Ramsey's theorem for pairs and proof size

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(joint work with Tin Lok Wong and Keita Yokoyama)

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The result

Theorem (Patey-Yokoyama 18)

Ramsey's theorem for pairs and two colours, RT^2_2*,*
is $\forall\Sigma^0_2$*-conservative over recursive comprehension,* RCA_0.

Question (Patey-Yokoyama)

Does RT^2_2 have significant proof speedup over RCA_0
w.r.t. $\forall\Sigma^0_2$ statements?

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Question (Patey-Yokoyama)

Does RT^2_2 have significant proof speedup over RCA_0
w.r.t. $\forall\Sigma^0_2$ statements?

Our theorem

No.

Glossary (1)

Language of second-order arithmetic has two sorts of variables:

- first-order sort $x, y, z, \ldots, i, j, k \ldots$ for natural numbers,
- second-order sort $X, Y, Z, \ldots$ for subsets of $\mathbb{N}$,
- extra-logical symbols: $+, \cdot, \leq, 0, 1; \in$.

Σ^0_n: class of formulas with n first-order quantifier blocks, beginning with $\exists$, then only bounded quantifiers $\exists x \leq t$, $\forall x \leq t$.
Π^0_n: dual class, beginning with $\forall$.
$\forall\Sigma^0_n$: formulas with arbitrary $\forall$ quantifiers followed by Σ^0_n.

Example: $\forall X\, \exists x\, \exists y\, \forall z\, [(z \in X \Rightarrow \exists w \leq x\, (z = w + y))]$ is $\forall\Sigma^0_2$.
(But so are e.g. P $\neq$ NP, Riemann's hypothesis, twin prime conjecture...)

Glossary (2)

RCA_0 is an axiomatic theory with the following axioms:

- $+, \cdot, \leq, 0, 1$ on first-order sort form non-negative part of discrete ordered ring,
- recursive comprehension: "for any Turing machine m and set X, if m^X halts on all inputs, then $\{i \in \mathbb{N} : m^X(i) = \mathsf{yes}\}$ exists".
- induction: $\forall X\,[0 \in X \wedge \forall k\,(k \in X \Rightarrow k+1 \in X) \Rightarrow \forall k\,(k \in X)]$.
- Σ^0_1 induction: "for any X and k, if X is infinite (i.e. has arbitrarily large elements), then X has a finite subset with k elements".

RCA_0 embodies "computable mathematics".

RT^2_2 is just a natural formulation of Ramsey's theorem
for pairs and two colours in this language.
(Using pairing to represent a 2-colouring of $[\mathbb{N}]^2$ as a subset of $\mathbb{N}$.)

Context: proof speedup for axiomatic theories

If $T \subseteq T^+$, then T^+ is Γ-conservative over T for class of sentences Γ if all $\varphi \in \Gamma$ provable in T^+ are provable in T.
Then we can ask if the proofs in T^+ can be much shorter than in T.

For reasonably strong theories one of two things usually happens:

- T^+ has at least iterated exponential speedup over T (w.r.t. Γ).
(E.g. GB over ZFC, ACA_0 over PA, RCA_0 over PRA.)
- T^+ is polynomially simulated by T: each proof (of $\varphi \in \Gamma$) in T^+ can be translated into T with at most polynomial blowup.
(E.g. WKL_0 over RCA_0, RCA_0 over $\mathsf{I}\Sigma_1$.)

Work in the area done (80's/90's) e.g. by Pudlák, Avigad, Ignjatović...
Small revival taking place in recent years.

The result, once more

Theorem (Patey-Yokoyama 18)

RT^2_2 *is* $\forall\Sigma^0_2$*-conservative over* RCA_0.

Question (Patey-Yokoyama)

Does RT^2_2 have significant proof speedup over RCA_0 w.r.t. $\forall\Sigma^0_2$ statements?

