Canonization on product and iterated perfect and large perfect sets

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Abstract

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Some canonization results, related to Borel equivalence relations modulo restriction to various categories of perfect sets, will be presented and commented.

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Section 1.
The canonization problem

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For instance, the theorem saying that every Borel real map is either a bijection or a constant on a perfect set, can be viewed as a canonization theorem, with

$$\mathscr{E}' = \{\text{bijections and constants}\} \quad \subseteq \quad \mathscr{E} = \{\text{Borel maps}\},$$

$$\mathscr{P} = \{\text{perfect sets of reals}\}.$$

V. Kanovei, M. Sabok, J. Zapletal,
Canonical Ramsey Theory on Polish Spaces,
Cambridge University Press, 2013.

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All results below belong to this book unless otherwise stated

Section 2.
Canonization: examples

Theorem (a corollary of Silver 1980)

If $\mathbf{E}$ *is a Borel equivalence relation on a perfect set* $P \subseteq \mathbb{R}$, *then there exists a perfect set* $Q \subseteq P$ *such that* $\mathbf{E} \restriction Q$ *is*:

either **the equality:** *so that* Q *is pairwise* $\mathbf{E}$*-inequivalent*;

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In the general canonization scheme, this can be codified as follows:

$$\mathscr{E}' = \{\mathbf{\Delta}, \ \mathbf{total}\} \quad \subseteq \quad \mathscr{E} = \{\text{all Borel equivalence relations}\},$$
$$\mathscr{P} = \{\text{perfect sets of reals}\},$$

where $\mathbf{\Delta}$ is the equality and **total** is the total equivalence (making all reals equivalent).

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Definition

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Theorem (K – Zapletal)

If $\mathbf{E}$ *is a Borel equivalence relation on a Vitali-large Borel set* $P \subseteq \mathbb{R}$*, then there is a perfect Vitali-large set* $Q \subseteq P$ *such that* $\mathbf{E}$ *is equal on* Q *to one of the following three relations*:

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compare to Silver

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- *the equality* $\mathbf{\Delta}$, **Silver's option**
- *the Vitali equivalence* **vit**, **new option !!**
- *the total equivalence* **total**. **Silver's option**

Section 3.
Application: degrees of equivalence classes

Let **F** be an equivalence relation on $\mathbb{R}$. If $x \in \mathbb{R}$ then let

$$[x]_{\mathbf{F}} = \{y \in \mathbb{R} : x \mathrel{\mathbf{F}} y\}, \quad \text{the } \mathbf{F}\text{-class of } x,$$

then $\mathbb{R}/\mathbf{F} = \{[x]_{\mathbf{F}} : x \in X\}$ is the quotient.

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- $\mathbb{V}$ is the background set universe,
- $\mathbf{F}$ is a Borel equivalence relation on $\mathbb{R}$.
- $\mathbb{P} \in \mathbb{V}$ is a forcing notion,
- $\mathbb{V}^+$ is a $\mathbb{P}$-generic extension of $\mathbb{V}$.

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Definition (reducibility)

In $\mathbb{V}^+$, if $X, Y \in \mathbb{R}/\mathbf{F}$ then $Y \leq_{\mathbb{V}} X$ ($\mathbb{V}$-reducibility) iff Y is **reduced** to X by an analytic graph Γ coded in $\mathbb{V}$.

