . . .
Structures may also have functions (operations) in addition to relations (boolean algebras, groups,. . .)
Ordered structures are having relation $\leq$ which is a linear order on vertices.

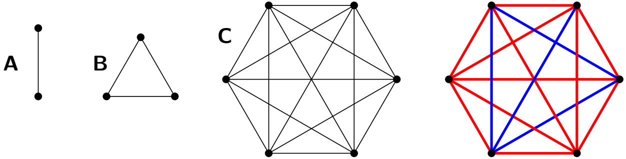
$\overrightarrow{Rel}(L)$ is a class of all finite ordered relational structures in given language L.

Theorem (Nešetřil-Rödl, 1977; Abramson-Harrington, 1978)

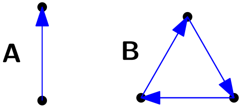
$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Rel}(L)} \exists_{\mathbf{C}\in\overrightarrow{Rel}(L)} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all substructures of **B** isomorphic to **A**.

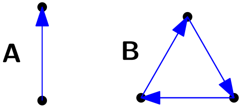
$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$: For every 2-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ is monochromatic.



Order is necessary

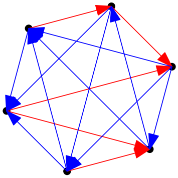


Order is necessary



Vertices of **C** can be linearly ordered and edges coloured accordingly:

- If edge is goes forward in linear order it is red
- blue otherwise.



Ramsey classes

Definition

A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

Ramsey classes

Definition

A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

Example (Linear orders — Ramsey Theorem, 1930)

The class of all finite linear orders is a Ramsey class.

Ramsey classes

Definition

A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

Example (Linear orders — Ramsey Theorem, 1930)

The class of all finite linear orders is a Ramsey class.

Example (Structures — Nešetřil-Rödl, 76; Abramson-Harrington, 78)

For every relational language L, $\overrightarrow{Rel}(L)$ is a Ramsey class.

Ramsey classes

Definition

A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

Example (Linear orders — Ramsey Theorem, 1930)

The class of all finite linear orders is a Ramsey class.

Example (Structures — Nešetřil-Rödl, 76; Abramson-Harrington, 78)

For every relational language L, $\overrightarrow{Rel}(L)$ is a Ramsey class.

Example (Partial orders — Nešetřil-Rödl, 84; Paoli-Trotter-Walker, 85)

The class of all finite partial orders with linear extension is Ramsey.

Ramsey classes

Definition

A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

Example (Linear orders — Ramsey Theorem, 1930)

The class of all finite linear orders is a Ramsey class.

Example (Structures — Nešetřil-Rödl, 76; Abramson-Harrington, 78)

For every relational language L, $\overrightarrow{Rel}(L)$ is a Ramsey class.

Example (Partial orders — Nešetřil-Rödl, 84; Paoli-Trotter-Walker, 85)

The class of all finite partial orders with linear extension is Ramsey.

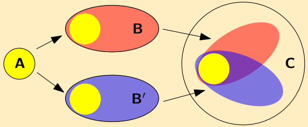
Example (Structures with functions — H.-Nešetřil, 2016)

For every language L, $\overrightarrow{Str}(L)$ is a Ramsey class.

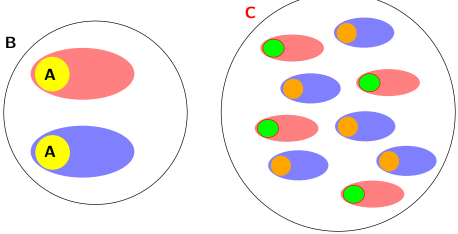
$\overrightarrow{Str}(L)$ = structures with functions and relations

Ramsey classes are amalgamation classes

Definition (Amalgamation)



Nešetřil, 80's: Under mild assumptions Ramsey classes have amalgamation property.



Nešetřil's Classification Programme, 2005

Classification Programme

$$\begin{array}{ccc}
\text{Ramsey classes} & \Longrightarrow & \text{amalgamation classes} \\
\Uparrow & & \Downarrow \\
\text{expansions of homogeneous} & \Longleftarrow & \text{homogeneous structures}
\end{array}$$

Nešetřil's Classification Programme, 2005

Classification Programme

$$\begin{array}{ccc}
\text{Ramsey classes} & \Longrightarrow & \text{amalgamation classes} \\
\Uparrow & & \Downarrow \\
\text{expansions of homogeneous} & \Longleftarrow & \text{homogeneous structures} \\
\Downarrow\Uparrow & & \Downarrow \\
\text{extremely amenable groups} & \Longrightarrow & \text{universal minimal flows}
\end{array}$$

Kechris, Pestov, Todorčevič: Fraïssé Limits, Ramsey Theory, and topological dynamics of automorphism groups (2005)

Nešetřil's Classification Programme, 2005

Classification Programme

$$\begin{array}{ccc} \text{Ramsey classes} & \Longrightarrow & \text{amalgamation classes} \\ \Uparrow & & \Downarrow \\ \text{expansions of homogeneous} & \Longleftarrow & \text{homogeneous structures} \\ \Downarrow\Uparrow & & \Downarrow \\ \text{extremely amenable groups} & \Longrightarrow & \text{universal minimal flows} \end{array}$$

Kechris, Pestov, Todorčevič: Fraïssé Limits, Ramsey Theory, and topological dynamics of automorphism groups (2005)

Definition

Let L' be language containing language L. A expansion (or lift) of L-structure $\mathbf{A}$ is L'-structure $\mathbf{A}'$ on the same vertex set such that all relations/functions in $L \cap L'$ are identical.

Nešetřil's Classification Programme, 2005

Classification Programme

Ramsey classes	$\Longrightarrow$	amalgamation classes
$\Uparrow$		$\Downarrow$
expansions of homogeneous	$\Longleftarrow$	homogeneous structures
$\Downarrow\Uparrow$		$\Downarrow$
extremely amenable groups	$\Longrightarrow$	universal minimal flows

Kechris, Pestov, Todorčevič: Fraïssé Limits, Ramsey Theory, and topological dynamics of automorphism groups (2005)

Definition

Let L' be language containing language L. A expansion (or lift) of L-structure $\mathbf{A}$ is L'-structure $\mathbf{A}'$ on the same vertex set such that all relations/functions in $L \cap L'$ are identical.

Theorem (Nešetřil, 1989)

All (countably infinite) homogeneous graphs have Ramsey expansion.

Proved using Lachlan—Woodrow catalogue of homogeneous graphs

Gower's Ramsey Theorem

Graham Rotschild Theorem: Parametric words

Milliken tree theorem: C-relations

Ramsey's theorem: rationals

Gower's Ramsey Theorem

Graham Rotschild Theorem: Parametric words

Milliken tree theorem: C-relations

Permutations

Equivalences

Ramsey's theorem: rationals

Product arguments

Gower's Ramsey Theorem

Graham Rotschild Theorem: Parametric words → Boolean algebras → Semilattices

Milliken tree theorem: C-relations

Permutations

Interval graphs

Equivalences → Cyclic orders

Ramsey's theorem: rationals

Product arguments   Interpretations

Gower's Ramsey Theorem

Graham Rotschild Theorem: Parametric words → Boolean algebras → Semilattices

Milliken tree theorem: C-relations

Permutations

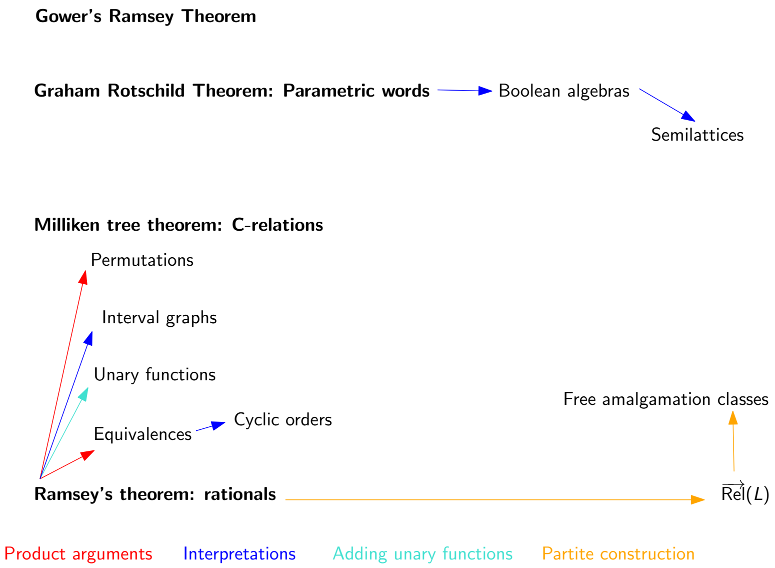
Interval graphs

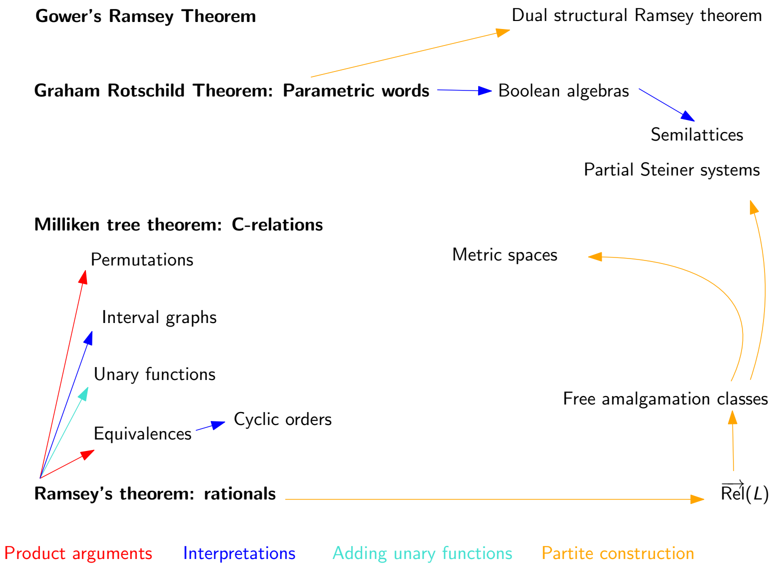
Unary functions

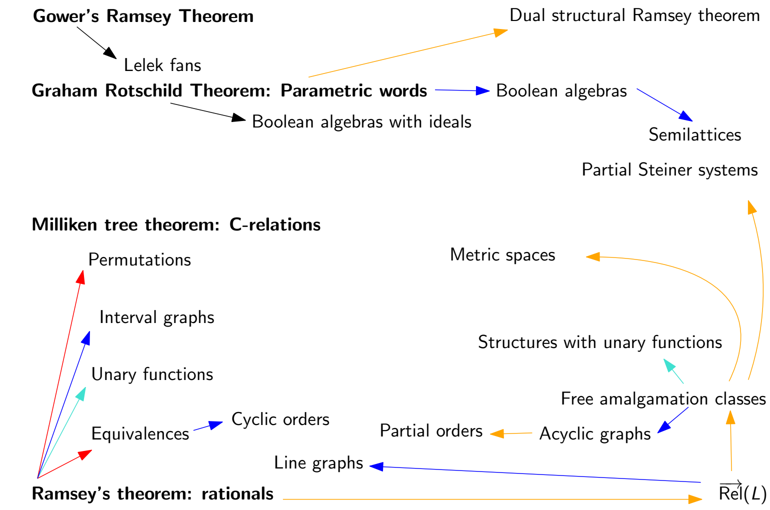
Equivalences → Cyclic orders

Ramsey's theorem: rationals

Product arguments   Interpretations   Adding unary functions


Gower's Ramsey Theorem
Graham Rotschild Theorem: Parametric words
Boolean algebras
Semilattices
Milliken tree theorem: C-relations
Permutations
Interval graphs
Unary functions
Equivalences
Cyclic orders
Free amalgamation classes
Ramsey's theorem: rationals
$\overrightarrow{\mathrm{Rel}}(L)$
Product arguments
Interpretations
Adding unary functions
Partite construction


Gower's Ramsey Theorem
Dual structural Ramsey theorem
Graham Rotschild Theorem: Parametric words
Boolean algebras
Semilattices
Partial Steiner systems
Milliken tree theorem: C-relations
Metric spaces
Permutations
Interval graphs
Unary functions
Free amalgamation classes
Equivalences
Cyclic orders
Ramsey's theorem: rationals
$\overrightarrow{\mathrm{Rel}}(L)$
Product arguments
Interpretations
Adding unary functions
Partite construction



Product arguments | Interpretations | Adding unary functions | Partite construction

The "Bertinoro 2011" question

Question (Bodirsky, Nešetřil, Nguyen Van Thé, Pinsker, Tsankov cca 2011)

Is there a Ramsey expansion for every amalgamation class?

