

RAMSEY CLASSES AND BEYOND

---

JAROSLAV NEŠETRIL  
CHARLES UNIVERSITY  
PRAGUE

---

RaTLoCC  
BERTINORO II. - 2018

RAMSEY CLASSES AND BEYOND

---

JAROSLAV NEŠETŘIL  
CHARLES UNIVERSITY  
PRAGUE

&  
DAVID EVANS  
LONDON

JAN HUBIČKA  
PRAGUE



---

RaTLoCC  
BERTINORO II - 2018

GRAPH RAMSEY THEOREM GRT

EVERY LARGE GRAPH  $G$  ( $|G| \geq N$ )

CONTAINS:

EITHER LARGE CLIQUE OR LARGE INDEPENDENT SET

$$\omega(G) \geq n$$

$$\alpha(G) \geq n.$$

GRAPH RAMSEY THEOREM

GRT

EVERY LARGE GRAPH  $G$  ( $|G| \geq N$ )

CONTAINS:

EITHER LARGE CLIQUE OR LARGE INDEPENDENT SET

$$\omega(G) \geq n$$

$$\alpha(G) \geq n.$$

GRT

FOR EVERY PARTITION  $\binom{[N]}{2} = a_1 \cup \dots \cup a_k$

THERE EXISTS  $Y \subseteq [N] = \{1, 2, \dots, N\}, i_0$

SUCH THAT  $|Y| = n$

$$\binom{Y}{2} \subseteq a_{i_0}.$$

$$N \longrightarrow \binom{n}{k}^2$$

ERDŐS-RADO PARTITION ARROW

FINITE RAMSEY THEOREM

FRT

$\forall p, k, n \exists N:$

FOR EVERY PARTITION  $\binom{[N]}{p} = a_1 \cup \dots \cup a_k$   
 THERE EXISTS  $Y \subseteq [N]$ ,  $i_0$  SUCH THAT

$$\binom{Y}{p} \subseteq a_{i_0}.$$

$$N \longrightarrow \binom{n}{k}^p.$$


---

OTHER EARLY EXAMPLES

(PIODGEON HOLE  $p=1$ )

VAN DER WAERDEN

SCHUR

HILBERT

## OTHER EARLY EXAMPLES

(PIGGEON HOLE  $p=1$ )

VAN DER WAERDEN

SCHUR

HILBERT

60-70ies  
COMBINATORIAL CUBES

$(m, p, c)$  SETS

HALES-JEWETT  
GRAHAM-ROTHSCHILD

(RADO CONJECTURE)

DEUBER, LEEB

FINITE VECTOR  
SPACES

(ROTA CONJECTURE)

GRAHAM, LEEB, ROTHSCHILD

$K_2$ -FREE GRAPHS

(ERDÖS, GALVIN, HAJNAL)

N. RÖDL



DEFINITION OF RAMSEY CLASS

(LEEB, N., RÖDL)

$\mathcal{K}$  CLASS OF STRUCTURES + SUBOBJECTS  
(EMBEDDINGS)

$A, B \in \mathcal{K}$   $\binom{B}{A}$  = ALL SUBOBJECTS OF B  
ISOMORPHIC TO A



INTERESTING HISTORY

$$\binom{n}{k}$$

A. ETTINGHAUSEN



AABAB

MENDEL

$$\binom{x}{k}$$

$$\binom{B}{A}$$

ANDREAS VON ETTINGHAUSEN:  
DE KOMBINATORISCHE ANALYSIS  
ALS  
VORBEREITUNGSLEHRE ZUM STUDIUM  
DER  
THEORETISCHEN HÖHEREN MATHEMATIK  
1826

Die  
**combinatorische Analysis**

als

**Vorbereitungslehre zum Studium**

der

**theoretischen höhern Mathematik,**

**dar gestellt**

von

**Andreas v. Ettingshausen,**

**Professor der höhern Mathematik an der k. k. Universität zu Wien.**

---

**W i e n , 1 8 2 6 .**

**Druck und Verlag von J. B. Wallishausser.**

402

# Versuche

über

# Pflanzen-Hybriden,

VON

**Gregor Mendel.**

---

(Separatdruck aus dem IV. Bande der Verhandlungen des naturforschenden Vereines.)

Im Verlage des Vereines.



---

Brünn, 1866.

Ans Georg Gasil's Buchdruckerei, Postgasse Nr. 446.

HELENA NEŠETŘILOVÁ, J.N. :  
MATHEMATICS OF MENDEL,  
MENDELIANUM, FOLIA MENDELIANA  
53,1-2 (2017), 5-22.

# DEFINITION OF RAMSEY CLASS

(LEEB, N., RÖDL)

$\mathcal{K}$  CLASS OF STRUCTURES + SUBOBJECTS  
(EMBEDDINGS)

$A, B \in \mathcal{K}$   $\binom{B}{A}$  = ALL SUBOBJECTS OF B ISOMORPHIC TO A



$\mathcal{K}$  IS A **RAMSEY CLASS**

IF

FOR EVERY  $A, B \in \mathcal{K}$ ,  $k \in \mathbb{N}$ ,  
THERE EXISTS  $C \in \mathcal{K}$  SUCH THAT

$$C \longrightarrow \binom{B}{k}^A$$

|||

FOR EVERY PARTITION  $\binom{C}{A} = a_1 \cup \dots \cup a_k$   
THERE EXISTS  $B' \in \binom{C}{B}$ ,  $i_0$   
SUCH THAT  $\binom{B'}{A} \subseteq a_{i_0}$ .

A GENERAL COMMENT

$$C \rightarrow (B)_2^A \Leftrightarrow \chi(\langle A, B, C \rangle) \geq 2$$

$$\langle A, B, C \rangle = (X, M)$$

$$X = \begin{pmatrix} C \\ A \end{pmatrix}$$

$$M \in M \Leftrightarrow \exists B' \in \begin{pmatrix} C \\ B \end{pmatrix}$$

$$M = \begin{pmatrix} B' \\ A \end{pmatrix}$$



WHAT ARE TOOLS FOR  
HIGH CHROMATIC #

2  
0



# WHAT ARE TOOLS FOR HIGH CHROMATIC #

?

- PIGEONHOLE

- PRODUCTS

- DENSITY  $(x \gg \frac{|X|}{\alpha})$

- EIGENVALUES

- ALG. GEOMETRY  
(KNESER-LOVA'SZ)

- PROBABILISTIC

⋮

WHAT ARE TOOLS FOR  
HIGH CHROMATIC #

?