Our theorem

RT^2_2 *is polynomially simulated by* RCA_0 *w.r.t. proofs of* $\forall\Sigma^0_2$ *statements.*

Plan for rest of talk

- State the combinatorial result at the heart of the proof: bound on "ordinal-valued Ramsey numbers" for colourings of *finite* sets.
- Explain the logic: how this combinatorial result implies the polynomial simulation.
- (As much as possible) Explain how the combinatorial result is proved.
- (If time permits) Say what happens without Σ^0_1 induction.

Measuring finite sets by ordinals: α-largeness

Ketonen-Solovay devised a way of using small countable ordinals to measure „size" of finite subsets of $\mathbb{N}$. For $\alpha < \omega^\omega$ it works like this:

- any finite subset of $\mathbb{N}$ is 0-large,
- X is $(\alpha + 1)$-large if $X \setminus \{\min X\}$ is α-large,
- X is $(\alpha + \omega^n)$-large iff $X \setminus \{\min X\}$ is $(\alpha + \omega^{n-1} \cdot \min X)$-large. (Where (each exponent in α) $\geq n \geq 1$.)

Examples:

- X is k-large iff $|X| \geq k$, for $k \in \mathbb{N}$.
- X is ω-large iff $|X| > \min X$.
- continued on next slide...

α-largeness: examples, cont'd

- X is $\omega + 2$-large, $X = \{x_0 < x_1 < x_2 < \ldots < x_k\}$,
  iff $\{x_2, \ldots, x_k\}$ is ω-large, thus iff $k - 1 > x_2$,
- X is $\omega + \omega$-large
  iff $X = X_1 \cup X_2$ with $X_1 < X_2$ and both X_i are ω-large,
- X is ω^2-large
  iff $X = \{\min X\} \cup X_1 \ldots \cup X_{\min X}$ with $X_i < X_{i+1}$, all X_i ω-large.

α-largeness: examples, cont'd

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α-largeness: examples, cont'd

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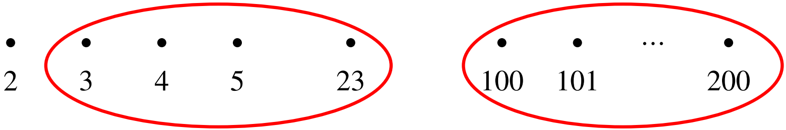


What is needed to prove non-speedup

Known from prior work:

- (Ketonen-Solovay 1981) ω^6-large $\to$ (ω-large$)^2_2$
- (Bigorajska-Kotlarski 2002) $\omega^{\omega^n \cdot 2}$-large $\to$ (ω^n-large$)^2_2$

Combinatorial core of Patey-Yokoyama

For every n there exists m such that $\mathsf{RCA}_0 \vdash \omega^m$-large $\to$ (ω^n-large$)^2_2$

Combinatorial core of our result

$\mathsf{RCA}_0 \vdash \forall n \left[\omega^{300n}\text{-large} \to (\omega^n\text{-large})^2_2\right]$.

- $m := n^2$ or even $m := n^{\log n}$ would suffice for non-speedup.
- m cannot be smaller than $2n$ (Kotlarski et al. 2007).

Finite consistency statements

$\mathrm{Con}(T)$:= there is no proof of contradiction in T.
$\mathrm{Con}_n(T)$:= there is no proof of contradiction of size $\leq n$ in T.

- $T \nvdash \mathrm{Con}(T)$,
- but $T \vdash \mathrm{Con}_n(T)$ with $\mathrm{poly}(n)$-size proofs,
- moreover, for each fixed k,
  $T \vdash \mathrm{Con}_n(T + \exists\Pi^0_k\text{-truth})$ with $\mathrm{poly}(n)$-size proofs.