Definition (reduction of Y to X)

Suppose that X, Y are non-$\varnothing$ sets of reals, and $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$ is a planar set (call it a *graph*). Say that Y is reduced to X by Γ, if, symbolically, $\varnothing \neq \Gamma[X] \subseteq Y$,

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1. $X \cap \mathtt{dom}\,\Gamma \neq \varnothing$, that is, $\exists x \in X \, \exists y \, (x \; \Gamma \; y)$;
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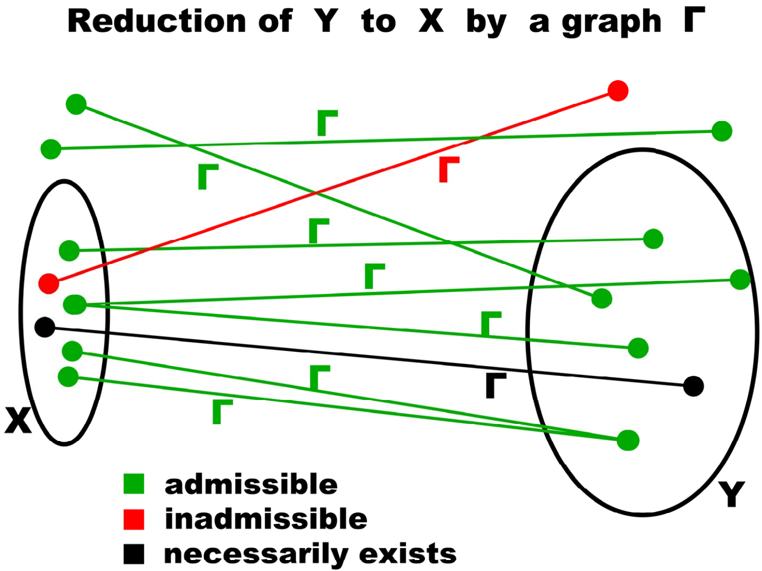
Reduction of Y to X by a graph Γ
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Comment (reduction of Y to X)

In this case, if $\mathbf{F}$ is an equivalence relation, and X, Y are $\mathbf{F}$-equivalence classes, then $Y = [\Gamma[X]]_{\mathbf{F}}$, where

$$\Gamma[X] = \{y : \exists x \in X \, (x \,\Gamma\, y)\}.$$

Thus **if we know Γ and X then we know Y as well**.

Let $\mathbf{F}$ be an equivalence relation on $\mathbb{R}$. If $x \in \mathbb{R}$ then let

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Goal: In $\mathbb{V}^+$, study the structure of $\mathbb{R}/\mathbf{F}$, under $\leq_{\mathbb{V}}$.

In the background set universe $\mathbb{V}$,

- let **VL** = all Vitali-large perfect sets $P \subseteq \mathbb{R}$ (the forcing),
- let **F** be a Borel equivalence relation in $\mathbb{V}$.

Theorem

*In any **VL**-generic extension $\mathbb{V}^+$ of $\mathbb{V}$, there exist at most 3 $\le_{\mathbb{V}}$-degrees of **F**-equivalence classes.*

Any **VL**-generic extension of the background set universe $\mathbb{V}$ has the form $\mathbb{V}^+ = \mathbb{V}[\mathbf{r}]$, where $\mathbf{r} \in \mathbb{R}$ is the principal $\mathbb{P}$-generic real.

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By the **canonization trichotomy theorem** there is a set $P \in \mathbf{VL}$, such that $\mathbf{r} \in P$ and $\mathbf{E}_f, \mathbf{E}_g$ are canonized on P into one of $\mathbf{\Delta}, \mathbf{vit}, \mathbf{total}$.

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Fact. If $\mathbf{E}_f, \mathbf{E}_g$ are canonized on P into **the same** relation $\mathbf{C}$ in the list $\{\mathbf{\Delta}, \mathbf{vit}, \mathbf{total}\}$, then $Y \le_{\mathbb{V}} X$ via the graph

$$\mathbf{\Gamma} = \{\langle x, y\rangle : \exists\, a\, (x \, \mathbf{F} \, f(a) \wedge y \, \mathbf{F} \, g(a))\},$$

and similarly $X \le_{\mathbb{V}} Y$.