- PIGEONHOLE

- PRODUCTS

- DENSITY  $(x \gg \frac{|X|}{\alpha})$

- EIGENVALUES

- ALG. GEOMETRY  
(KNESER-LOVA'SZ)

⋮

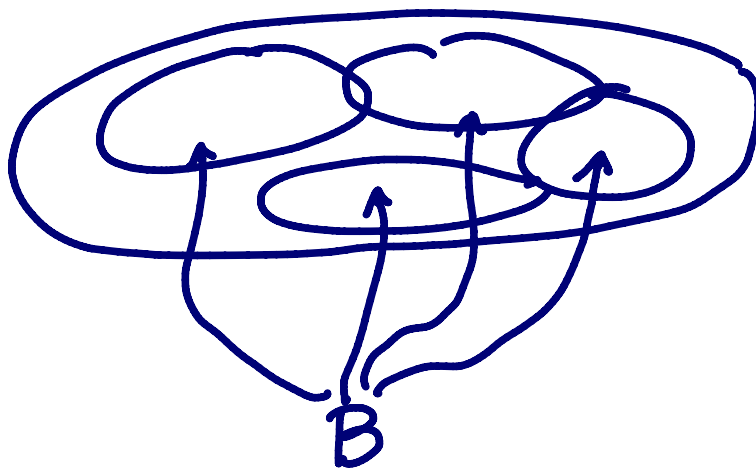
- ULTRAFILTERS

$(X, \mathcal{U})$  ULTRAFILTER

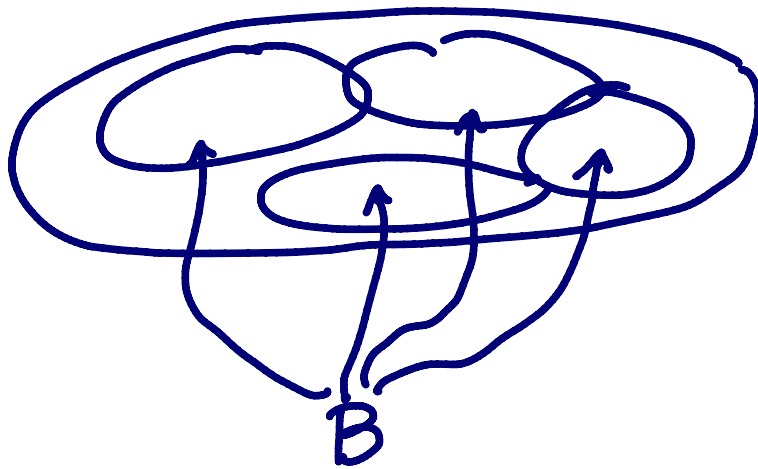
↓

$x(X, \mathcal{U}) > 2$

WHY NOT (RANDOM)  
INSERTION METHOD ?



WHY NOT (RANDOM)  
INSERTION METHOD ?

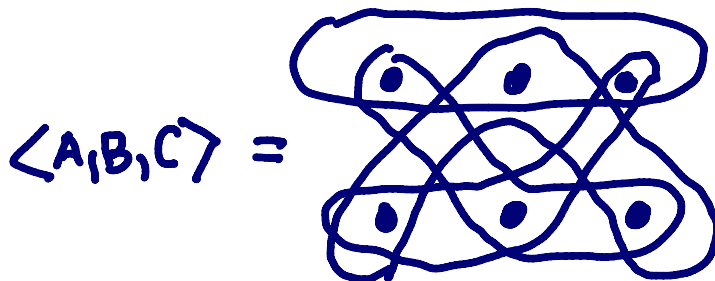


---

USEFULL FOR  $|A|=1$   
FOR  $|A|>1$   $\langle A, B, C \rangle$   
VERY STRUCTURED

EXAMPLE

$A = I, B = \Delta, C = \boxtimes$



# RAMSEY CLASSES

||

TOP OF THE LINE OF RAMSEY PROPERTIES

"ONE CAN COLOUR ANYTHING (POINTS, EDGES, ...)  
IN ANY NUMBER OF COLOURS AND  
YET OBTAIN ANY GIVEN OBJECT  
MONOCHROMATIC"

# EXAMPLES OF RAMSEY CLASSES

ARITHMETIC PROGRESSIONS	$(\subseteq \mathbb{Z})$	NO
————— —————	$(\subseteq \mathbb{Z}^n)$	YES
GRAPHS		NO
ORDERED GRAPHS		YES
ORDERED $\Delta$ -FREE		YES
ORDERED $\square$ -FREE		NO

---

# EXAMPLES OF RAMSEY CLASSES

ARITHMETIC PROGRESSIONS	$(\subseteq \mathbb{Z})$	NO
	$(\subseteq \mathbb{Z}^n)$	YES
GRAPHS		NO
ORDERED GRAPHS		YES
ORDERED $\Delta$ -FREE		YES
ORDERED $\square$ -FREE		NO

