The k-colour Ramsey number of odd cycles via non-linear optimisation

Jozef Skokan

London School of Economics

July 16, 2018

Joint work with Matthew Jenssen

Background

Definition

For a graph G and an integer $k \geq 2$, let $R_k(G)$ denote the smallest integer N for which any edge-coloring of the complete graph K_N by k colors contains a monochromatic copy of G.

Background

Definition

For a graph G and an integer $k \geq 2$, let $R_k(G)$ denote the smallest integer N for which any edge-coloring of the complete graph K_N by k colors contains a monochromatic copy of G.

Theorem (Ramsey, 1930)

For any graph G, $R_k(G)$ is finite.

Background

Theorem (Erdős, 1947; Erdős, Szekeres, 1935)

$$2^{n/2} \leq R_2(K_n) \leq 2^{2n}$$

Famous problem: improve these bounds

Background

Theorem (Erdős, 1947; Erdős, Szekeres, 1935)

$$2^{n/2} \leq R_2(K_n) \leq 2^{2n}$$

Famous problem: improve these bounds

Questions: (Erdős et al. 70's)

- What controls the growth of $R_2(G)$ as a function of $|G|$?
- Can we strengthen Ramsey's theorem to show that the monochromatic clique has some additional structure?
- How large monochromatic set exists in edge-colorings of K_N satisfying certain restrictions ?

Background

The Ramsey properties of sparse graphs (of bounded maximum degree, bounded degeneracy, bounded average degree) have been extensively studied early on, for example:

Theorem (Gyarfás, Gerencsér, '67)

$$R_2(P_n) = \left\lfloor \frac{3n}{2} \right\rfloor - 1.$$

Theorem (Bondy and Erdős, Faudree and Schelp, Rosta, '73)

$$R_2(C_n) = \begin{cases} 2n - 1, & \textit{if } n \geq 5 \textit{ is odd,} \\ \frac{3n}{2} - 1, & \textit{if } n \geq 6 \textit{ is even.} \end{cases}$$

Background

The Ramsey properties of sparse graphs (of bounded maximum degree, bounded degeneracy, bounded average degree) have been extensively studied early on, for example:

Theorem (Gyarfás, Gerencsér, '67)

$$R_2(P_n) = \left\lfloor \frac{3n}{2} \right\rfloor - 1.$$

Theorem (Bondy and Erdős, Faudree and Schelp, Rosta, '73)

$$R_2(C_n) = \begin{cases} 2n-1, & \text{if } n \geq 5 \text{ is odd,} \\ \frac{3n}{2} - 1, & \text{if } n \geq 6 \text{ is even.} \end{cases}$$

For more than two colors much less is known!

Problem History

For more than three colors much less is known:

Theorem (Cockayne, Lorimer, '75)

$R_k(nK_2) = n + 1 + k(n-1)$

Problem History

For more than three colors much less is known:

Theorem (Cockayne, Lorimer, '75)

$R_k(nK_2) = n + 1 + k(n-1)$

Conjecture (Bondy, Erdős, '73)

For all k and odd $n > 3$, $R_k(C_n) = 2^{k-1}(n-1) + 1$.

4. COMMENTS

We have not been able to evaluate $R(G_1, ..., G_k)$ for $k > 2$ even in the case of cycles. It is easy to see that, when $G_i \cong C_n$, $1 \leqslant i \leqslant k$, and n is odd,

$$R(G_1, ..., G_k) \geqslant 2^{k-1}(n-1) + 1.$$

On the other hand we can show that, in this case,

$$R(G_1, ..., G_k) \leqslant (k+2)!n.$$

Also of interest would be to find $R(C_n, C_r)$, $R(C_n, K_r)$, and $R(C_n, K_r^2)$ for all values of n and r. Since, by [4], $R(C_4, K_4) = 10$ it is possible that

A Lower Bound Construction

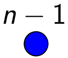
Conjecture (Bondy, Erdős, '73)

For all k and odd $n > 3$, $R_k(C_n) = 2^{k-1}(n-1) + 1$.

A Lower Bound Construction

Conjecture (Bondy, Erdős, '73)

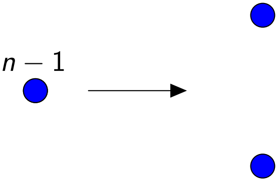
For all k and odd $n > 3$, $R_k(C_n) = 2^{k-1}(n-1) + 1$.



A Lower Bound Construction

Conjecture (Bondy, Erdős, '73)

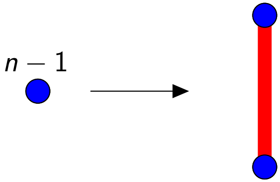
For all k and odd $n > 3$, $R_k(C_n) = 2^{k-1}(n-1) + 1$.