Section 4.
Canonization on products

By **Silver's dichotomy canonization theorem**, the canonization spectrum of Borel equivalence relations on perfect sets $P \subseteq \mathbb{R}$ consists of just two equivalence relations:

- the equality $\boldsymbol{\Delta}$, and
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Theorem (canonization on finite perfect products)

If $\mathbf{E}$ *is a Borel equivalence relation on a finite perfect product* $P = P_1 \times \ldots \times P_n \subseteq \mathbb{R}^n$ (n factors), *then there is a perfect product* $Q = Q_1 \times \ldots \times Q_n \subseteq P$, *such that* $\mathbf{E}$ *is equal on* Q *to a product of the form* $\mathbf{E}_1 \times \ldots \times \mathbf{E}_n$, *where each* $\mathbf{E}_k$ *is* $\mathbf{\Delta}$ *or* **total**.

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No result like this is known for infinite products.

Similarly, the **Vitali-large trichotomy canonization** naturally extends to finite products. Let a Vitali-large perfect product be a set of the form $P = P_1 \times \ldots \times P_n$, where each $P_i \subseteq \mathbb{R}$ is a Vitali-large perfect set.

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Theorem (canonization on finite Vitali-large products)

If $\mathbf{E}$ is a Borel equivalence relation on a finite Vitali-large perfect product $P = P_1 \times \ldots \times P_n \subseteq \mathbb{R}^n$, then there is a perfect Vitali-large product $Q = Q_1 \times \ldots \times Q_n \subseteq P$, such that $\mathbf{E}$ is equal on Q to a product of the form $\mathbf{E}_1 \times \ldots \times \mathbf{E}_n$, where each $\mathbf{E}_k$ is one of $\mathbf{\Delta}$, $\mathbf{vit}$, $\mathbf{total}$.

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Definition

A Borel equivalence relation $\mathbf{E}$ on a (Borel) set X is smooth, if there is a Borel map $\vartheta : X \to \mathbb{R}$ such that $x \mathbin{\mathbf{E}} y \iff \vartheta(x) = \vartheta(y)$.

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Theorem (reduction to smooth)

If $\mathbf{E}$ is an equivalence relation of certain type, then for any infinite perfect product $P = \prod_k P_k \subseteq \mathbb{R}^\omega$ there is an infinite perfect product $Q \subseteq P$ such that $\mathbf{E} \restriction Q$ is smooth.

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Here certain type $=$ those classifiable by countable structures, and those Borel reducible to analytic P-ideals.

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Conjecture

If $\mathbf{E}$ is a smooth Borel equivalence relation on an infinite perfect product $P = \prod_{k<\omega} P_k \subseteq \mathbb{R}^\omega$ (n factors),
then there is a perfect product $Q = \prod_{k<\omega} Q_k \subseteq P$,
such that $\mathbf{E}$ is equal on Q to a product of the form $\prod_{k<\omega} \mathbf{E}_k$,
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Still there is a partial result.

Theorem (K)

Suppose that **E** *and* **F** *are smooth Borel equivalence relations on an infinite perfect product* $P = \prod_{k<\omega} P_k \subseteq \mathbb{R}^\omega$.
Then there is a perfect product $Q = \prod_{k<\omega} Q_k \subseteq P$, *such that*

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1. **F** *is* *independent of the* k*-th co-ordinate on* Q, *so that if sequences* $\vec{x} = \{x_n\}_{n<\omega}$ *and* $\vec{y} = \{y_n\}_{n<\omega}$ *belong to* Q *and* $x_n = y_n$ *for all* $n \neq k$, *then* $\vec{x}\,\mathbf{F}\,\vec{y}$, *but*
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1 + 2 imply that $\mathbf{F} \not\subseteq \mathbf{E}$ on any smaller perfect product $Q' \subseteq Q$