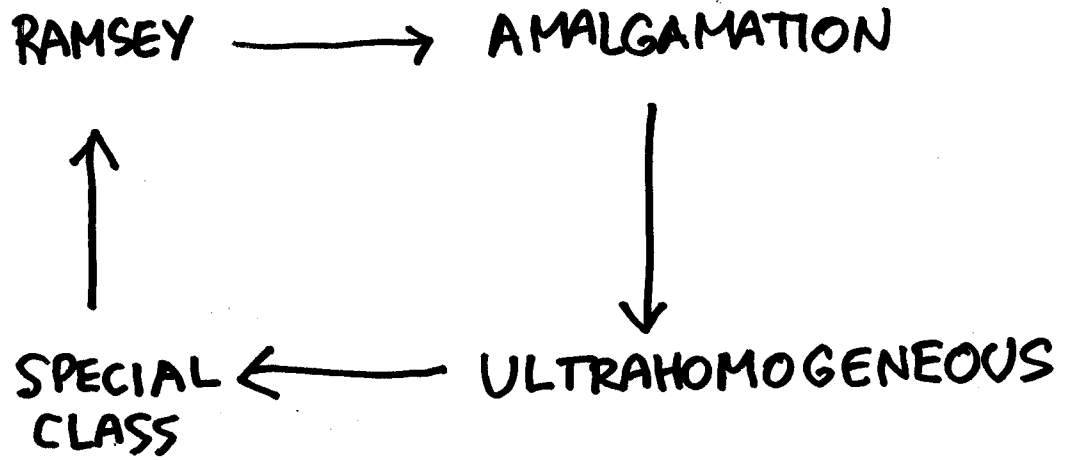
---

COMBINATORIAL ZOO

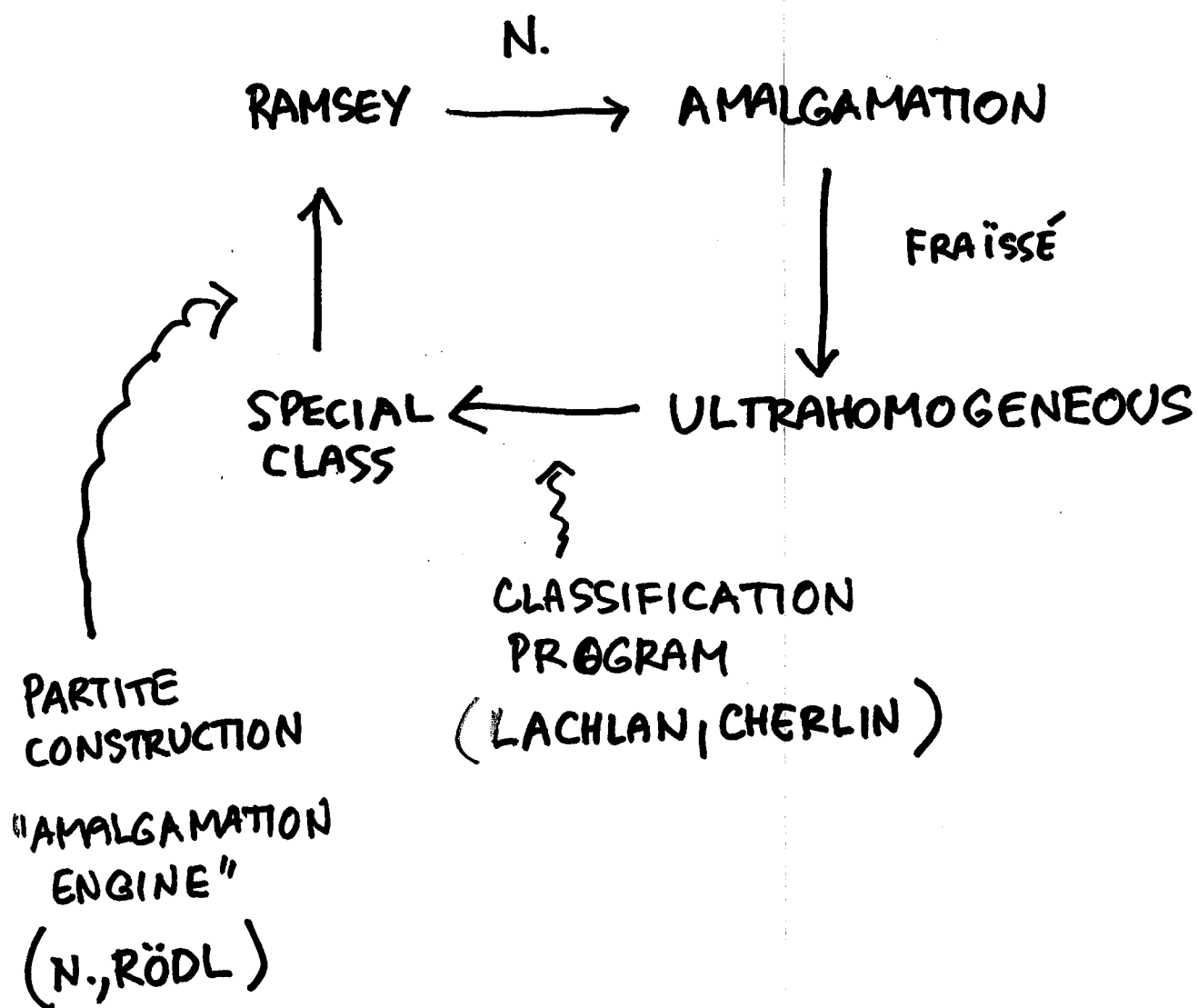
# CHARACTERISATION OF RAMSEY CLASSES ?



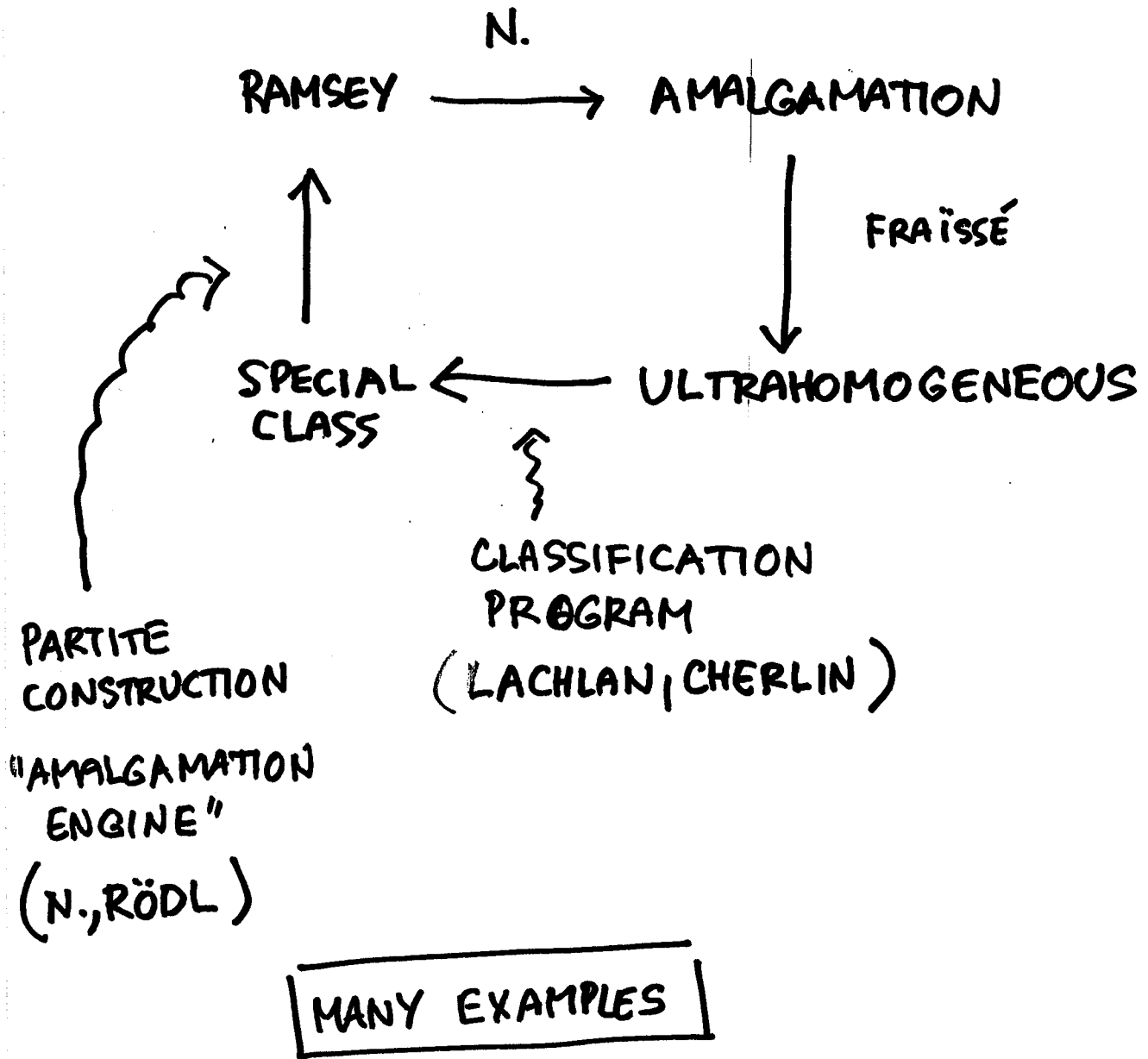
## I. AMALGAMATION CHARACTERIZATION



# I. AMALGAMATION CHARACTERIZATION



# I. AMALGAMATION CHARACTERIZATION



GRAPHS, RELATIONAL STRUCTURES, POSETS  
METRIC SPACES, ...