Yes: extend language by infinitely many unary relations; assign every vertex to unique relation.

The "Bertinoro 2011" question

Question (Bodirsky, Nešetřil, Nguyen Van Thé, Pinsker, Tsankov cca 2011)

Is there a Ramsey expansion for every amalgamation class?

Yes: extend language by infinitely many unary relations; assign every vertex to unique relation.

Definition (Nguyen Van Thé)

Let $\mathcal{K}$ be class of L-structures and $\mathcal{K}'$ be class of expansions of $\mathcal{K}$.

- $\mathcal{K}$ is precompact if for every $\mathbf{A} \in \mathcal{K}$ there are only finitely many expansions of $\mathbf{A}$ in $\mathcal{K}'$.

The “Bertinoro 2011” question

Question (Bodirsky, Nešetřil, Nguyen Van Thé, Pinsker, Tsankov cca 2011)

Is there a Ramsey expansion for every amalgamation class?

Yes: extend language by infinitely many unary relations; assign every vertex to unique relation.

Definition (Nguyen Van Thé)

Let $\mathcal{K}$ be class of L-structures and $\mathcal{K}'$ be class of expansions of $\mathcal{K}$.

- $\mathcal{K}$ is precompact if for every $\mathbf{A} \in \mathcal{K}$ there are only finitely many expansions of $\mathbf{A}$ in $\mathcal{K}'$.
- $\mathcal{K}'$ has expansion property if for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ such that every expansion of $\mathbf{B}$ in $\mathcal{K}'$ contains every expansion of $\mathbf{A}$ in $\mathcal{K}'$.

The "Bertinoro 2011" question

Question (Bodirsky, Nešetřil, Nguyen Van Thé, Pinsker, Tsankov cca 2011)

Is there a Ramsey expansion for every amalgamation class?

Yes: extend language by infinitely many unary relations; assign every vertex to unique relation.

Definition (Nguyen Van Thé)

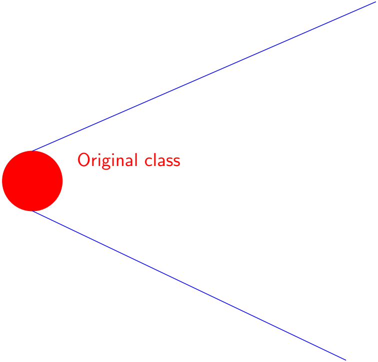
Let $\mathcal{K}$ be class of L-structures and $\mathcal{K}'$ be class of expansions of $\mathcal{K}$.

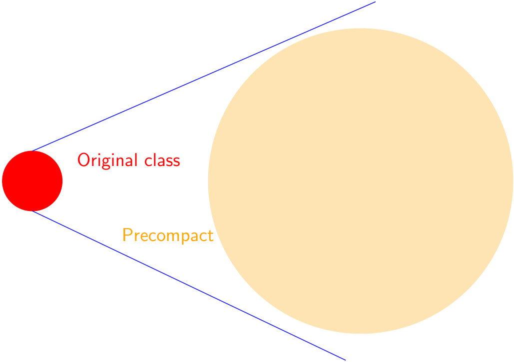
- $\mathcal{K}$ is precompact if for every $\mathbf{A} \in \mathcal{K}$ there are only finitely many expansions of $\mathbf{A}$ in $\mathcal{K}'$.
- $\mathcal{K}'$ has expansion property if for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ such that every expansion of $\mathbf{B}$ in $\mathcal{K}'$ contains every expansion of $\mathbf{A}$ in $\mathcal{K}'$.

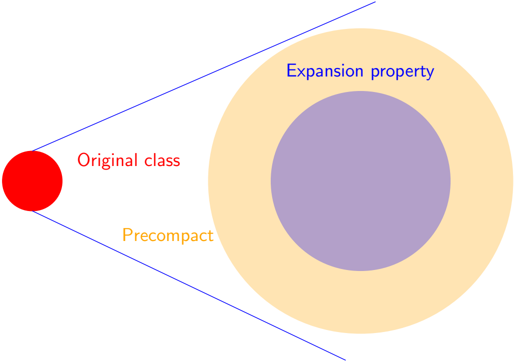
Theorem (Kechris, Pestov, Todorčevič 2005, Nguyen Van Thé 2012)

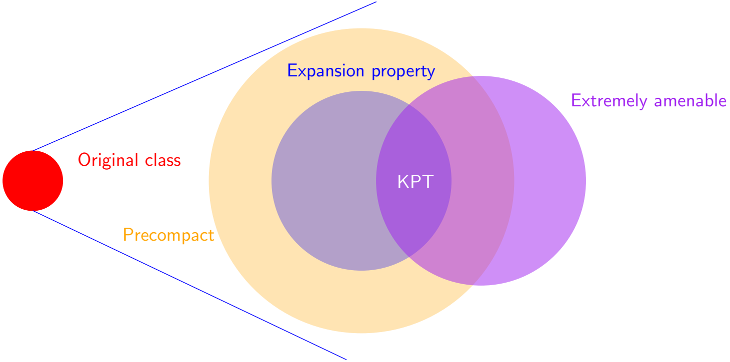
For every amalgamation class $\mathcal{K}$ there exists, up to bi-definability, at most one Ramsey class $\mathcal{K}'$ of expansions of $\mathcal{K}$ with expansion property.

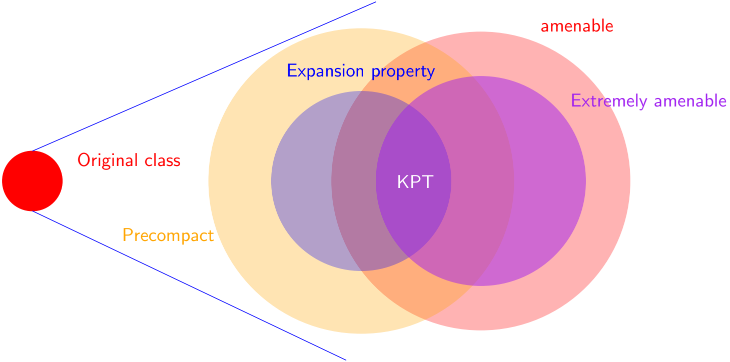

Original class

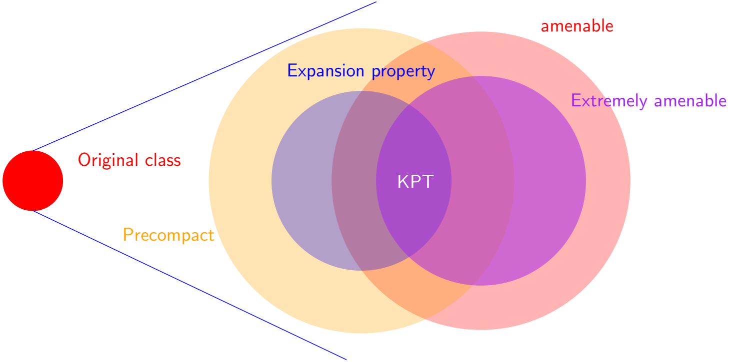

Original class


Original class
Precompact


Expansion property
Original class
Precompact


Expansion property
Extremely amenable
Original class
KPT
Precompact


amenable
Expansion property
Extremely amenable
Original class
KPT
Precompact

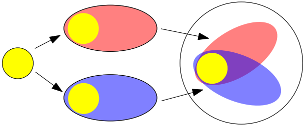


Theorem (Jasiński,Laflamme,Nguyen Van Thé,Woodrow, 2014)

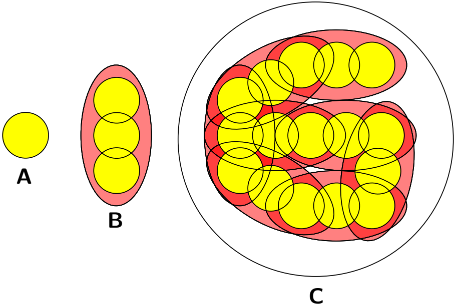
All homogeneous digraphs have precompact Ramsey lift with expansion property.

Proved case by case using Cherlin's catalogue

Why Ramsey objects are hard to construct?



The Nešetřil-Rödl partite construction of Ramsey object demands more complicated (multi)amalgamations.



Structural condition

Theorem (H.-Nešetřil, 2016)

Let L be language with relations and (partial) functions. Let $\mathcal{R}$ be a Ramsey class of irreducible finite structures and let $\mathcal{K}$ be a strong amalgamation subclass of $\mathcal{R}$. If $\mathcal{K}$ is locally finite subclass of $\mathcal{R}$ then $\mathcal{K}$ is Ramsey.

Structural condition

Theorem (H.-Nešetřil, 2016)

Let L be language with relations and (partial) functions. Let $\mathcal{R}$ be a Ramsey class of irreducible finite structures and let $\mathcal{K}$ be a strong amalgamation subclass of $\mathcal{R}$. If $\mathcal{K}$ is locally finite subclass of $\mathcal{R}$ then $\mathcal{K}$ is Ramsey.

Schematically

Recall:

$$\begin{array}{ccc} \text{Ramsey classes} & \Longrightarrow & \text{amalgamation classes} \\ \Uparrow & & \Downarrow \\ \text{expansions of homogeneous} & \Longleftarrow & \text{homogeneous structures} \end{array}$$

Structural condition

Theorem (H.-Nešetřil, 2016)

Let L be language with relations and (partial) functions. Let $\mathcal{R}$ be a Ramsey class of irreducible finite structures and let $\mathcal{K}$ be a strong amalgamation subclass of $\mathcal{R}$. If $\mathcal{K}$ is locally finite subclass of $\mathcal{R}$ then $\mathcal{K}$ is Ramsey.

Schematically

Recall:

$$\begin{array}{ccc} \text{Ramsey classes} & \Longrightarrow & \text{amalgamation classes} \\ \Uparrow & & \Downarrow \\ \text{expansions of homogeneous} & \Longleftarrow & \text{homogeneous structures} \end{array}$$

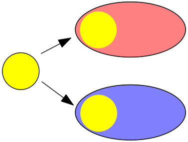
We get:

strong amalgamation + order + local finiteness $\Longrightarrow$ Ramsey

What is local finiteness?