A Lower Bound Construction

Conjecture (Bondy, Erdős, '73)

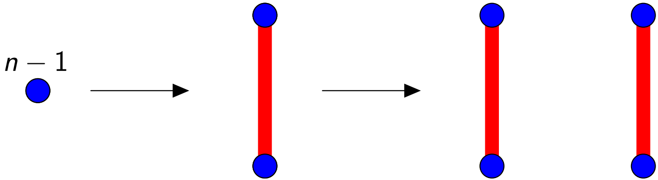
For all k and odd $n > 3$, $R_k(C_n) = 2^{k-1}(n-1) + 1$.



A Lower Bound Construction

Conjecture (Bondy, Erdős, '73)

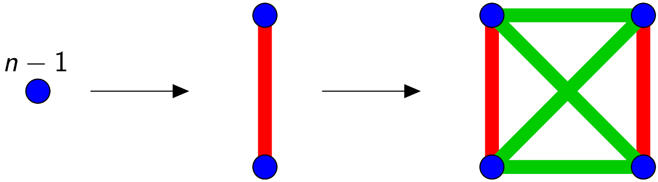
For all k and odd $n > 3$, $R_k(C_n) = 2^{k-1}(n-1) + 1$.



A Lower Bound Construction

Conjecture (Bondy, Erdős, '73)

For all k and odd $n > 3$, $R_k(C_n) = 2^{k-1}(n-1) + 1$.



Progress Toward Conjecture ($k = 3$)

First breakthrough/ early use of the Regularity Method.

Theorem (Łuczak, '99)

$$R_3(C_n) = 4n + o(n) \text{ as } n \to \infty.$$

Progress Toward Conjecture ($k = 3$)

First breakthrough/ early use of the Regularity Method.

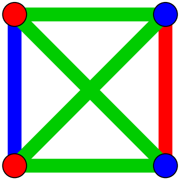
Theorem (Łuczak, '99)

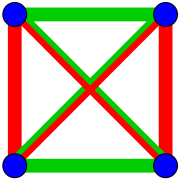
$R_3(C_n) = 4n + o(n)$ *as* $n \to \infty$.

Subsequently improved using stability-type arguments.

Theorem (Kohayakawa, Simonovits, Skokan, '05)

$R_3(C_n) = 4n - 3$ *for sufficiently large odd* n.





Progress Toward Conjecture (general case)

Theorem (Erdős, Graham, '75)

For all k and odd n, $R_k(C_n) \leq 2(k+2)!n$.

Progress Toward Conjecture (general case)

Theorem (Erdős, Graham, '75)

For all k and odd n, $R_k(C_n) \leq 2(k+2)!n$.

Theorem (Łuczak, Simonovits, Skokan, '10)

For all k and odd n, $R_k(C_n) \leq k2^k n + o(n)$ as $n \to \infty$.

Progress Toward Conjecture (general case)

Theorem (Erdős, Graham, '75)

For all k and odd n, $R_k(C_n) \leq 2(k+2)!n$.

Theorem (Łuczak, Simonovits, Skokan, '10)

For all k and odd n, $R_k(C_n) \leq k2^k n + o(n)$ as $n \to \infty$.

Theorem (Jenssen, S. '16+)

For all k, if n is odd and sufficiently large then

$$R_k(C_n) = 2^{k-1}(n-1) + 1.$$

Act 1: there are more extremal colourings than we imagined

Subcubes of Q_k

Think of an element of $\tau \in \{0,1,*\}^k$ as a subset of $\{0,1\}^k$ by considering a '$*$' as a 'missing bit' and considering all possible ways of filling in the missing bits.