## CHARACTERIZATION OF

- ALL RAMSEY CLASSES OF GRAPHS  
WITH FREE ORDERING (N. 89)
- ALL RAMSEY CLASSES OF POSETS  
WITH LINEAR EXTENSION (N. 05)
- ALL RAMSEY CLASSES OF TOURNAMENTS  
WITH FREE ORDERING (N. 05)
- ALL RAMSEY CLASSES OF DIRECTED  
GRAPHS WITH ORDERING AND LABELING  
(JASINSKI, LAFLAMME, NGUYEN VAN THE,  
WOODROW 2013)

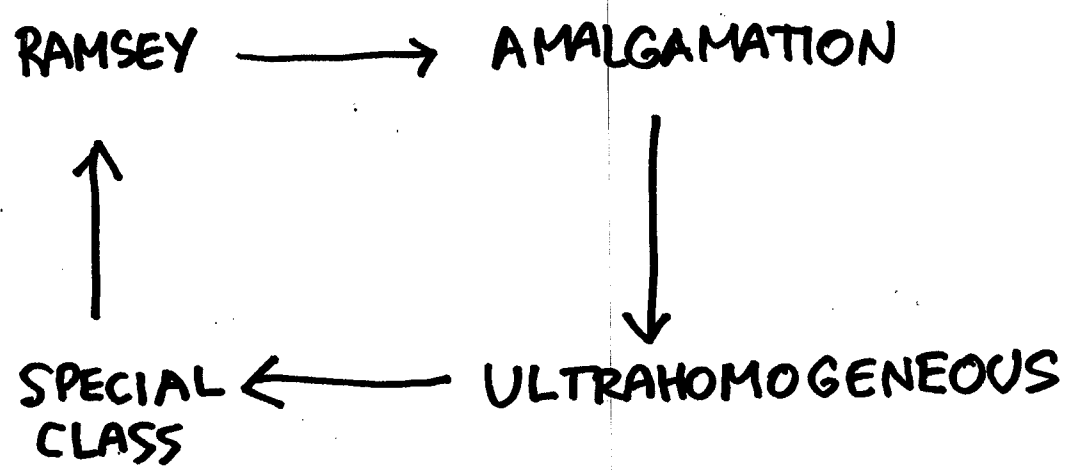
---

CHARACTERISATION OF UTRAHOMOGENEOUS  
(LACHLAN+WOODROW, CHERLIN )

+

COMBINATORIAL THEOREMS (N.+RÖDL)

# I. AMALGAMATION CHARACTERIZATION



FREE AMALGAMATION

**THM** (N., RÖDL 77)

ANY CLASS  $\mathcal{K}$  OF RELATIONAL STRUCTURES  
 CLOSED ON FREE AMALGAMATION AND  
 FREE ORDERINGS IS A RAMSEY CLASS

**THM** (EVANS, HUBIČKA, N. 17)

LET  $L$  BE LANGUAGE (INVOLVING  
RELATION SYMBOLS AND PARTIAL FUNCTIONS).

LET  $\mathcal{K}$  BE A FREE AMALGAMATION  
CLASS OF  $L$ -STRUCTURES. THEN  $\mathcal{K}$   
IS A RAMSEY CLASS.


**THM** (EVANS, HUBIČKA, N. 17)

LET  $L$  BE LANGUAGE (INVOLVING  
RELATION SYMBOLS AND PARTIAL FUNCTIONS).

LET  $\mathcal{K}$  BE A FREE AMALGAMATION  
CLASS OF  $L$ -STRUCTURES. THEN  $\mathcal{K}$   
IS A RAMSEY CLASS.

**COROLLARY**

BOWTIE FREE GRAPHS HAVE FINITE  
RAMSEY EXPANSION

(FORB() )

**THM** (EVANS, HUBIČKA, N. 17)

LET  $L$  BE LANGUAGE (INVOLVING  
RELATION SYMBOLS AND PARTIAL FUNCTIONS).

LET  $\mathcal{K}$  BE A FREE AMALGAMATION  
CLASS OF  $L$ -STRUCTURES. THEN  $\mathcal{K}$   
IS A RAMSEY CLASS.

**COROLLARY**

BOWTIE FREE GRAPHS HAVE FINITE  
RAMSEY EXPANSION

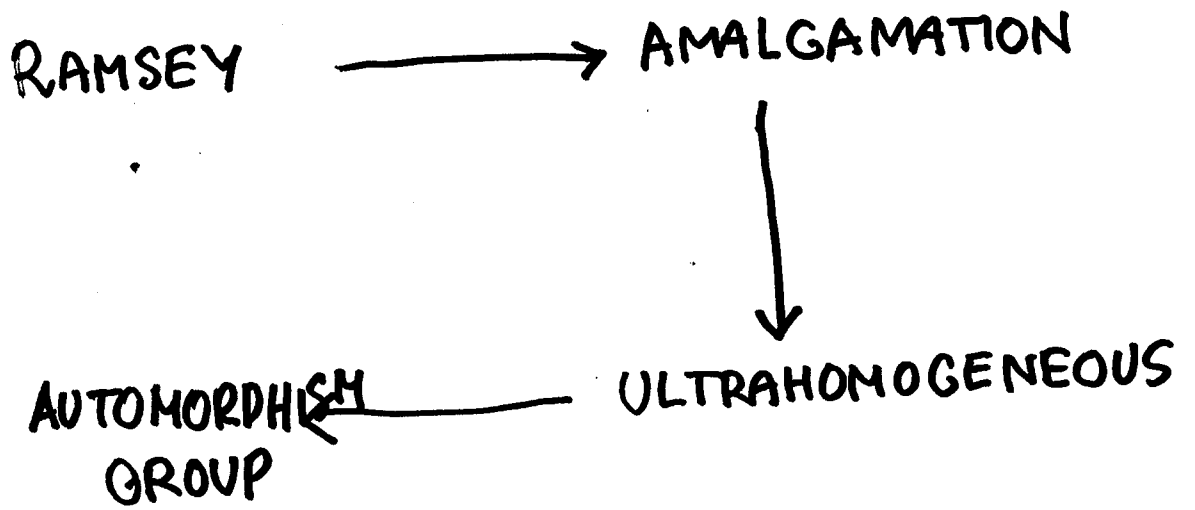
(FORB( ~~$D$~~ ))