Multiamalgams as structures with holes

Representing multiamalgams as “completion of structures with holes”:



An L-structure **A** is irreducible if it can not be created as a free amalgamation of its two proper substructures.

Amalgamation of irreducible structures is

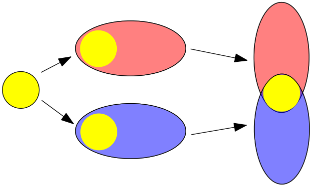
1. free amalgamation,
2. completion.

Definition

Irreducible structure $\mathbf{C}'$ is a completion of $\mathbf{C}$ if it has the same vertex set and every irreducible substructure of $\mathbf{C}$ is also (induced) substructure of $\mathbf{C}'$.

Multiamalgams as structures with holes

Representing multiamalgams as “completion of structures with holes”:



An L-structure **A** is irreducible if it can not be created as a free amalgamation of its two proper substructures.

Amalgamation of irreducible structures is

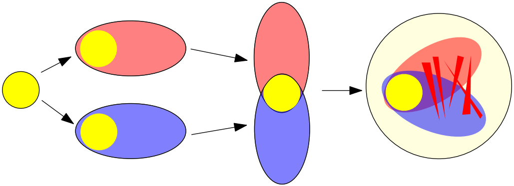
1. free amalgamation,
2. completion.

Definition

Irreducible structure $\mathbf{C}'$ is a completion of $\mathbf{C}$ if it has the same vertex set and every irreducible substructure of $\mathbf{C}$ is also (induced) substructure of $\mathbf{C}'$.

Multiamalgams as structures with holes

Representing multiamalgams as “completion of structures with holes”:



An L-structure $\mathbf{A}$ is irreducible if it can not be created as a free amalgamation of its two proper substructures.

Amalgamation of irreducible structures is

1. free amalgamation,
2. completion.

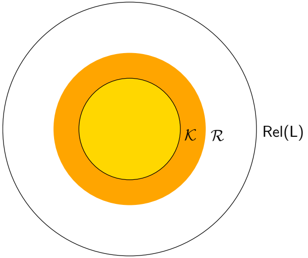
Definition

Irreducible structure $\mathbf{C}'$ is a completion of $\mathbf{C}$ if it has the same vertex set and every irreducible substructure of $\mathbf{C}$ is also (induced) substructure of $\mathbf{C}'$.

Multiamalgamation which locally is amalgamation

Intuition

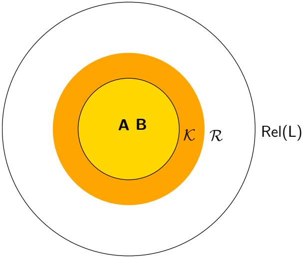
$\mathcal{K}$ is locally finite subclass of (Ramsey class) $\mathcal{R}$ if for every $\mathbf{C}_0$ in $\mathcal{R}$ there exists a finite bound on size of minimal obstacles which prevents a structure with homomorphism to $\mathbf{C}_0$ from being completed to $\mathcal{K}$.



Multiamalgamation which locally is amalgamation

Intuition

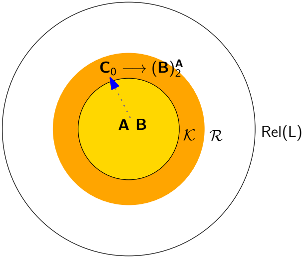
$\mathcal{K}$ is locally finite subclass of (Ramsey class) $\mathcal{R}$ if for every $\mathbf{C}_0$ in $\mathcal{R}$ there exists a finite bound on size of minimal obstacles which prevents a structure with homomorphism to $\mathbf{C}_0$ from being completed to $\mathcal{K}$.



Multiamalgamation which locally is amalgamation

Intuition

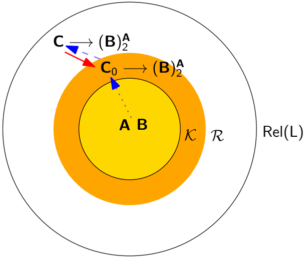
$\mathcal{K}$ is locally finite subclass of (Ramsey class) $\mathcal{R}$ if for every $\mathbf{C}_0$ in $\mathcal{R}$ there exists a finite bound on size of minimal obstacles which prevents a structure with homomorphism to $\mathbf{C}_0$ from being completed to $\mathcal{K}$.



Multiamalgamation which locally is amalgamation

Intuition

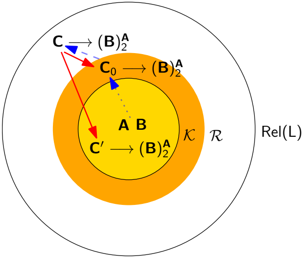
$\mathcal{K}$ is locally finite subclass of (Ramsey class) $\mathcal{R}$ if for every $\mathbf{C}_0$ in $\mathcal{R}$ there exists a finite bound on size of minimal obstacles which prevents a structure with homomorphism to $\mathbf{C}_0$ from being completed to $\mathcal{K}$.



Multiamalgamation which locally is amalgamation

Intuition

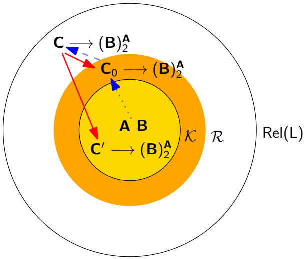
$\mathcal{K}$ is locally finite subclass of (Ramsey class) $\mathcal{R}$ if for every $\mathbf{C}_0$ in $\mathcal{R}$ there exists a finite bound on size of minimal obstacles which prevents a structure with homomorphism to $\mathbf{C}_0$ from being completed to $\mathcal{K}$.



Multiamalgamation which locally is amalgamation

Intuition

$\mathcal{K}$ is locally finite subclass of (Ramsey class) $\mathcal{R}$ if for every $\mathbf{C}_0$ in $\mathcal{R}$ there exists a finite bound on size of minimal obstacles which prevents a structure with homomorphism to $\mathbf{C}_0$ from being completed to $\mathcal{K}$.

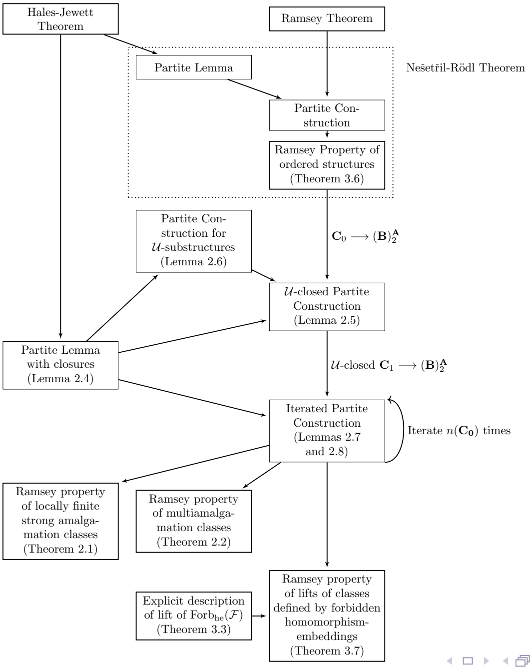


Definition

Let $\mathcal{R}$ be a class of finite irreducible structures and $\mathcal{K}$ a subclass of $\mathcal{R}$. We say that the class $\mathcal{K}$ is locally finite subclass of $\mathcal{R}$ if for every $\mathbf{C}_0 \in \mathcal{R}$ there is $n = n(\mathbf{C}_0)$ such that every structure $\mathbf{C}$ has completion in $\mathcal{K}$ providing that it satisfies the following:

1. there is a homomorphism-embedding from $\mathbf{C}$ to $\mathbf{C}_0$
2. every substructure of $\mathbf{C}$ with at most n vertices has a completion in $\mathcal{K}$.

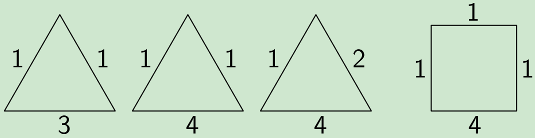
homomorphism-embedding is a homomorphism which is an embedding on every irreducible substructure.


Hales-Jewett Theorem
Ramsey Theorem
Partite Lemma
Nešetřil-Rödl Theorem
Partite Construction
Ramsey Property of ordered structures (Theorem 3.6)
Partite Construction for $\mathcal{U}$-substructures (Lemma 2.6)
$\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$
$\mathcal{U}$-closed Partite Construction (Lemma 2.5)
Partite Lemma with closures (Lemma 2.4)
$\mathcal{U}$-closed $\mathbf{C}_1 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$
Iterated Partite Construction (Lemmas 2.7 and 2.8)
Iterate $n(\mathbf{C_0})$ times
Ramsey property of locally finite strong amalgamation classes (Theorem 2.1)
Ramsey property of multiamalgamation classes (Theorem 2.2)
Explicit description of lift of $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ (Theorem 3.3)
Ramsey property of lifts of classes defined by forbidden homomorphism-embeddings (Theorem 3.7)

Locally finite subclass, an example

Example

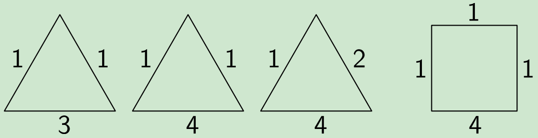
Consider class of metric spaces with distances $\{1, 2, 3, 4\}$. Graph with edges labelled by $\{1, 2, 3, 4\}$ can be completed to a metric space if and only if it does not contain one of:



Locally finite subclass, an example

Example

Consider class of metric spaces with distances $\{1, 2, 3, 4\}$. Graph with edges labelled by $\{1, 2, 3, 4\}$ can be completed to a metric space if and only if it does not contain one of:

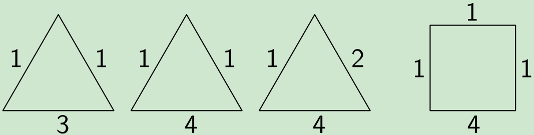


The class $\overrightarrow{\mathcal{M}}_k$ of all ordered metric spaces with integer distances at most k is Ramsey.

Locally finite subclass, an example

Example

Consider class of metric spaces with distances $\{1,2,3,4\}$. Graph with edges labelled by $\{1,2,3,4\}$ can be completed to a metric space if and only if it does not contain one of:



The class $\overrightarrow{\mathcal{M}}_k$ of all ordered metric spaces with integer distances at most k is Ramsey.

Theorem (Nešetřil, 2007)

The class $\overrightarrow{\mathcal{M}}_{\mathbb{Q}}$ of all metric spaces with rational distances is Ramsey.

Generalisations

Theorem (Aranda, H., Hng, Karamanlis, Kompatscher, Konečný, Pawliuk, Bradley-Williams, 2017)

All known metrically homogeneous graphs have precompact Ramsey expansion with expansion property.

Generalisations

Theorem (Aranda, H., Hng, Karamanlis, Kompatscher, Konečný, Pawliuk, Bradley-Williams, 2017)

All known metrically homogeneous graphs have precompact Ramsey expansion with expansion property.

Generalisations

Theorem (Aranda, H., Hng, Karamanlis, Kompatscher, Konečný, Pawliuk, Bradley-Williams, 2017)

All known metrically homogeneous graphs have precompact Ramsey expansion with expansion property.

Theorem (H.,Konečný, Nešetřil, 2017)

Conant's generalized metric spaces have Ramsey expansion.