Fact

T^+ is polynomially simulated by T w.r.t. $\forall\Sigma^0_k$ sentences $\iff$
$T \vdash \mathrm{Con}_n(T^+ + \exists\Pi^0_k\text{-truth})$ *with* $\mathrm{poly}(n)$*-size proofs.*

α-largeness and consistency statements

Fact

For each fixed n, $\mathsf{RCA}_0 \vdash$ *"every infinite set has an ω^n-large subset".*

Fact

$\mathsf{RCA}_0 \vdash$ *"for every x, if every infinite set has an ω^x-large subset, then* $\mathrm{Con}_{\log^*(x)}(\mathsf{RCA}_0 + \exists\Pi^0_2\text{-truth})$*."*

$\rightsquigarrow$ Proved by interpreting terms in size-x *cut-free* proof from RCA_0 as some subsets of the large set A *resp*. elements bounded by $\min A$.
$\rightsquigarrow$ Σ^0_1-induction dealt with using: if A is ω^x-large and $|B| < \min A$, then there is ω^{x-1}-large $A_1 \subseteq A$ such that $B \cap [\min A_1, \max A_1] = \varnothing$.

α-largeness and consistency statements (cont'd)

Our combinatorial bound gives:

Lemma

$\mathsf{RCA}_0 \vdash$ *"for every x, if every infinite set has an ω^{300^x}-large subset, then* $\mathrm{Con}_{\log^*(x)}(\mathsf{RT}^2_2 + \exists\Pi^0_2\text{-truth})$*."*

But we get the following by standard arguments:

Fact

$\mathsf{RCA}_0 \vdash$ *"every infinite set has an $\omega^{2^{2^{\cdot^{\cdot^{2}}}}}$-large subset" (stack of n exponents) with* $\mathrm{poly}(n)$*-size proofs.*

Theorem

$\mathsf{RCA}_0 \vdash \mathrm{Con}_n(\mathsf{RT}^2_2 + \exists\Pi^0_2\text{-truth})$ *with* $\mathrm{poly}(n)$*-size proofs.*

Splitting Ramsey

RT^2_2 splits into $\mathsf{EM} + \mathsf{ADS}$, where:

EM:= Every $f : [\mathbb{N}]^2 \to 2$ is transitive on some infinite set.
ADS:=Every transitive $f : [\mathbb{N}]^2 \to 2$ has an infinite homogeneous set.

(f is transitive if $i < j < k$ and $f(i,j) = f(j,k)$ implies $f(i,k) = f(i,j)$.)

- Already in P-Y (implicitly): If A is ω^{4n+4}-large, every transitive $f : [A]^2 \to 2$ has an ω^n-large homogeneous set.
- Our new result: If A is ω^{36n+3}-large, every $f : [A]^2 \to 2$ is transitive on some ω^n-large set.

Reduction to groupings

To be proved: if A is ω^{36n+3}-large,
then every $f\colon [A]^2 \to 2$ is transitive on some ω^n-large set.

Also in P-Y: this reduces to a statement about *groupings*.

Definition

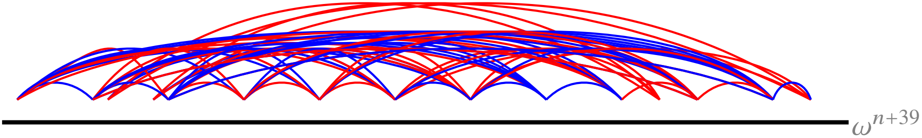
An (α, β)-grouping w.r.t. f is a family of sets $G_1 < \ldots < G_\ell$ such that:

- each G_i is α-large,
- $\{\max G_1, \ldots, \max G_\ell\}$ is β-large ,
- $f\restriction_{G_i \times G_j}$ is constant for each pair $i \neq j$.