**COROLLARY**

STEINER SYSTEMS FORM A RAMSEY  
CLASS



## II. TOPOLOGICAL CHARACTERISATION



## II. TOPOLOGICAL CHARACTERISATION

RAMSEY  $\longrightarrow$  AMALGAMATION



EXTREME  
AMENABILITY



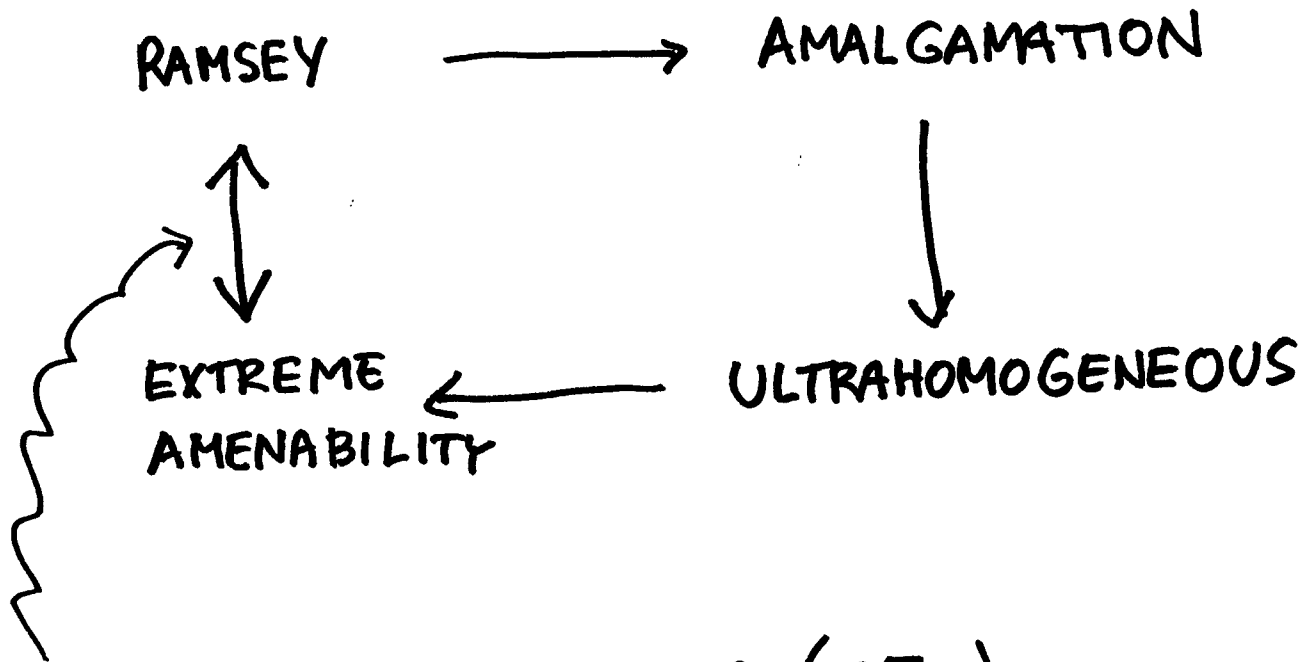
ULTRAHOMOGENEOUS



KECHRIS, PESTOV, TODORCEVIC (05)

"KPT CORRESPONDENCE"

## II. TOPOLOGICAL CHARACTERISATION



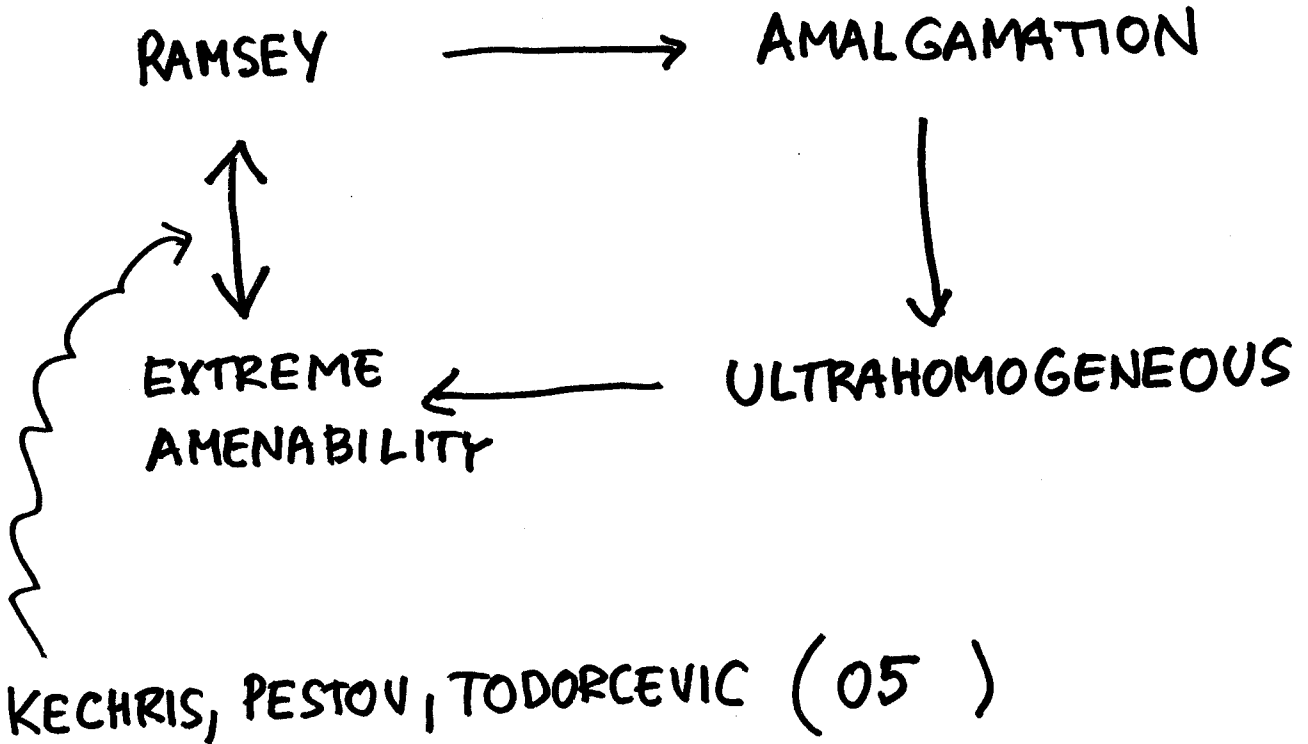
KECHRIS, PESTOV, TODORCEVIC (05)

"KPT CORRESPONDENCE

---

EXTREM AMENABILITY VIA RAMSEY  
(GLASNER, WEISS)

