The Hrushovski property for hypertournaments and profinite topologies

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Theorem (Hrushovski)

If G is a finite graph and

$$\varphi_1, \ldots, \varphi_n : G \to G$$

are partial isomorphisms, then there exists a finite graph G' containing G as an induced subgraph such that all φ_i extend to automorphisms of G'.

Definition

A class K of finite structures has the **Hrushovski property** if for any M in K and partial isomorphisms

$$\varphi_1, \ldots, \varphi_n : G \to G$$

there exists M' in K extending M such that all φ_i extend to automorphisms of M'.

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Question (Herwig and Lascar)

Does the class of finite tournaments have the Hrushovski property?

Definition

Let P be a set of prime numbers. The **pro-**P topology on the free group F_n is the one with neighborhoods of 1 consisting of those $N \triangleleft F_n$ such that

$$[F_n : N] \text{ is finite and divisible only by numbers in } P$$

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Examples

- if P is the set of all primes, then the above is called the profinite topology,
- if $P = \{p\}$, then the above is the pro-p topology,
- if $P = \{\text{all primes} > 2\}$, then this is called the pro-odd topology

Definition

A subgroup $G < F_n$ is **closed under q-th roots** if

$$g^q \in G \quad \text{implies} \quad g \in G.$$

When $q = 2$, we refer to this as **closed under square roots**.

Note

If G is pro-P closed, then it must be closed under q-th roots for every prime $q \notin P$.

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If $g^q \in G$ but $g \notin G$, then g cannot be separated from G in any finite quotient of size not divisible by q because in such fintie quotient $\pi(g)$ is a power of $\pi(g^q)$.

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In particular, a group which is pro-odd closed must be closed under square roots

Fact (Herwig and Lascar)

TFAE

- The class of finite tournaments has the Hrushovski property,
- For any n TFAE for any finitely generated $G < F_n$:
  - G is closed in the pro-odd topology,
  - G is closed under square roots.

Question 2

Is it true that for any set of primes P TFAE for any finitely generated $G < F_n$.

- G is closed in the pro-P topology,
- G is closed under q-th roots for all prime $q \notin P$.

Definition

Let P be a set of primes. An P-**hypertournament** is a relational structure $(A, R_i : i \in P$ where R_i is an i-ary relation such that for every $i \in P$:

- for every $(x_1, \ldots, x_i) \in A^i$ there exists a permutation σ such that
$$A \models R_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)})$$
- there does not exist $(x_1, \ldots, x_i) \in A^i$ such that
$$A \models R_i(x_1, \ldots, x_i),$$
$$A \models R_i(x_2, \ldots, x_i, x_1),$$
$$\ldots,$$
$$A \models R_i(x_i, x_1, \ldots, x_{i-1})$$

Fact

For general set of primes P, Question 2 is equivalent to the Hrushovski property for the class of P-hypertournaments.

A positive answer (HPSW)

For any set of primes P and any subgroup $G < F_n$ of rank 1 (i.e. cyclic) TFAE:

- G is closed in the pro-P topology,
- G is closed under q-th roots for all primes $q \notin P$

To understand, what this means in terms of the Hrushovski property, we need to introduce some terminology

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Immersed graphs

An **immersed graph** is a directed graph with edges labelled with letters $a_1, \ldots, a_n$ such that every vertex has at most incoming and at most one outgoing edge labelled with a_i.

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Immersed graphs

An **immersed graph** is a directed graph with edges labelled with letters $a_1, \ldots, a_n$ such that every vertex has at most incoming and at most one outgoing edge labelled with a_i.

Note that a family of partial isomorhpisms of X $a_1, \ldots, a_n$ naturally induce an immereed graph on X.

Definition

A graph is a **subtadpole** if in each connected component there is at most one vertex of degree 3 and all other have degree at most 2

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Definition

A family of partial isomorhpisms **forms a subtadpole** if the induced immersed graph is a subtadpole (treated as a graph).

Fact

For any set of primes P the following are equivalent:

- The class of finite P-hypertournaments has the Hrushovski property for familis of partial isometries which form subtadpoles.
- for any subgroup $G < F_n$ of rank 1 (i.e. cyclic) TFAE:
  - G is closed in the pro-P topology,
  - G is closed under q-th roots for all primes $q \notin P$

Remark

Note that if $\varphi_1, \ldots, \varphi_n$ are partial isomorphisms which have pariwise disjoint domains and pairwise disjoint ranges, then they form a subtadpole (every vertex in the immersed graph has degree at most 2)

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Corollary

The class of finite tournaments has the Hrushovski property for families of partial isomorphisms with pariwise disjoint domains and pairwise disjoint ranges.

A negative answer (HPSW)

There exists a finitely generated subgroup $G < F_2$ which is closed under q-th roots for all q and is not closed in any p-adic topology.

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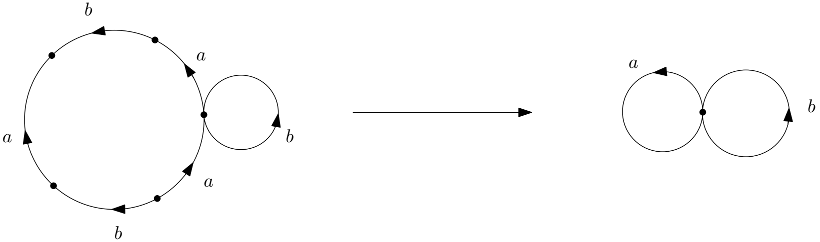
The group has a nice presentation

$$G = \langle aba^{-1}b^{-1}a, b \rangle$$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To see that this group is closed under q-th roots, one can either introduce the structure of an hypertournament on the set of vertices or argue more abstractly

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Definition

A subgroup $G < F_n$ is **malnormal** if for every $g \notin G$ we have $G \cap gGg^{-1} = \{1\}$

Fact

If $G < F_n$ is malnormal, then it is closed under q-th roots for all prime q

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proof

Suppose $g \notin G$ but $g^q \in G$. Then both G and gGg^{-1} contain the infinite cyclic subgroup generated by g^q and so $G \cap gGg^{-1} \neq \{1\}$.

Fact

The group $G = \langle aba^{-1}b^{-1}a, b \rangle$ is malnormal in F_2.

Theorem (Adams, Gersten)

Let p be a prime and $\pi_X : \hat{X} \to X$, $\pi_Y : \hat{Y} \to Y$ be $\mathbb{Z}/p\mathbb{Z}$ covers. Suppose $f : X \to Y$ is a continuous map and $\hat{f} : \hat{X} \to \hat{Y}$ is a lift

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{f}} & \hat{Y} \\ \big\downarrow \pi_X & & \big\downarrow \pi_Y \\ X & \xrightarrow{f} & Y \end{array}$$

If $f_* : H_1(X, \mathbb{Z}/p\mathbb{Z}) \to H_1(Y, \mathbb{Z}/p\mathbb{Z})$ is 1-1,
then $\hat{f}_* : H_1(\hat{X}, \mathbb{Z}/p\mathbb{Z}) \to H_1(\hat{Y}, \mathbb{Z}/p\mathbb{Z})$ is 1-1 too.

This is used to show that in a sequence of $Z/p\mathbb{Z}$ covers of the immeresed graph of

$$G = \langle aba^{-1}b^{-1}a, b \rangle$$

the induced maps on first homology are always isomorphisms.

Thank you