Example

We think of $\tau = (1,0,*)$ as the set

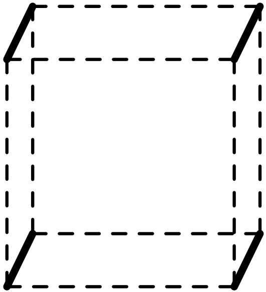
$$Q_\tau = \{(1,0,0),(1,0,1)\}.$$

Example

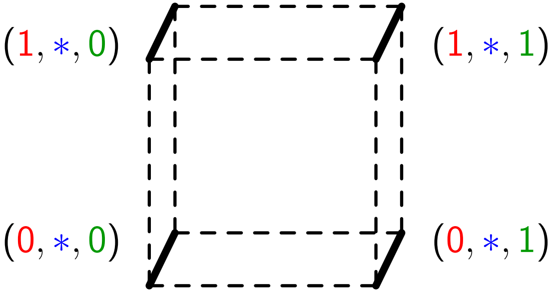
We think of $\tau = (0,*,*)$ as the set

$$Q_\tau = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\}.$$

Extremal Colorings from Matchings in Q_3

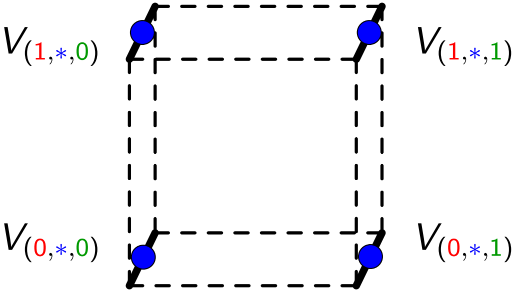


Extremal Colorings from Matchings in Q_3



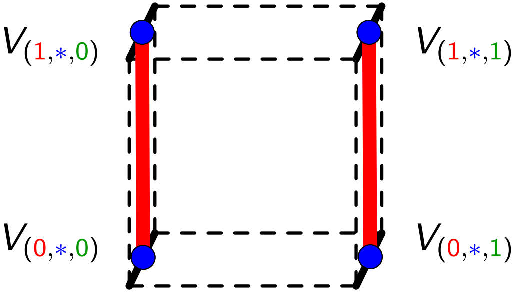
Extremal Colorings from Matchings in Q_3

Sets $V_{(0,*,0)}$, $V_{(1,*,0)}$, $V_{(0,*,1)}$, $V_{(1,*,1)}$, have size $n-1$ each



Extremal Colorings from Matchings in Q_3

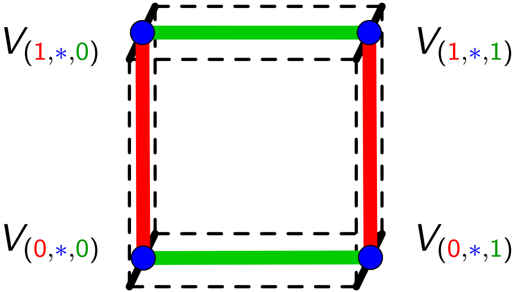
Sets $V_{(0,*,0)}$, $V_{(1,*,0)}$, $V_{(0,*,1)}$, $V_{(1,*,1)}$, have size $n-1$ each



We allow colour i between V_τ and V_σ iff $\tau_i = 1 - \sigma_i$.

Extremal Colorings from Matchings in Q_3

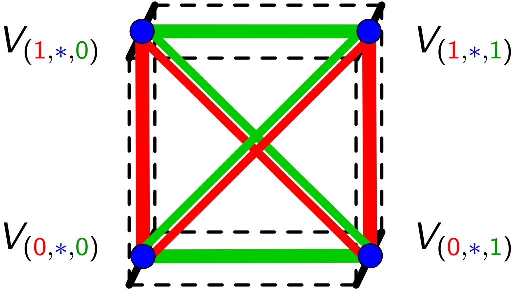
Sets $V_{(0,*,0)}$, $V_{(1,*,0)}$, $V_{(0,*,1)}$, $V_{(1,*,1)}$, have size $n-1$ each



We allow colour i between V_τ and V_σ iff $\tau_i = 1 - \sigma_i$.

Extremal Colorings from Matchings in Q_3

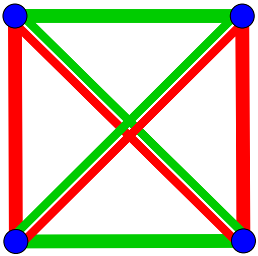
Sets $V_{(0,*,0)}$, $V_{(1,*,0)}$, $V_{(0,*,1)}$, $V_{(1,*,1)}$, have size $n-1$ each there is no monochromatic C_n if n is odd.



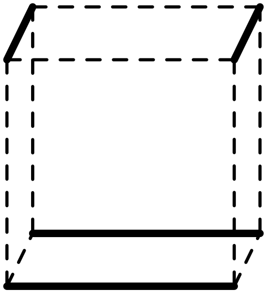
We allow colour i between V_τ and V_σ iff $\tau_i = 1 - \sigma_i$.

Extremal Colorings from Matchings in Q_3

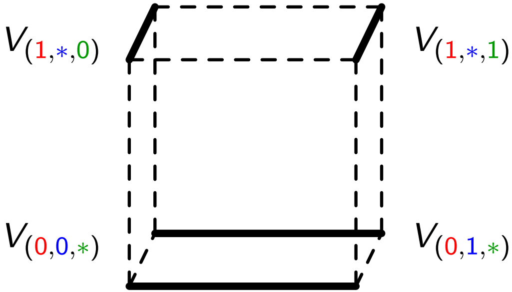
Colouring of $K_{2^2(n-1)}$ without monochromatic C_n.