Generalisations

Theorem (Aranda, H., Hng, Karamanlis, Kompatscher, Konečný, Pawliuk, Bradley-Williams, 2017)

All known metrically homogeneous graphs have precompact Ramsey expansion with expansion property.

Theorem (H.,Konečný, Nešetřil, 2017)

Conant's generalized metric spaces have Ramsey expansion.

Theorem (Branunfeld, 2017)

Λ-ultrametric spaces have Ramsey expansion.

Generalisations

Theorem (Aranda, H., Hng, Karamanlis, Kompatscher, Konečný, Pawliuk, Bradley-Williams, 2017)

All known metrically homogeneous graphs have precompact Ramsey expansion with expansion property.

Theorem (H.,Konečný, Nešetřil, 2017)

Conant's generalized metric spaces have Ramsey expansion.

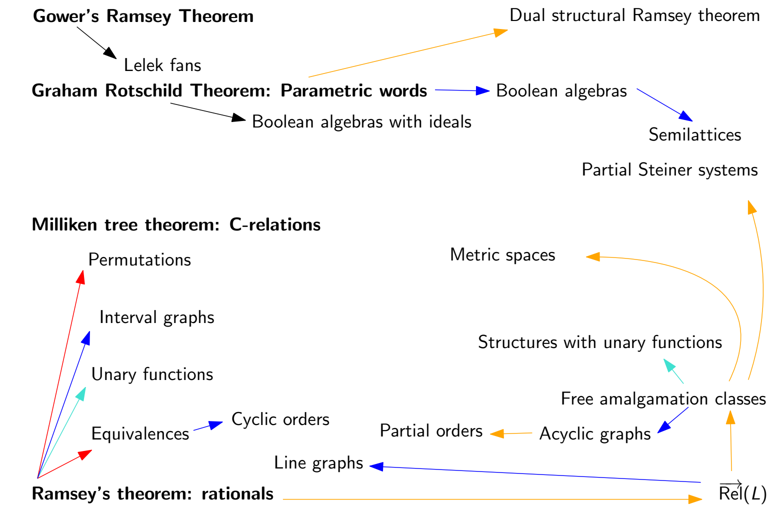
Theorem (Branunfeld, 2017)

Λ-ultrametric spaces have Ramsey expansion.

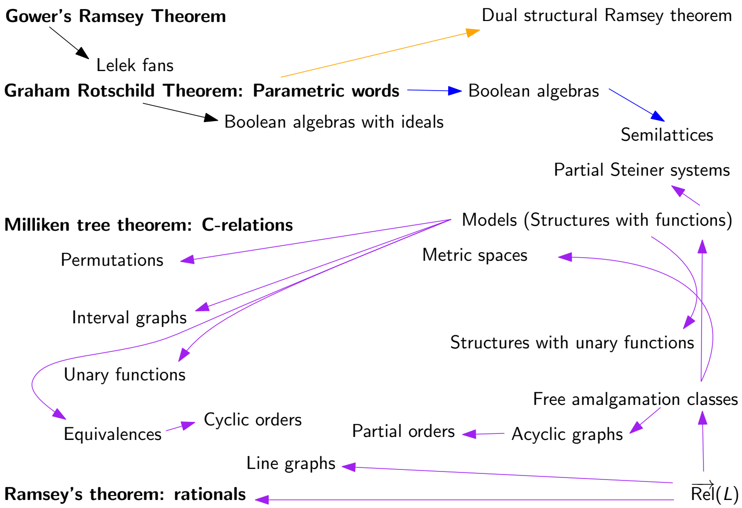
Theorem (H.,Konečný, Nešetřil, 2018+)

Semigroup-valued metric spaces omitting disobedient cycles have Ramsey expansion.

1. Common generalisation of all known symmetric binary Ramsey classes with strong amalgamation
2. May cover all homogeneous symmetric binary structures in the finite language

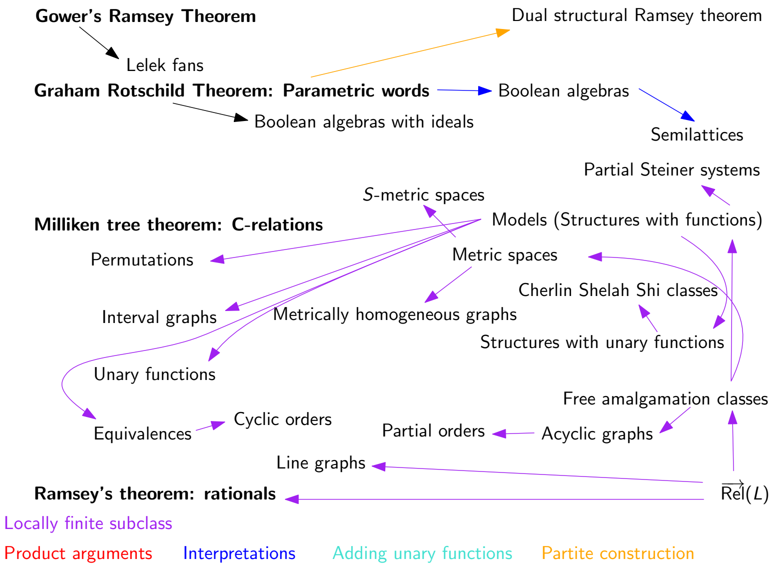


Product arguments Interpretations Adding unary functions Partite construction



Locally finite subclass

Product arguments | Interpretations | Adding unary functions | Partite construction



Gower's Ramsey Theorem
Dual structural Ramsey theorem
Lelek fans
Graham Rotschild Theorem: Parametric words
Boolean algebras
Boolean algebras with ideals
Semilattices
Partial Steiner systems
S-metric spaces
Milliken tree theorem: C-relations
Models (Structures with functions)
Permutations
Metric spaces
Cherlin Shelah Shi classes
Interval graphs
Metrically homogeneous graphs
Structures with unary functions
Unary functions
Free amalgamation classes
Cyclic orders
Partial orders
Acyclic graphs
Equivalences
Line graphs
Ramsey's theorem: rationals
$\overrightarrow{\mathrm{Rel}}(L)$
Locally finite subclass
Product arguments
Interpretations
Adding unary functions
Partite construction

Open problems and future work

Amalgamation classes where Ramsey expansion is not known:

1. Graphs omitting (induced or non-induced) 4-cycles $\implies$ Rank 3 matroids
2. Steiner systems omitting short odd cycles and/or 4-cycle
3. Affinely independent Euclidian metric spaces
4. "Dual-type" structures, such as finite measure algebras

Dual (projective) variant of our main theorem is work in progress.

Extension property for partial automorphisms is implied by local finiteness + automorphism preserving completion.

Negative result

Theorem (Evans, 2015+)

There is a countable, ω-categorical structure $\mathbf{M}_F$ no precompact Ramsey expansion.

Counter-example was given by Hrushovski construction.

Three variants of David's example

- $\mathcal{C}_0$: The easy example
- $\mathcal{C}_1$: The kindergarten example
- $\mathcal{C}_F$: The actual counter-example

Hrushovski construction

- Predimension of a graph $\mathbf{G} = (V, E)$ is

$$\delta(G) = 2|V| - |E|.$$

Example

$\delta(K_1) = 2$ $\quad$ $\delta(K_2) = 4 - 1 = 3$ $\quad$ $\delta(K_3) = 6 - 3 = 3$

$\delta(K_4) = 8 - 6 = 2$ $\quad$ $\delta(K_5) = 10 - 10 = 0$ $\quad$ $\delta(K_6) = 12 - 30 = -18.$

Hrushovski construction

- Predimension of a graph $\mathbf{G} = (V, E)$ is
$$\delta(G) = 2|V| - |E|.$$

Example

$\delta(K_1) = 2$ $\quad$ $\delta(K_2) = 4 - 1 = 3$ $\quad$ $\delta(K_3) = 6 - 3 = 3$
$\delta(K_4) = 8 - 6 = 2$ $\quad$ $\delta(K_5) = 10 - 10 = 0$ $\quad$ $\delta(K_6) = 12 - 30 = -18.$

- Finite graph $\mathbf{G}$ is in $\mathcal{C}_0$ iff $\forall_{H \subseteq G} \delta(H) \geq 0$.

Hrushovski construction

- Predimension of a graph $\mathbf{G} = (V, E)$ is
$$\delta(G) = 2|V| - |E|.$$

Example

$\delta(K_1) = 2$ $\quad$ $\delta(K_2) = 4 - 1 = 3$ $\quad$ $\delta(K_3) = 6 - 3 = 3$

$\delta(K_4) = 8 - 6 = 2$ $\quad$ $\delta(K_5) = 10 - 10 = 0$ $\quad$ $\delta(K_6) = 12 - 30 = -18.$

- Finite graph $\mathbf{G}$ is in $\mathcal{C}_0$ iff $\forall_{H \subseteq G} \delta(H) \geq 0$.
- $\mathbf{G} \subseteq \mathbf{H}$ is self-sufficient, $\mathbf{G} \leq_s \mathbf{H}$, iff $\forall_{\mathbf{G} \subseteq \mathbf{G}' \subseteq \mathbf{H}} \delta(\mathbf{G}) \leq \delta(\mathbf{G}')$.

Hrushovski construction

- Predimension of a graph $\mathbf{G} = (V, E)$ is

$$\delta(G) = 2|V| - |E|.$$

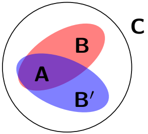
Example

$\delta(K_1) = 2$ $\quad$ $\delta(K_2) = 4 - 1 = 3$ $\quad$ $\delta(K_3) = 6 - 3 = 3$
$\delta(K_4) = 8 - 6 = 2$ $\quad$ $\delta(K_5) = 10 - 10 = 0$ $\quad$ $\delta(K_6) = 12 - 30 = -18.$

- Finite graph $\mathbf{G}$ is in $\mathcal{C}_0$ iff $\forall_{H \subseteq G} \delta(H) \geq 0$.
- $\mathbf{G} \subseteq \mathbf{H}$ is self-sufficient, $\mathbf{G} \leq_s \mathbf{H}$, iff $\forall_{\mathbf{G} \subseteq \mathbf{G}' \subseteq \mathbf{H}} \delta(\mathbf{G}) \leq \delta(\mathbf{G}')$.

Lemma

$\mathcal{C}_0$ is closed for free amalgamation over self-sufficient substructures.



Proof.

$\delta(\mathbf{C}) = \delta(\mathbf{B}) + \delta(\mathbf{B}') - \delta(\mathbf{A})$. □

Hrushovski class $\mathcal{C}_0$

- Predimension of a graph $\mathbf{G} = (V, E)$ is $\delta(\mathbf{G}) = 2|V| - |E|$.
- Finite graph $\mathbf{G}$ is in $\mathcal{C}_0$ iff $\forall_{\mathbf{H} \subseteq \mathbf{G}} \delta(\mathbf{H}) \geq 0$.

Lemma (By marriage theorem)

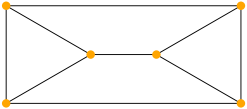
- $\mathbf{G} \in \mathcal{C}_0$ *iff it has 2-orientation (out-degrees at most 2).*
- $\mathbf{H} \leq_s \mathbf{G}$ *iff* $\mathbf{G}$ *can be 2-oriented with no edge from* $\mathbf{H}$ *to* $\mathbf{G} \setminus \mathbf{H}$.

Hrushovski class $\mathcal{C}_0$

- Predimension of a graph $\mathbf{G} = (V, E)$ is $\delta(\mathbf{G}) = 2|V| - |E|$.
- Finite graph $\mathbf{G}$ is in $\mathcal{C}_0$ iff $\forall_{\mathbf{H} \subseteq \mathbf{G}} \delta(\mathbf{H}) \geq 0$.