## II. TOPOLOGICAL CHARACTERISATION

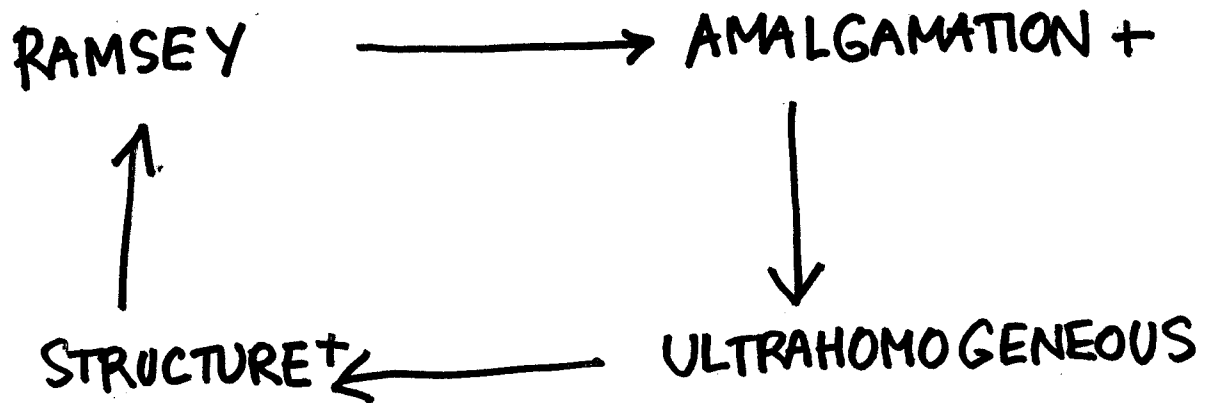


"KPT CORRESPONDENCE

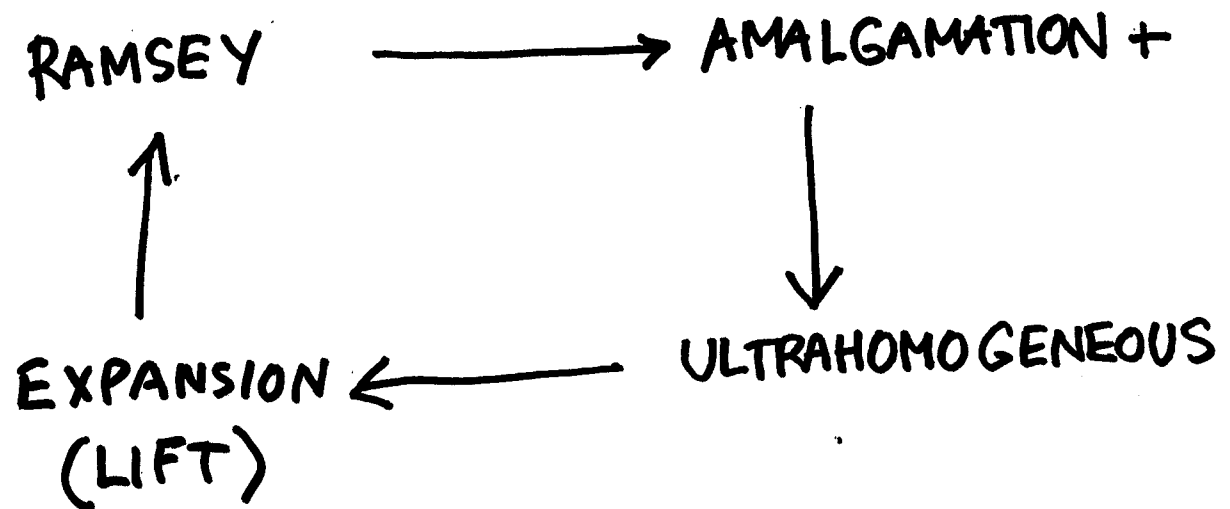
EXTREM AMENABILITY VIA RAMSEY  
(GLASNER, WEISS)

QUEST FOR RAMSEY CLASSES

### III. EXPANSIONS & LIFTS



### III. EXPANSIONS & LIFTS




---

LANGUAGE

L



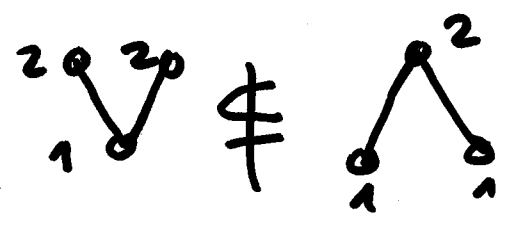
EXTENDED LANGUAGE

L U

ADDITIONAL  
RELATIONS  
FUNCTIONS

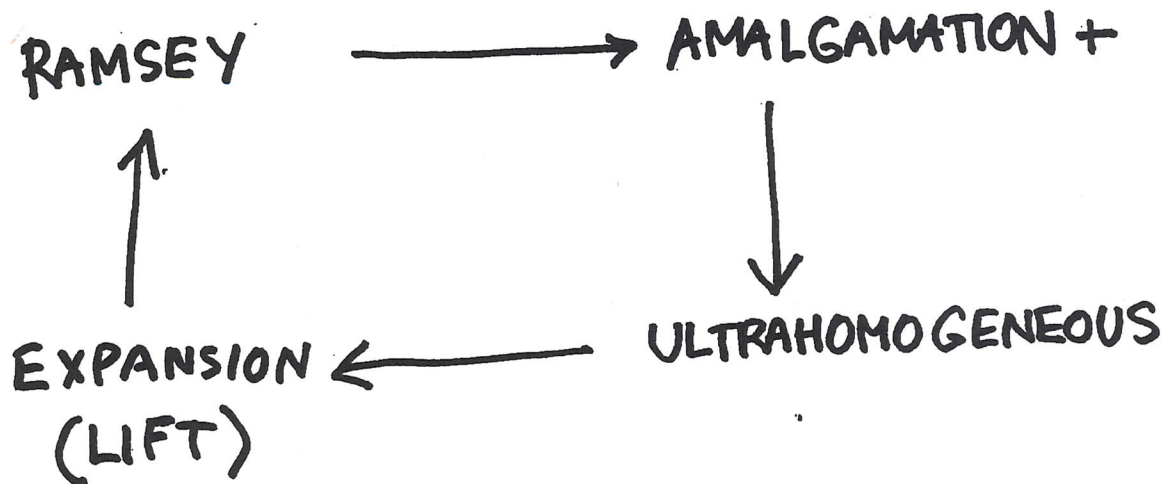
EXAMPLES

BIPARTITE GRAPHS  $\rightsquigarrow$  BIPARTITE GRAPHS WITH MARKED PARTS



GRAPHS  $\rightsquigarrow$  GRAPHS WITH ORDERED VERTICES

### III. EXPANSIONS & LIFTS




---

FOR WHICH  $\omega$ -CATEGORICAL STRUCTURE  
 THE AGE HAS FINITE  
 PRECOMPACT

RAMSEY EXPANSION ?

(BERTINORO QUESTION 11 )  
 BODIRSKY, VAN THE, N.  
 PINSKER, TSANKOV,



## GROUP FORMULATION

SUPPOSE  $G$  IS CLOSED, OLIGOMORPHIC  
PERMUTATION GROUP ON A COUNTABLE  
SET. DOES THERE EXIST A CLOSED,  
EXTREMELY AMENABLE, OLIGOMORPHIC  
SUBGROUP OF  $G$  ?

## GROUP FORMULATION

SUPPOSE  $G$  IS CLOSED, OLIGOMORPHIC PERMUTATION GROUP ON A COUNTABLE SET. DOES THERE EXIST A CLOSED, EXTREMELY AMENABLE, OLIGOMORPHIC SUBGROUP OF  $G$  ?