Section 5.
Canonization on perfect iterations

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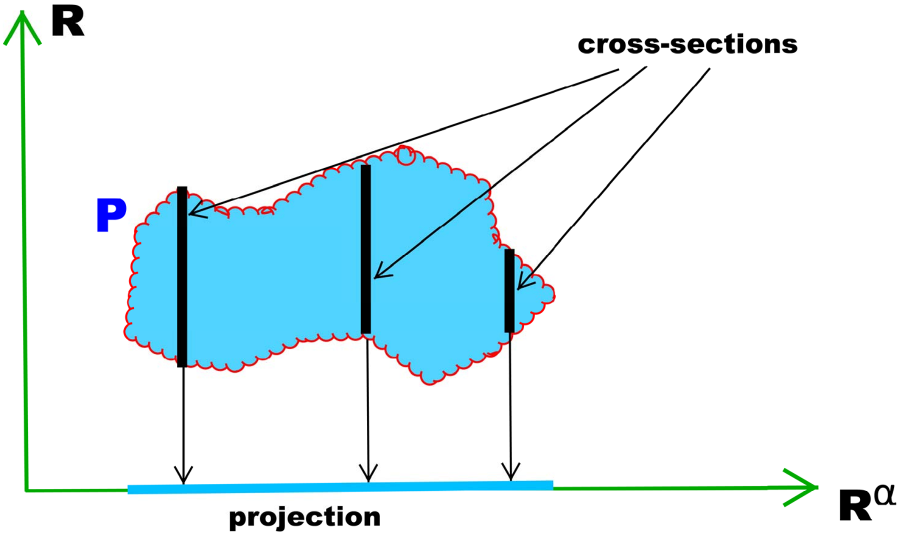
R
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Perfect products in $\mathbb{R}^\alpha$ belong to $\mathbf{PI}(\alpha)$, but sets in $\mathbf{PI}(\alpha)$ are not necessarily products.

Let $\xi \le \alpha < \omega_1$, and $\vec{x} = \{x_\xi\}$, $\vec{y} = \{y_\xi\} \in \mathbb{R}^\alpha$. Define

$\vec{x}\ \mathbf{\Delta}_\xi\ \vec{y}$ iff $\mathbf{pr}_{<\xi}(\vec{x}) = \mathbf{pr}_{<\xi}(\vec{y})$, so that $x_\eta = y_\eta$ for all $\eta < \xi$.

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Theorem (canonization on perfect iterations)

Let $\alpha < \omega_1$. If $\mathbf{E}$ is a Borel equivalence relation on a perfect α-iteration $P \subseteq \mathbb{R}^\alpha$, then there is a perfect α-iteration $Q \subseteq P$, such that $\mathbf{E}$ is equal on Q to one of $\mathbf{\Delta}_\xi$, $\xi \le \alpha$.

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Comparing to the canonization on perfect products theorem, one may ask why there is no arbitrary products of coordinate-wise equivalence relations in the canonization scheme here?

Answer: α-products of the form $\mathbf{E} = \prod_{\xi<\alpha} \mathbf{E}_\xi$, where each $\mathbf{E}_\xi$ is **total** or the equality, admit further canonization on perfect α-iterations, according to the following:

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If $\alpha < \omega_1$, and $P \subseteq \mathbb{R}^\alpha$ is a perfect α-iteration, then there is a perfect α-iteration $Q \subseteq P$, such that for all α-strings $\vec{x} = \{x_\xi\}, \vec{y} = \{y_\xi\} \in Q$, and all indices $\eta < \xi < \alpha$, we have:

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In other words, the coordinate equalities, $\vec{x}\;\mathbf{E}_\xi\;\vec{y}$ iff $x_\xi = y_\xi$, are not necessarily independent on perfect α-iterations. (As they are on perfect products.)

Section 6.
Canonization on Vitali-large iterations

Let $1 \le \alpha < \omega_1$. The notion of Vitali-large (VL , for brevity) α-iterations $P \subseteq \mathbb{R}^\alpha$ is defined by induction of α.

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The speaker thanks **the organizers** for the opportunity to present this talk

The speaker thanks **everybody** for patience