Lemma (By marriage theorem)

- *$\mathbf{G} \in \mathcal{C}_0$ iff it has 2-orientation (out-degrees at most 2).*
- *$\mathbf{H} \leq_s \mathbf{G}$ iff $\mathbf{G}$ can be 2-oriented with no edge from $\mathbf{H}$ to $\mathbf{G} \setminus \mathbf{H}$.*

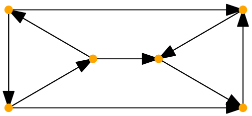


Hrushovski class $\mathcal{C}_0$

- Predimension of a graph $\mathbf{G} = (V, E)$ is $\delta(\mathbf{G}) = 2|V| - |E|$.
- Finite graph $\mathbf{G}$ is in $\mathcal{C}_0$ iff $\forall_{\mathbf{H} \subseteq \mathbf{G}} \delta(\mathbf{H}) \geq 0$.

Lemma (By marriage theorem)

- $\mathbf{G} \in \mathcal{C}_0$ *iff it has 2-orientation (out-degrees at most 2).*
- $\mathbf{H} \leq_s \mathbf{G}$ *iff* $\mathbf{G}$ *can be 2-oriented with no edge from* $\mathbf{H}$ *to* $\mathbf{G} \setminus \mathbf{H}$.

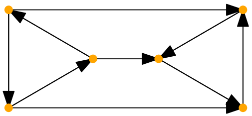


Hrushovski class $\mathcal{C}_0$

- Predimension of a graph $\mathbf{G} = (V, E)$ is $\delta(\mathbf{G}) = 2|V| - |E|$.
- Finite graph $\mathbf{G}$ is in $\mathcal{C}_0$ iff $\forall_{\mathbf{H} \subseteq \mathbf{G}} \delta(\mathbf{H}) \geq 0$.

Lemma (By marriage theorem)

- *$\mathbf{G} \in \mathcal{C}_0$ iff it has 2-orientation (out-degrees at most 2).*
- *$\mathbf{H} \leq_s \mathbf{G}$ iff $\mathbf{G}$ can be 2-oriented with no edge from $\mathbf{H}$ to $\mathbf{G} \setminus \mathbf{H}$.*



Corollary

$\mathcal{C}_0$ is a class of all finite 2-orientations $\mathcal{D}_0$ with directions forgotten.

$\mathcal{D}_0$ is closed for free amalgamation over successor-closed substructures.

Ramsey expansions of $\mathcal{C}_0$ and orientations

Theorem (Kechris, Pestov, Todorčevič, 2005)

Let **F** *be a Fraïssé limit, then the following are equivalent.*

- *Automorphism group of* **F** *is extremely amenable;*
- Age(**F**) *has the Ramsey property.*

Ramsey expansions of $\mathcal{C}_0$ and orientations

Theorem (Kechris, Pestov, Todorčevič, 2005)

Let **F** *be a Fraïssé limit, then the following are equivalent.*

- *Automorphism group of* **F** *is extremely amenable;*
- $\mathrm{Age}(\mathbf{F})$ *has the Ramsey property.*

Denote by $\mathbf{M}_0$ the generalised Fraïssé limit of $\mathcal{C}_0$.

Theorem (Evans 2015)

If $\mathbf{M}_0^+$ *is a Ramsey expansion of* $\mathbf{M}_0$*, then* $\mathrm{Aut}(\mathbf{M}_0^+)$ *fixes a* 2*-orientation.*

Ramsey expansions of $\mathcal{C}_0$ and orientations

Theorem (Kechris, Pestov, Todorčevič, 2005)

Let $\mathbf{F}$ *be a Fraïssé limit, then the following are equivalent.*

- *Automorphism group of* $\mathbf{F}$ *is extremely amenable;*
- $\mathrm{Age}(\mathbf{F})$ *has the Ramsey property.*

Denote by $\mathbf{M}_0$ the generalised Fraïssé limit of C_0.

Theorem (Evans 2015)

If $\mathbf{M}_0^+$ *is a Ramsey expansion of* $\mathbf{M}_0$*, then* $\mathrm{Aut}(\mathbf{M}_0^+)$ *fixes a 2-orientation.*

Proof.

- Consider G acting on the space $X(M_0)$ of 2-orientations of $\mathbf{M}_0$ (a G-flow).
- As $\mathrm{Aut}(\mathbf{M}_0^+)$ is extremely amenable, there is some $\mathbf{S} \in X(\mathbf{M}_0)$ which is fixed by $\mathrm{Aut}(\mathbf{M}_0^+)$.
- $\mathrm{Aut}(\mathbf{M}_0^+)$ is a subgroup of $\mathrm{Aut}(\mathbf{S})$.

□

No precompact Ramsey expansions of $\mathcal{C}_0$

Theorem (Evans 2016)

There is no precompact Ramsey expansion of $(\mathcal{C}_0; \leq_s)$.

No precompact Ramsey expansions of $\mathcal{C}_0$

Theorem (Evans 2016)

There is no precompact Ramsey expansion of $(\mathcal{C}_0; \leq_s)$.

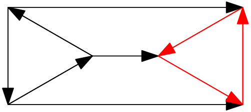
- Let $(\mathcal{C}_0^+, \sqsubseteq)$ be a Ramsey expansion of $(\mathcal{C}_0, \leq_s)$, then every $\mathbf{A} \in C_0$ has infinitely many expansions in $(\mathcal{C}_0^+; \sqsubseteq)$.

No precompact Ramsey expansions of $\mathcal{C}_0$

Theorem (Evans 2016)

There is no precompact Ramsey expansion of $(\mathcal{C}_0; \leq_s)$.

- Let $(\mathcal{C}_0^+, \sqsubseteq)$ be a Ramsey expansion of $(\mathcal{C}_0, \leq_s)$, then every $\mathbf{A} \in C_0$ has infinitely many expansions in $(\mathcal{C}_0^+; \sqsubseteq)$.
- Given two 2-orientations $\mathbf{A} \subseteq \mathbf{B}$, we write $\mathbf{A} \sqsubseteq_s \mathbf{B}$ if there is no edge from $\mathbf{A}$ to $\mathbf{B} \setminus \mathbf{A}$.
- $\sqsubseteq$ is coarser than $\sqsubseteq_s$ for 2-orientation fixed by $(\mathcal{C}_0^+, \sqsubseteq)$.

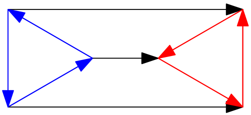


No precompact Ramsey expansions of $\mathcal{C}_0$

Theorem (Evans 2016)

There is no precompact Ramsey expansion of $(\mathcal{C}_0; \leq_s)$.

- Let $(\mathcal{C}_0^+, \sqsubseteq)$ be a Ramsey expansion of $(\mathcal{C}_0, \leq_s)$, then every $\mathbf{A} \in C_0$ has infinitely many expansions in $(\mathcal{C}_0^+; \sqsubseteq)$.
- Given two 2-orientations $\mathbf{A} \subseteq \mathbf{B}$, we write $\mathbf{A} \sqsubseteq_s \mathbf{B}$ if there is no edge from $\mathbf{A}$ to $\mathbf{B} \setminus \mathbf{A}$.
- $\sqsubseteq$ is coarser than $\sqsubseteq_s$ for 2-orientation fixed by $(\mathcal{C}_0^+, \sqsubseteq)$.

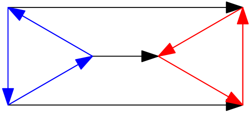


No precompact Ramsey expansions of $\mathcal{C}_0$

Theorem (Evans 2016)

There is no precompact Ramsey expansion of $(\mathcal{C}_0; \leq_s)$.

- Let $(\mathcal{C}_0^+, \sqsubseteq)$ be a Ramsey expansion of $(\mathcal{C}_0, \leq_s)$, then every $\mathbf{A} \in C_0$ has infinitely many expansions in $(\mathcal{C}_0^+; \sqsubseteq)$.
- Given two 2-orientations $\mathbf{A} \subseteq \mathbf{B}$, we write $\mathbf{A} \sqsubseteq_s \mathbf{B}$ if there is no edge from $\mathbf{A}$ to $\mathbf{B} \setminus \mathbf{A}$.
- $\sqsubseteq$ is coarser than $\sqsubseteq_s$ for 2-orientation fixed by $(\mathcal{C}_0^+, \sqsubseteq)$.



Proof.

- Every vertex $v \in \mathbf{M}_0^+$ has out-degree at most 2, but infinite in-degree.
- Oriented path $v_1 \to v_2 \to v_2 \ldots v_n$ always extends by a vertex v_0 to $v_0 \to v_1 \to v_2 \to v_2 \ldots v_n$.

□

$\mathcal{D}_0^{\prec}$ is Ramsey

Denote by $\mathcal{D}_0^{\prec}$ the class of all finite ordered 2-orientations.

Theorem (Evans, H., Nešetřil, 2018)

$\mathcal{D}_0^{\prec}$ is a Ramsey class.

$\mathcal{D}_0^{\prec}$ is Ramsey

Denote by $\mathcal{D}_0^{\prec}$ the class of all finite ordered 2-orientations.

Theorem (Evans, H., Nešetřil, 2018)

$\mathcal{D}_0^{\prec}$ is a Ramsey class.

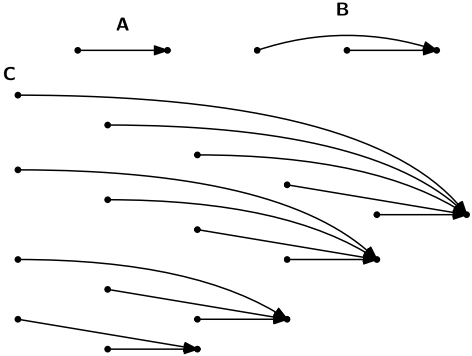
Proof.

- Given $\mathbf{A}, \mathbf{B} \in \mathcal{D}_0^{\prec}$ put $N \longrightarrow (|B|)_2^{|A|}$.
- Extend language by unary predicates $R_1, R_2, \ldots R_N$.
- Given $|B|$ tuple $\vec{b} = (b_1, b_2, \ldots b_{|B|})$, denote by $\mathbf{B}_{\vec{b}}$ expansion of $\mathbf{B}$ where i-th vertex is in relation R_{b_i}.
- $\mathbf{P}_0$ is a disjoint union of $\mathbf{B}_{\vec{v}}$, $v \in \binom{n}{|B|}$.
- Put $u \sim v$ if successor-closure of u is isomorphic to v.
- $C = P_0/\sim$. $C \longrightarrow (B)_2^A$.