---

NO (EVANS 16)

BUT MAXIMAL EXTREMELY AMENABLE SUBGROUPS EXIST



OPTIMAL RAMSEY EXPANSIONS EXIST

# HRUSHOVSKI CLASSES $\mathcal{C}_F$

$k$ -ORIENTABLE GRAPHS

+

$$k \cdot |A| - |E \cap \binom{A}{2}| \geq F(|A|)$$

FOR ALL  $A \subseteq V$ .

THM

(EVANS, HUBIČKA, N. 17)

1.  $\mathcal{C}_F$  OLIGOMORPHIC
2. EVERY EXTREMELY AMENABLE SUBGROUP HAS INFINITELY MANY ORBITS ON  $[\mathbb{N}]^2$ .
3. THERE EXIST REASONABLE EXPANSION OF  $\mathcal{C}_F$  WHICH IS RAMSEY CLASS

# POSITIVE SOLUTIONS

① FORBIDDEN SET OF  
IRREDUCIBLE STRUCTURES

(IRREDUCIBLE  $\equiv$  FREE AMALGAM IRREDUCIBLE)  
 $\equiv$  ANY PAIR OF DISTINCT  
POINTS COVERED



RAMSEY LIFT = ORDERING  
(N., RÖDL 78)

# POSITIVE SOLUTIONS

- ① FORBIDDEN SET OF  
IRREDUCIBLE STRUCTURES

(IRREDUCIBLE  $\equiv$  FREE AMALGAM IRREDUCIBLE)  
 $\equiv$  ANY PAIR OF DISTINCT  
POINTS COVERED



RAMSEY LIFT = ORDERING  
(N., RÖDL 78)

- ② FORBIDDEN HOMOMORPHISMS  
FROM A FINITE SET OF  
CONNECTED STRUCTURES  
(N. 10)

RAMSEY LIFT = ORDERING  
+  
PIECES

(NEEDED: NEW COMBINATORIAL PROOF  
OF CHERLIN, SHELAH, SHE)

(3)

FORBIDDEN HOMOMORPHISMS  
FROM AN INFINITE REGULAR  
FAMILY  $\mathcal{F}$  WHICH IS LOCALLY  
FINITE

RAMSEY LIFT = ORDERING  
+  
MAXIMAL  
(J. HUBIČKA, N. 15)


③

FORBIDDEN HOMOMORPHISMS  
FROM AN INFINITE REGULAR  
FAMILY  $\mathcal{F}$  WHICH IS LOCALLY  
FINITE

RAMSEY LIFT = ORDERING  
+  
MAXIMAL

(J. HUBIČKA, N. 15)

EXAMPLE


FORB<sub>HOM</sub> ()

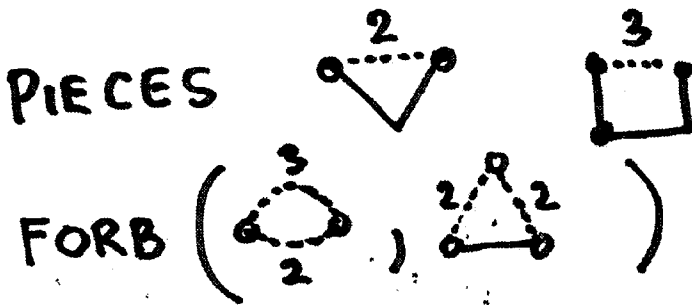
③ FORBIDDEN HOMOMORPHISMS  
 FROM AN INFINITE REGULAR  
 FAMILY  $\mathcal{F}$  WHICH IS LOCALLY  
 FINITE

RAMSEY LIFT = ORDERING  
 +  
 MAXIMAL

(J. HUBIČKA, N. 15)

EXAMPLE

FORB<sub>HOM</sub> (  )



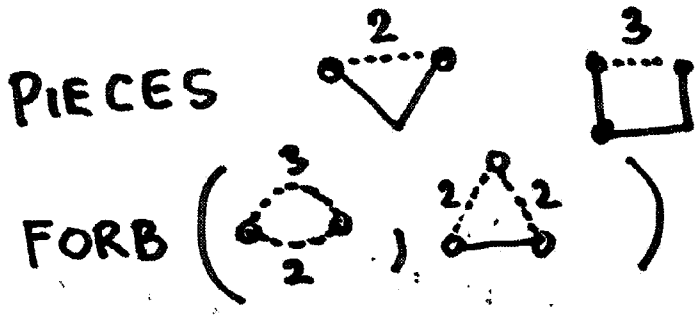


③ FORBIDDEN HOMOMORPHISMS  
 FROM AN INFINITE REGULAR  
 FAMILY  $\mathcal{F}$  WHICH IS LOCALLY  
 FINITE

RAMSEY LIFT = ORDERING  
 +  
 MAXIMAL  
 (J. HUBIČKA, N. 15)

EXAMPLE

$\text{FORB}_{\text{HOM}}(\text{pentagon})$



$\text{FORB}_{\text{HOM}}(\text{pentagon})$  NOT RAMSEY



③ FORBIDDEN HOMOMORPHISMS FROM AN INFINITE REGULAR FAMILY  $\mathcal{F}$  WHICH IS LOCALLY FINITE

RAMSEY LIFT = ORDERING + MAXIMAL

(J. HUBIČKA, N. 15)

EXAMPLE

$\text{FORB}_{\text{HOM}}(\text{pentagon})$

PIECES  

$\text{FORB}(\text{triangle with dashed line and labels 2, 3}, \text{triangle with dashed line and labels 2, 2})$

$\text{FORB}_{\text{HOM}}(\text{pentagon})$  NOT RAMSEY

$\xrightarrow{\text{RAMSEY LIFT}} \text{FORB}_{\text{HOM}}(\text{pentagon}, \text{triangle with dashed line and labels 2, 3}, \text{triangle with dashed line and labels 2, 2})$

RAMSEY

$\cap \text{REL}(E_1, \dots, E_n)$

④

**METRIC SPACES**

①

**FINITE METRIC SPACES****RAMSEY LIFT = LINEAR ORDER****N. 05**

④

**METRIC SPACES**

①

**FINITE METRIC SPACES****RAMSEY LIFT = LINEAR ORDER**

N. 05

②

**GRAPHS WITH ISOMETRIC  
EMBEDDINGS****RAMSEY LIFT = LINEAR ORDER****DELLAMONICA, RÖDL 12**

④

# METRIC SPACES

①

FINITE METRIC SPACES

RAMSEY LIFT = LINEAR ORDER

N. 05

②

GRAPHS WITH ISOMETRIC EMBEDDINGS

RAMSEY LIFT = LINEAR ORDER

DELLAMONICA, RÖDL 12

③

 $S$ -METRIC SPACES(SPACES WITH DISTANCES IN  $S$ )

THM
-----

FOR  $S$  THE FOLLOWING IS EQUIV.① FINITE CONVEXLY ORDERED  $S$ -METRIC SPACES ARE RAMSEY② THERE EXISTS  $S$ -URYSOHN SPACE  
HUBIČKA, N. 15+ (USING SAUER 12)

d) ALL ULTRAHOMOGENEOUS  
METRIC SPACES (CHERLIN  
LIST)

HAVE RAMSEY LIFT

(ARANDA, BRADLEY-WILLIAMS,  
HUBIČKA, KARAMANLIS,  
KOMPATSCHER, KONEČNÝ,  
PAWLIUK)

# A MASTER THEOREM

(HUBIČKA, N. 16)

$\mathcal{R}$  RAMSEY CLASS,  $\mathcal{K} \subseteq \mathcal{R}$

SATISFIES:

- $\mathcal{K}$  HEREDITARY
- $\mathcal{K}$  CLOSED ON STRONG AMALGAMATION
- ALL STRUCTURES IN  $\mathcal{R}$  WHICH CANNOT BE COMPLETED IN  $\mathcal{K}$  ARE LOCALLY FINITE.