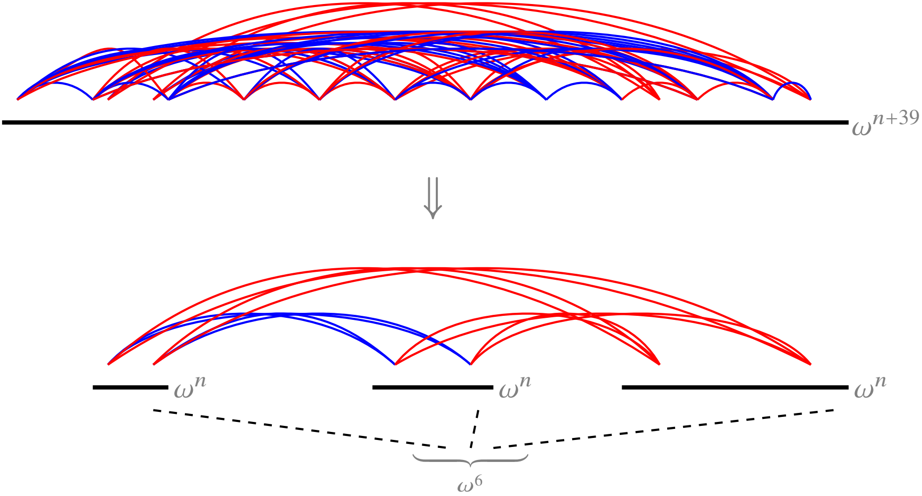
Main lemma

If A is ω^{n+39}-large, then every $f\colon [A]^2 \to 2$ has an (ω^n, ω^6)-grouping.

Main lemma, pictured



Main lemma, pictured



Proof of simple case

We sketch a proof of: if A is ω^{n+6}-large and $\min A \geq d$,
then every $f\colon [A]^2 \to 2$ has an (ω^n, d)-grouping.

- First thin out A so that it is ω^{n+3}-large but *exp-sparse*: for $x, y \in A$, if $x < y$ then $4^x < y$.
- Split A into $\{\min A\} < A_1 < \ldots < A_d$ with each A_i ω^{n+2}-large.
- General fact $(*)$: if you divide $\omega^m \cdot 4k$-large set into k pieces, at least one of them will be ω^m-large.
- Using $(*)$, take ω^{n+1}-large $B_1 \subseteq A_1, \ldots, B_d \subseteq A_d$ so that $f\restriction_{\{x\}\times B_j}$ constant for each $x \in A_i$, $i < j$.
- Using $(*)$, take ω^n-large $C_d \subseteq B_d, \ldots, C_1 \subseteq B_1$ so that $f\restriction_{C_i \times \{\max B_j\}}$ constant for each $i < j$. □

The theory RCA_0^*

It makes perfect sense to consider: $\mathsf{RCA}_0^* := \mathsf{RCA}_0 \setminus \{\Sigma^0_1\text{-induction}\}$.
One just has to add the axiom "2^k exists for every k".

RT^2_2 remains $\forall\Sigma^0_2$-conservative over RCA_0^* (Yokoyama 2013).

Fact

For each fixed n, $\mathsf{RCA}_0^* \vdash$ *"every infinite set has an n-element subset".*
$\mathsf{RCA}_0^* \vdash$ *"for every x, if every infinite set has an x-element subset, then* $\mathrm{Con}_{\log^*(x)}(\mathsf{RCA}_0^* + \exists\Pi^0_2\text{-truth})$*."*

The theory RCA^*_0

Using the exponential lower bounds on $R(n,n)$ we get:

Lemma

$\mathsf{RCA}^*_0 + \mathsf{RT}^2_2 \vdash$ *"for every x, if every infinite set has an x-element subset, then every infinite set has a 2^x-element subset".*

This gives short proofs that infinite sets contain very large finite subsets. Combining this with the implication to consistency, we get:

Lemma

For $m = 2^{2^{\cdot^{\cdot^{2}}}}$ (stack of n exponents), $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ proves $\mathrm{Con}_m(\mathsf{RCA}^*_0)$ with proofs of size $\mathrm{poly}(n)$.

Theorem

$\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ has iterated exponential speedup over RCA^*_0 w.r.t. proofs of Π^0_1 sentences.

References

Patey, Yokoyama, *The proof-theoretic strength of Ramsey's theorem for pairs and two colors*, Adv. Math. 330(2018), 1034-1070.

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