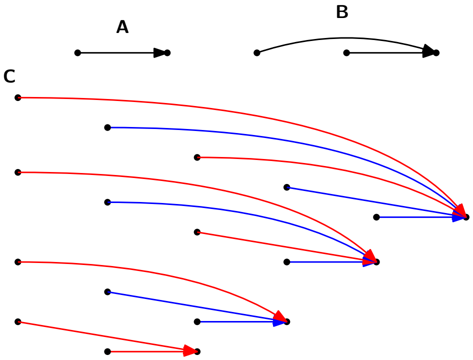
□

$\mathcal{D}_0^{\prec}$ is Ramsey

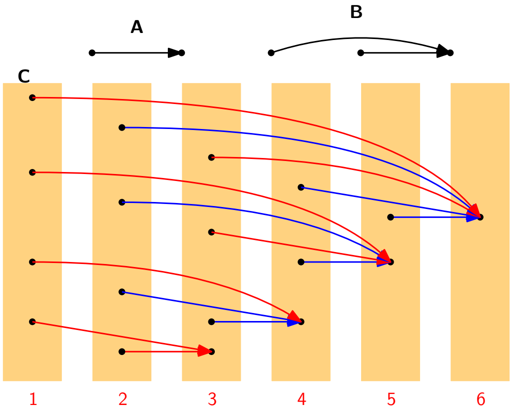
$\mathcal{D}_0^{\prec}$ is Ramsey



$\mathcal{D}_0^{\prec}$ is Ramsey

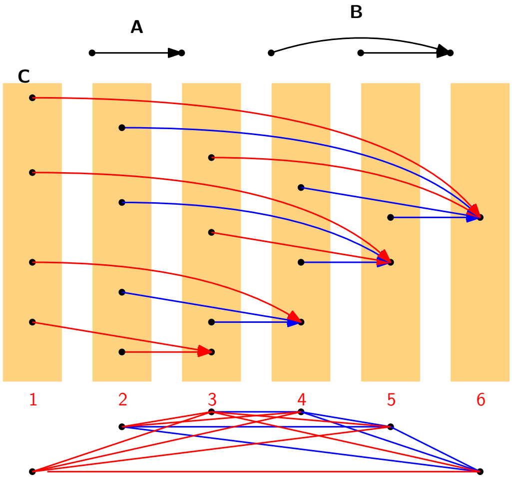


$\mathcal{D}_0^{\prec}$ is Ramsey



$$6 \longrightarrow (|B|)_2^{|A|}$$

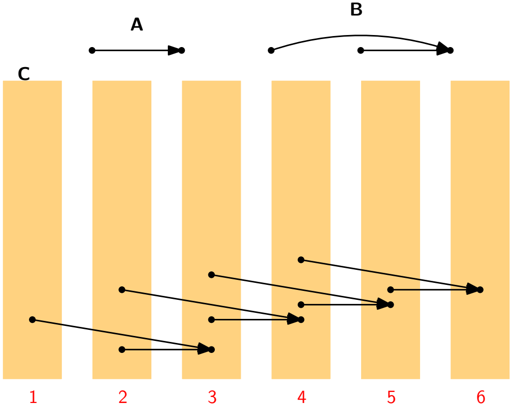
$\mathcal{D}_0^{\prec}$ is Ramsey



$$6 \longrightarrow (|B|)_2^{|A|}$$

Every $|B|$-tuple of parts corresponds to a copy of B
Every $|A|$-tuple corresponds to at most one copy of A

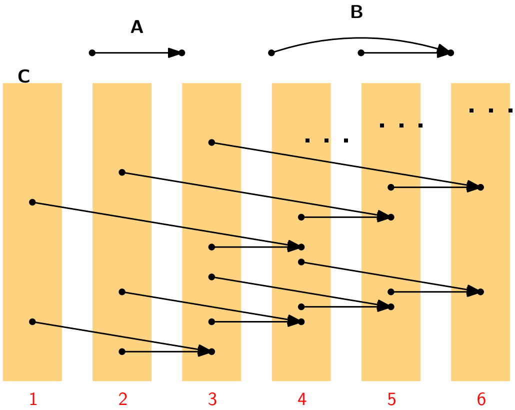
$\mathcal{D}_0^{\prec}$ is Ramsey



$$6 \longrightarrow (|B|)_2^{|A|}$$

Every $|B|$-tuple of parts corresponds to a copy of B
Every $|A|$-tuple corresponds to at most one copy of A

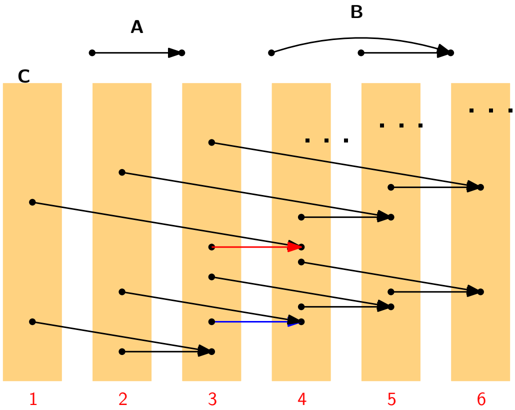
$\mathcal{D}_0^{\prec}$ is Ramsey



$$6 \longrightarrow (|B|)_2^{|A|}$$

Every $|B|$-tuple of parts corresponds to a copy of B
Every $|A|$-tuple corresponds to at most one copy of A

$\mathcal{D}_0^{\prec}$ is Ramsey

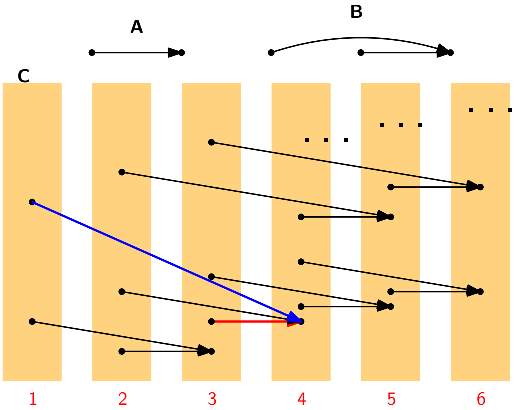


$$6 \longrightarrow (|B|)_2^{|A|}$$

Every $|B|$-tuple of parts corresponds to a copy of B

Every $|A|$-tuple corresponds to at most one copy of A

$\mathcal{D}_0^{\prec}$ is Ramsey



$$6 \longrightarrow (|B|)_2^{|A|}$$

Every $|B|$-tuple of parts corresponds to a copy of B
Every $|A|$-tuple corresponds to at most one copy of A

Optimality of Ramsey expansion

Question: (Tsankov)

Is $(\mathcal{D}_0^{\prec}; \sqsubseteq_s)$ any better than the trivial Ramsey expansion?

Optimality of Ramsey expansion

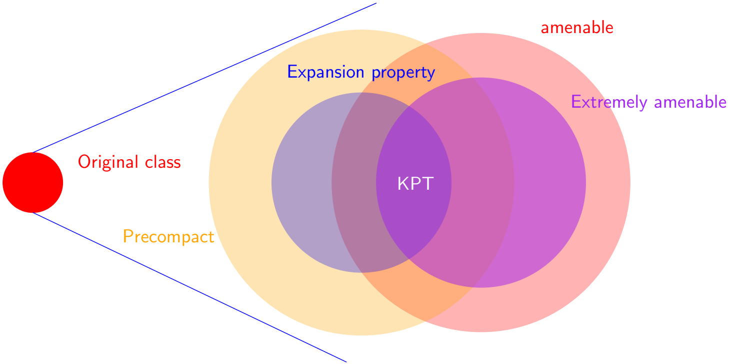
Question: (Tsankov)

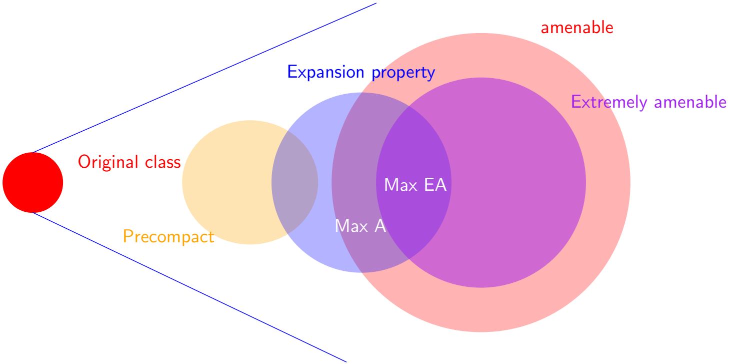
Is $(\mathcal{D}_0^{\prec}; \sqsubseteq_s)$ any better than the trivial Ramsey expansion?

Theorem (Evans, H., Nešetřil, 2018)

There exists $\mathcal{G}_0 \subset \mathcal{D}_0^{\prec}$ such that

- $(\mathcal{G}_0; \sqsubseteq_s)$ *is strong expansion of* $(\mathcal{C}_0; \leq_s)$,
- $(\mathcal{G}_0; \sqsubseteq_s)$ *is Ramsey classes,*
- $N_{\mathcal{G}_0}$*, the group of automorphisms of Fraïssé limit of* $(\mathcal{G}_0; \sqsubseteq_s)$ *is maximal amongst extremely amenable subgroups of* $\mathrm{Aut}(\mathbf{M}_0)$.
- *Class of all self-sufficient substructures of* $\mathcal{G}_0$ *has an Expansion Property with respect to* $\mathcal{C}_0$ *and thus give a minimal* $\mathrm{Aut}(\mathbf{M}_0)$ *flow.*


amenable
Expansion property
Extremely amenable
Original class
KPT
Precompact


amenable
Expansion property
Extremely amenable
Original class
Max EA
Max A
Precompact

Expansion property of non-precompact expansion

Definition

$\mathcal{K}'$ has expansion property wrt $\mathcal{K}$ if for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ such that every expansion of $\mathbf{B}$ in $\mathcal{K}'$ contains every expansion of $\mathbf{A}$ in $\mathcal{K}'$.

Expansion property of non-precompact expansion

Definition

$\mathcal{K}'$ has expansion property wrt $\mathcal{K}$ if for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ such that every expansion of $\mathbf{B}$ in $\mathcal{K}'$ contains every expansion of $\mathbf{A}$ in $\mathcal{K}'$.

Denote by $(\mathcal{D}_1; \sqsubseteq_s)$ the class of all finite acyclic orientations.
Denote by $(\mathcal{C}_1; \sqsubseteq_s)$ unoriented reduct of $(\mathcal{D}_1; \sqsubseteq_s)$.
(Kindergarten example)

Expansion property of non-precompact expansion

Definition

$\mathcal{K}'$ has expansion property wrt $\mathcal{K}$ if for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ such that every expansion of $\mathbf{B}$ in $\mathcal{K}'$ contains every expansion of $\mathbf{A}$ in $\mathcal{K}'$.

Denote by $(\mathcal{D}_1; \sqsubseteq_s)$ the class of all finite acyclic orientations.
Denote by $(\mathcal{C}_1; \sqsubseteq_s)$ unoriented reduct of $(\mathcal{D}_1; \sqsubseteq_s)$.
(Kindergarten example)

Theorem (Evans, H., Nešetřil, 2018)

For every $\mathbf{A}^+ \in \mathcal{D}_1$ there exists $\mathbf{B} \in \mathcal{C}_1$ such that every expansion $\mathbf{B}^+ \in \mathcal{D}_1$ contains $\mathbf{A}^+$ as a self-sufficient substructure.

Expansion property of non-precompact expansion

Definition

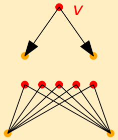
$\mathcal{K}'$ has expansion property wrt $\mathcal{K}$ if for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ such that every expansion of $\mathbf{B}$ in $\mathcal{K}'$ contains every expansion of $\mathbf{A}$ in $\mathcal{K}'$.