THEN

$\mathcal{K}$  IS RAMSEY CLASS.

# A MASTER THEOREM

(HUBIČKA, N. 16)

$\mathcal{R}$  RAMSEY CLASS,  $\mathcal{K} \subseteq \mathcal{R}$

SATISFIES:

- $\mathcal{K}$  HEREDITARY
- $\mathcal{K}$  CLOSED ON STRONG AMALGAMATION
- ALL STRUCTURES IN  $\mathcal{R}$  WHICH CANNOT BE COMPLETED IN  $\mathcal{K}$  ARE LOCALLY FINITE.

THEN  $\mathcal{K}$  IS RAMSEY CLASS.

(LOCAL FINITE : FOR SOME  $f$   
 $F \rightarrow A \Rightarrow \exists F' \in \mathcal{K} \text{ (IF } |F'| \leq f(|A|)\text{)}$   
 $F' \rightarrow A$ )



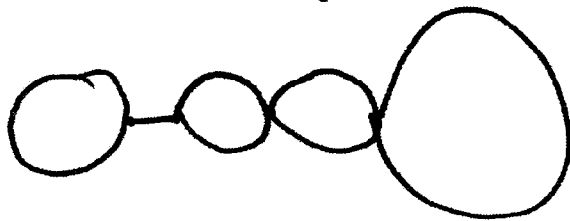
ALSO GENERAL RESULTS FOR  
STRUCTURES WITH CLOSURES  
( $\sim$  ALGEBRAIC RANK)

**EXAMPLE**

$\text{FORB}_{\text{MONO}}(\{F\})$  UNIVERSAL



CHERLIN CONJ.



CHAIN OF  
COMPLETE  
GRAPHS

**THM**

ALL CHERLIN CLASSES  
HAVE RAMSEY LIFT.

(HUBIČKA, N. 16)

FOR EXAMPLE:



NO CHARACTERISATION

BUT

(SURPRISING) RICH  
VARIETY

AND

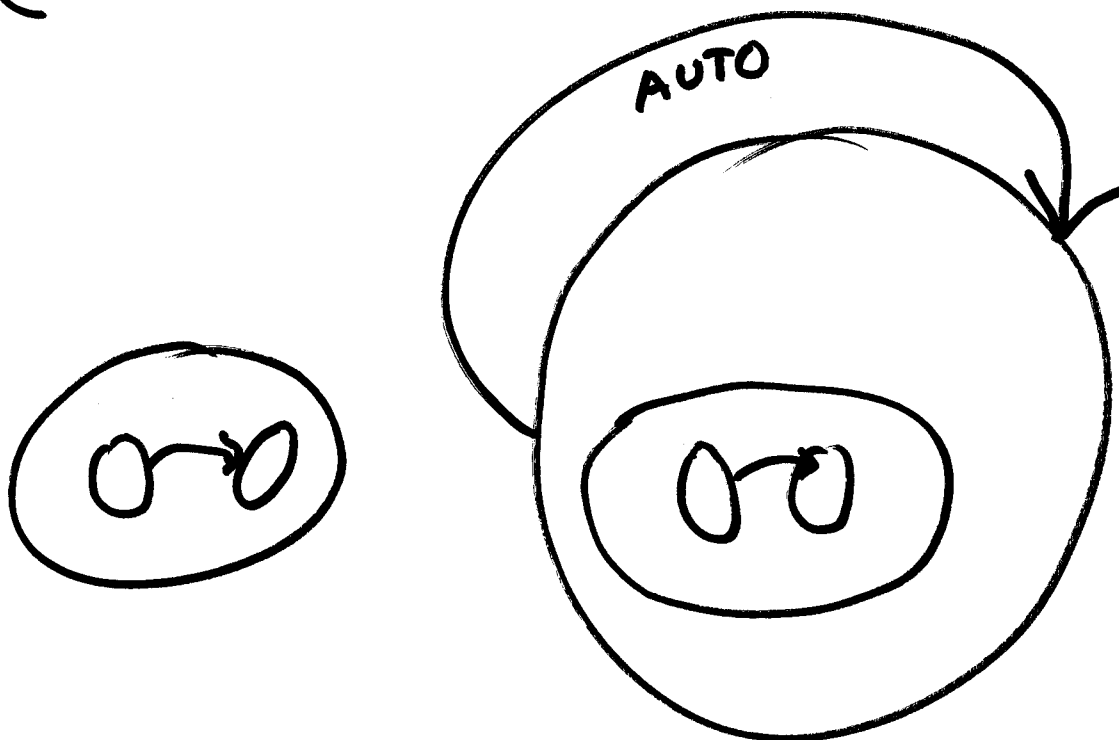
VERSATILITY OF

RAMSEY CLASSES

# CONCLUDING REMARKS

SIMILAR RESULTS FOR  
EXTENSION PROPERTY FOR  
PARTIAL ISOMORPHISMS } EPPA

(HRUSHOVSKI PROPERTY)



**THM** (EVANS, HUBIČKA, N. 17)

LET  $L$  BE LANGUAGE SUCH THAT  
EVERY FUNCTION  $f \in L$  IS UNARY.

THEN EVERY FREE AMALGAMATION  
CLASS  $\mathcal{K}$  OF  $L$ -STRUCTURES HAS  
EPPA.

(EXTENDS HODKINSON-OTTO, HERWIG-LASKAR)

---

BINARY ?

DO STEINER SYSTEM HAVE EPPA ?

**Placed Image**

# DOCCOURSE "SPARSITY"

NOV 4 — DEC 15  
2018

AN INTENSIVE PROGRAM  
FOR STUDENTS AND POSTDOCS

LECTURES & EXERCISES  
SPEAKERS

I. ADLER, B. SULLIVAN, F. REIDL,  
M. PILIPCZUK, S. SIEBERTZ,  
P. SIMON, P. OSSONA DE MENDEZ,  
D. KRÁL', ...



DOCCOURSE @ IUK.MFF.CUNI.CZ



THANK YOU FOR  
YOUR ATTENTION