Denote by $(\mathcal{D}_1; \sqsubseteq_s)$ the class of all finite acyclic orientations.
Denote by $(\mathcal{C}_1; \sqsubseteq_s)$ unoriented reduct of $(\mathcal{D}_1; \sqsubseteq_s)$.
(Kindergarten example)

Theorem (Evans, H., Nešetřil, 2018)

For every $\mathbf{A}^+ \in \mathcal{D}_1$ *there exists* $\mathbf{B} \in \mathcal{C}_1$ *such that every expansion* $\mathbf{B}^+ \in \mathcal{D}_1$ *contains* $\mathbf{A}^+$ *as a self-sufficient substructure.*

Proof by induction on $|A^+|$.



Expansion property of non-precompact expansion

Definition

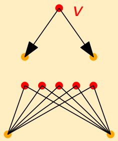
$\mathcal{K}'$ has expansion property wrt $\mathcal{K}$ if for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ such that every expansion of $\mathbf{B}$ in $\mathcal{K}'$ contains every expansion of $\mathbf{A}$ in $\mathcal{K}'$.

Denote by $(\mathcal{D}_1; \sqsubseteq_s)$ the class of all finite acyclic orientations.
Denote by $(\mathcal{C}_1; \sqsubseteq_s)$ unoriented reduct of $(\mathcal{D}_1; \sqsubseteq_s)$.
(Kindergarten example)

Theorem (Evans, H., Nešetřil, 2018)

For every $\mathbf{A}^+ \in \mathcal{D}_1$ there exists $\mathbf{B} \in \mathcal{C}_1$ such that every expansion $\mathbf{B}^+ \in \mathcal{D}_1$ contains $\mathbf{A}^+$ as a self-sufficient substructure.

Proof by induction on $|A^+|$.



- Every $\mathbf{A} \in \mathcal{D}_1$ has vertex v of in-degree 0.
- $\mathbf{A}^0 = \mathbf{A} \setminus \{v\}$.

Expansion property of non-precompact expansion

Definition

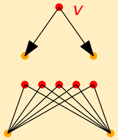
$\mathcal{K}'$ has expansion property wrt $\mathcal{K}$ if for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ such that every expansion of $\mathbf{B}$ in $\mathcal{K}'$ contains every expansion of $\mathbf{A}$ in $\mathcal{K}'$.

Denote by $(\mathcal{D}_1; \sqsubseteq_s)$ the class of all finite acyclic orientations.
Denote by $(\mathcal{C}_1; \sqsubseteq_s)$ unoriented reduct of $(\mathcal{D}_1; \sqsubseteq_s)$.
(Kindergarten example)

Theorem (Evans, H., Nešetřil, 2018)

For every $\mathbf{A}^+ \in \mathcal{D}_1$ there exists $\mathbf{B} \in \mathcal{C}_1$ such that every expansion $\mathbf{B}^+ \in \mathcal{D}_1$ contains $\mathbf{A}^+$ as a self-sufficient substructure.

Proof by induction on $|A^+|$.



- Every $\mathbf{A} \in \mathcal{D}_1$ has vertex v of in-degree 0.
- $\mathbf{A}^0 = \mathbf{A} \setminus \{v\}$.
- Construct $\mathbf{B}^0$ by induction hypothesis.
- Extend every copy of $\mathbf{A}^0$ in $\mathbf{B}^0$ to $\mathbf{A}$ by 5 copies of v.

□

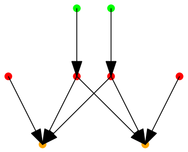
Extension property of non-precompact expansion

Definition

Suppose $\mathbf{A} \in \mathcal{D}_1$ we put $\mathbf{A} \in \mathcal{E}_1$ iff:

1. If $l(a) \prec l(b)$.
2. If $l(a) = l(b)$ then order is defined lexicographically by descending chains of their successors

$l(a)$ denote the level of vertex a.



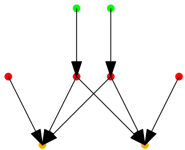
Extension property of non-precompact expansion

Definition

Suppose $\mathbf{A} \in \mathcal{D}_1$ we put $\mathbf{A} \in \mathcal{E}_1$ iff:

1. If $l(a) \prec l(b)$.
2. If $l(a) = l(b)$ then order is defined lexicographically by descending chains of their successors

$l(a)$ denote the level of vertex a.



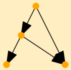
Theorem (Evans, H., Nešetřil, 2018)

For every $\mathbf{A}^+ \in \mathcal{E}_1$ there exists $\mathbf{B} \in \mathcal{C}_1$ such that every expansion $\mathbf{B}^+ \in \mathcal{E}_1$ contains $\mathbf{A}$ as self-sufficient substructure.

Theorem (Evans, H., Nešetřil, 2018)

For every $\mathbf{A}^+ \in \mathcal{E}_1$ *there exists* $\mathbf{B} \in \mathcal{C}_1$ *such that every expansion* $\mathbf{B}^+ \in \mathcal{E}_1$ *contains* $\mathbf{A}$ *as self-sufficient substructure.*

Proof.

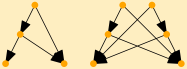


□

Theorem (Evans, H., Nešetřil, 2018)

For every $\mathbf{A}^+ \in \mathcal{E}_1$ *there exists* $\mathbf{B} \in \mathcal{C}_1$ *such that every expansion* $\mathbf{B}^+ \in \mathcal{E}_1$ *contains* $\mathbf{A}$ *as self-sufficient substructure.*

Proof.

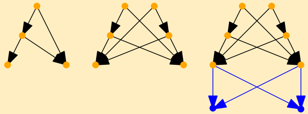


□

Theorem (Evans, H., Nešetřil, 2018)

For every $\mathbf{A}^+ \in \mathcal{E}_1$ there exists $\mathbf{B} \in \mathcal{C}_1$ such that every expansion $\mathbf{B}^+ \in \mathcal{E}_1$ contains $\mathbf{A}$ as self-sufficient substructure.

Proof.

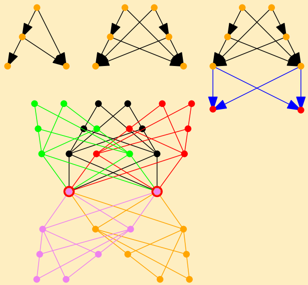


□

Theorem (Evans, H., Nešetřil, 2018)

For every $\mathbf{A}^+ \in \mathcal{E}_1$ *there exists* $\mathbf{B} \in \mathcal{C}_1$ *such that every expansion* $\mathbf{B}^+ \in \mathcal{E}_1$ *contains* $\mathbf{A}$ *as self-sufficient substructure.*

Proof.



□

The ω-categorical case

- $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$

The ω-categorical case

- $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$
- $\mathcal{C}_0 = \{\mathbf{B} : \delta(\mathbf{A}) \geq 0 \text{ for all } \mathbf{A} \subseteq \mathbf{B}\}$.
  $\mathcal{C}_F = \{\mathbf{B} : \delta(\mathbf{A}) \geq F(|\mathbf{A}|) \text{ for all } \mathbf{A} \subseteq \mathbf{B}\}$.

The ω-categorical case

- $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$
- $\mathcal{C}_0 = \{\mathbf{B} : \delta(\mathbf{A}) \geq 0 \text{ for all } \mathbf{A} \subseteq \mathbf{B}\}$.
  $\mathcal{C}_F = \{\mathbf{B} : \delta(\mathbf{A}) \geq F(|\mathbf{A}|) \text{ for all } \mathbf{A} \subseteq \mathbf{B}\}$.
- $\mathbf{A} \leq_s \mathbf{B}$ iff $\delta(\mathbf{A}) \leq \delta(\mathbf{B}')$ for all finite $\mathbf{B}'$ with $\mathbf{A} \subset \mathbf{B}' \subseteq \mathbf{B}$.
  $\mathbf{A} \leq_d \mathbf{B}$ iff $\delta(\mathbf{A}) < \delta(\mathbf{B}')$ for all finite $\mathbf{B}'$ with $\mathbf{A} \subset \mathbf{B}' \subseteq \mathbf{B}$.

The ω-categorical case

- $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$
- $\mathcal{C}_0 = \{\mathbf{B} : \delta(\mathbf{A}) \geq 0 \text{ for all } \mathbf{A} \subseteq \mathbf{B}\}$.
  $\mathcal{C}_F = \{\mathbf{B} : \delta(\mathbf{A}) \geq F(|\mathbf{A}|) \text{ for all } \mathbf{A} \subseteq \mathbf{B}\}$.
- $\mathbf{A} \leq_s \mathbf{B}$ iff $\delta(\mathbf{A}) \leq \delta(\mathbf{B}')$ for all finite $\mathbf{B}'$ with $\mathbf{A} \subset \mathbf{B}' \subseteq \mathbf{B}$.
  $\mathbf{A} \leq_d \mathbf{B}$ iff $\delta(\mathbf{A}) < \delta(\mathbf{B}')$ for all finite $\mathbf{B}'$ with $\mathbf{A} \subset \mathbf{B}' \subseteq \mathbf{B}$.

Lemma

Put $F(x) = \ln(x)$. Then $(\mathcal{C}_F; \leq_d)$ is a free amalgamation class.

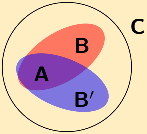
The ω-categorical case

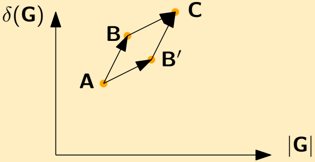
- $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$
- $\mathcal{C}_0 = \{\mathbf{B} : \delta(\mathbf{A}) \geq 0 \text{ for all } \mathbf{A} \subseteq \mathbf{B}\}$.
  $\mathcal{C}_F = \{\mathbf{B} : \delta(\mathbf{A}) \geq F(|\mathbf{A}|) \text{ for all } \mathbf{A} \subseteq \mathbf{B}\}$.
- $\mathbf{A} \leq_s \mathbf{B}$ iff $\delta(\mathbf{A}) \leq \delta(\mathbf{B}')$ for all finite $\mathbf{B}'$ with $\mathbf{A} \subset \mathbf{B}' \subseteq \mathbf{B}$.
  $\mathbf{A} \leq_d \mathbf{B}$ iff $\delta(\mathbf{A}) < \delta(\mathbf{B}')$ for all finite $\mathbf{B}'$ with $\mathbf{A} \subset \mathbf{B}' \subseteq \mathbf{B}$.

Lemma

Put $F(x) = \ln(x)$. Then $(\mathcal{C}_F; \leq_d)$ is a free amalgamation class.

Proof.





□

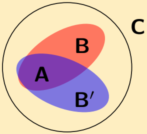
The ω-categorical case

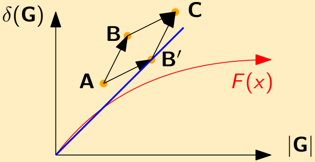
- $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$
- $\mathcal{C}_0 = \{\mathbf{B} : \delta(\mathbf{A}) \geq 0 \text{ for all } \mathbf{A} \subseteq \mathbf{B}\}$.
  $\mathcal{C}_F = \{\mathbf{B} : \delta(\mathbf{A}) \geq F(|\mathbf{A}|) \text{ for all } \mathbf{A} \subseteq \mathbf{B}\}$.
- $\mathbf{A} \leq_s \mathbf{B}$ iff $\delta(\mathbf{A}) \leq \delta(\mathbf{B}')$ for all finite $\mathbf{B}'$ with $\mathbf{A} \subset \mathbf{B}' \subseteq \mathbf{B}$.
  $\mathbf{A} \leq_d \mathbf{B}$ iff $\delta(\mathbf{A}) < \delta(\mathbf{B}')$ for all finite $\mathbf{B}'$ with $\mathbf{A} \subset \mathbf{B}' \subseteq \mathbf{B}$.

Lemma

Put $F(x) = \ln(x)$. Then $(\mathcal{C}_F; \leq_d)$ is a free amalgamation class.

Proof.





□

Successor-d-closure

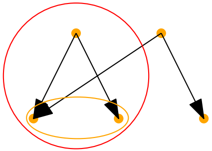
$\text{roots}_{\mathbf{A}}(\mathbf{B})$ is set of all roots of $\mathbf{A}$ reachable from $\mathbf{B} \subseteq \mathbf{A}$

Lemma (Evans, H., Nešetřil, 2018)

Let $\mathbf{B} \subseteq \mathbf{A}$ be an 2-orientations. Then $\mathbf{B}$ is both d-closed and successor-closed in $\mathbf{A}$ iff

$$\mathbf{B} = \{v : \text{roots}_{\mathbf{A}}(v) \subseteq \text{roots}_{\mathbf{A}}(\mathbf{B})\}.$$

Recall: $\mathbf{B}$ is d-closed in $\mathbf{A}$ iff $\delta(\mathbf{B}) < \delta(\mathbf{B}')$ for all $\mathbf{B}'$ s.t. $\mathbf{B} \subset \mathbf{B}' \subseteq \mathbf{A}$.



Successor-d-closure

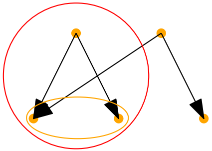
$\text{roots}_{\mathbf{A}}(\mathbf{B})$ is set of all roots of $\mathbf{A}$ reachable from $\mathbf{B} \subseteq \mathbf{A}$

Lemma (Evans, H., Nešetřil, 2018)

Let $\mathbf{B} \subseteq \mathbf{A}$ be an 2-orientations. Then $\mathbf{B}$ is both d-closed and successor-closed in $\mathbf{A}$ iff

$$\mathbf{B} = \{v : \text{roots}_{\mathbf{A}}(v) \subseteq \text{roots}_{\mathbf{A}}(\mathbf{B})\}.$$

Recall: $\mathbf{B}$ is d-closed in $\mathbf{A}$ iff $\delta(\mathbf{B}) < \delta(\mathbf{B}')$ for all $\mathbf{B}'$ s.t. $\mathbf{B} \subset \mathbf{B}' \subseteq \mathbf{A}$.

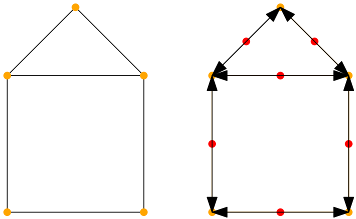


Proof.

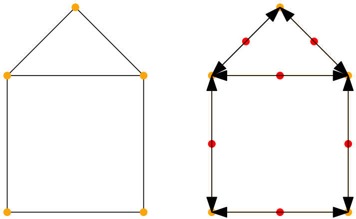
- Given $\mathbf{B} \sqsubseteq_s \mathbf{A}$, $\delta(\mathbf{B})$ is the number of roots of out-degree 1 + twice number of roots of out-degree 0.
- Extending $\mathbf{B}$ by all vertices v such that $\text{roots}_{\mathbf{A}}(v) \subseteq \text{roots}_{\mathbf{A}}(\mathbf{B})$ keeps δ.
- Extending $\mathbf{B}$ by any other vertex increases δ.

□

$\mathcal{C}_F$ is harder

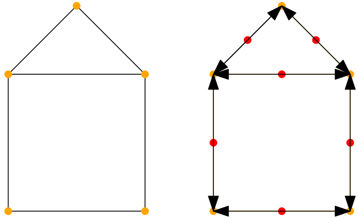


$\mathcal{C}_F$ is harder



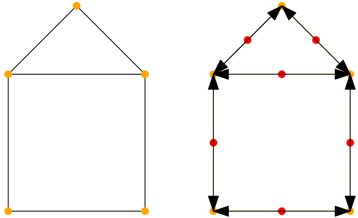
- $(\mathcal{C}_F; \leq_d)$ contains subclass interpreting undirected graphs

$\mathcal{C}_F$ is harder



- $(\mathcal{C}_F; \leq_d)$ contains subclass interpreting undirected graphs
- successor-d-closure is not unary: it is not true that successor-d-closure of a set is union of successor-d-closures of its vertices.

$\mathcal{C}_F$ is harder

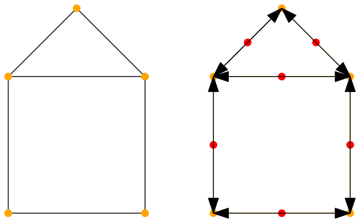


- $(\mathcal{C}_F; \leq_d)$ contains subclass interpreting undirected graphs
- successor-d-closure is not unary: it is not true that successor-d-closure of a set is union of successor-d-closures of its vertices.

$\mathcal{C}_F$ is harder but partly solved by big hammers (for specific choices of F)

- Ramsey property of $(\mathcal{D}_F^{\prec}; \sqsubseteq_d)$ as locally finite subclass.
- Expansion property is a combination of expansion property for $(\mathcal{C}_0; \leq_s)$ and ordering property for graphs (via Ramsey property).

$\mathcal{C}_F$ is harder

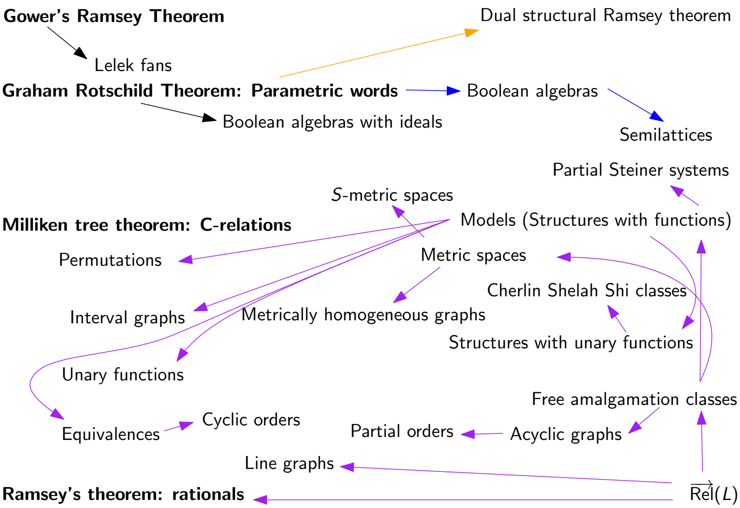


- $(\mathcal{C}_F; \leq_d)$ contains subclass interpreting undirected graphs
- successor-d-closure is not unary: it is not true that successor-d-closure of a set is union of successor-d-closures of its vertices.

$\mathcal{C}_F$ is harder but partly solved by big hammers (for specific choices of F)

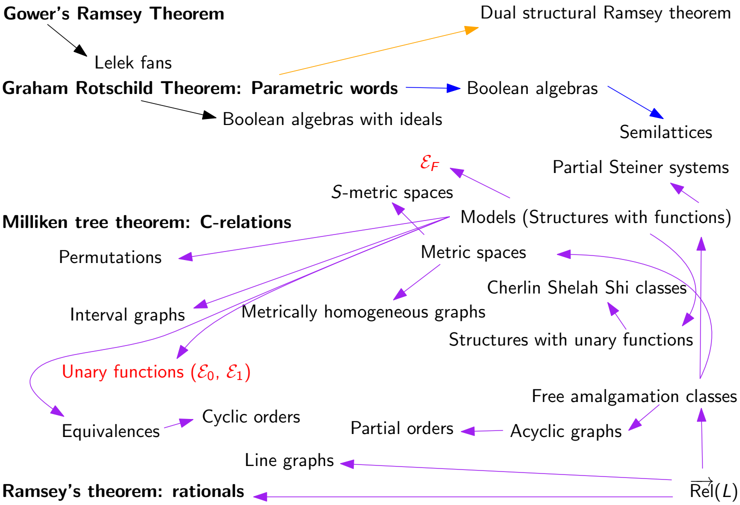
- Ramsey property of $(\mathcal{D}_F^{\prec}; \sqsubseteq_d)$ as locally finite subclass.
- Expansion property is a combination of expansion property for $(\mathcal{C}_0; \leq_s)$ and ordering property for graphs (via Ramsey property).

EPPA and big Ramsey degree currently open (WIP).



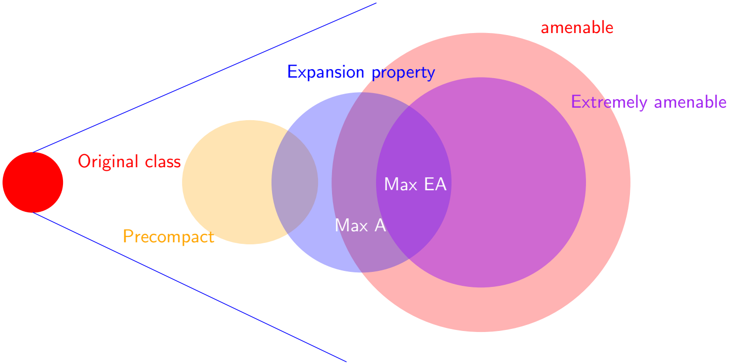
Locally finite subclass

Product arguments   Interpretations   Adding unary functions   Partite construction



Locally finite subclass

Product arguments   Interpretations   Adding unary functions   Partite construction



In ω-categorical case Ramsey argument is difficult. EPPA is work in progress.

We know the maximal extremely amenable subgroup. We conjecture what the maximal amenable subgroup is.

Thank you for the attention

- J.H., J. Nešetřil: All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms). Submitted (arXiv:1606.07979), 2016, 59 pages.
- D. Evans, J. H., J. Nešetřil: Ramsey properties and extending partial automorphisms for classes of finite structures. Submitted (arXiv:1705.02379), 2017, 33 pages.
- D. Evans, J.H., J. Nešetřil: Automorphism groups and Ramsey properties of sparse graphs. Submitted (arXiv:1801.01165), 2018, 47 pages.

- J.H., J. Nešetřil: Bowtie-free graphs have a Ramsey lift. Advances in Applied Mathematics 96 (2018), 286–311.
- J.H., M. Konečný, J. Nešetřil: Conant's generalised metric spaces are Ramsey. To appear in Contributions to Discrete Mathematics (arXiv:1710.04690), 20 pages.
- J.H., J. Nešetřil: Ramsey Classes with Closure Operations (Selected Combinatorial Applications). Connections in Discrete Mathematics: A Celebration of the Work of Ron Graham, 240–258.

- A. Aranda, J. H., M. Karamanlis, M. Kompatscher, M. Konečný, M. Pawliuk, D. Bradley-Williams: Ramsey expansions of metrically homogeneous graphs. Submitted (arXiv:1706.00295), 57 pages.
- A. Aranda, K. E. Hng, J. H., M. Karamanlis, M. Kompatscher, M. Konečný, M. Pawliuk, D. Bradley-Williams: Completing graphs to metric spaces, Submitted (arXiv:1707.02612), 19 pages.
- M. Konečný: Combinatorial Properties of Metrically Homogeneous Graphs, Bachelor thesis

- J.H., M. Konečný, J. Nešetřil: Semigroup-valued metric spaces. To appear.
- M. Konečný: Semigroup-valued metric spaces, Master thesis to appear
- R. Coulson, J. H., M. Kompatcher, M. Konečný: Forbidden cycles in metrically homogeneous graphs. To appear.