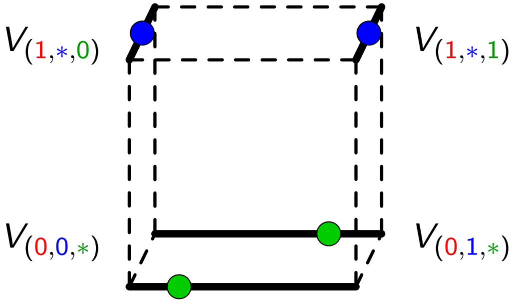
Extremal Colorings from Matchings in Q_3



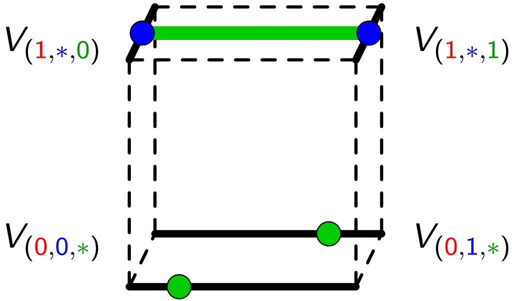
Extremal Colorings from Matchings in Q_3



Extremal Colorings from Matchings in Q_3

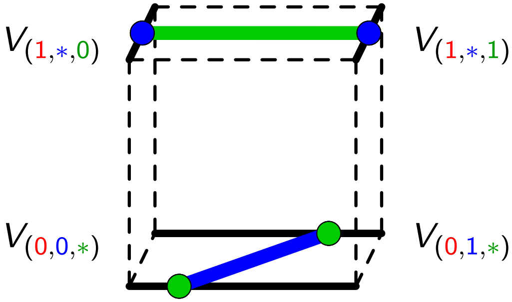


Extremal Colorings from Matchings in Q_3



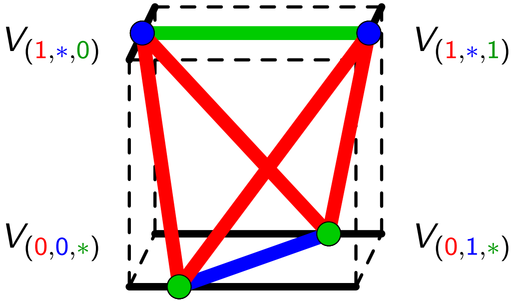
We allow colour i between V_τ and V_σ iff $\tau_i = 1 - \sigma_i$.

Extremal Colorings from Matchings in Q_3



We allow colour i between V_τ and V_σ iff $\tau_i = 1 - \sigma_i$.

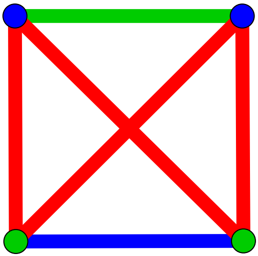
Extremal Colorings from Matchings in Q_3



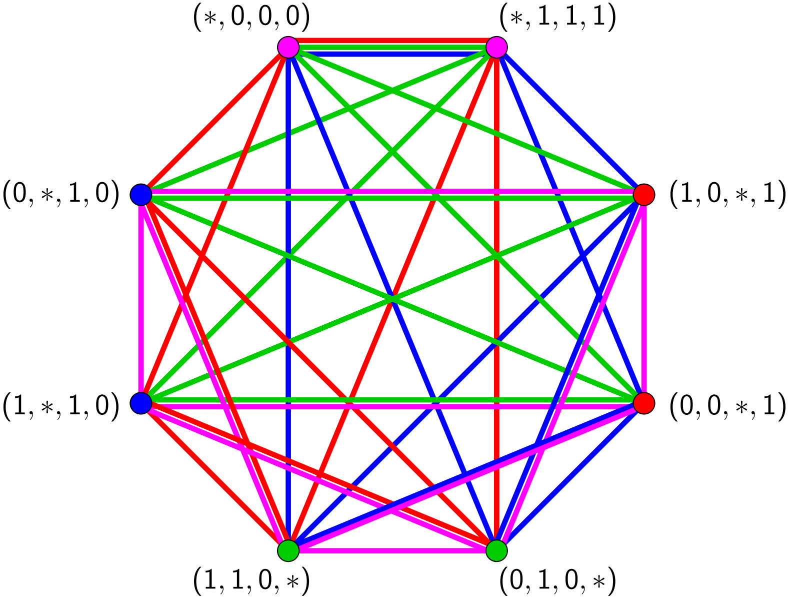
We allow colour i between V_τ and V_σ iff $\tau_i = 1 - \sigma_i$.

Extremal Colorings from Matchings in Q_3

Another colouring of $K_{2^2(n-1)}$ without monochromatic C_n.



New Extremal Constructions



Act 2: how to exploit the characterization of (all) extremal colourings

An Asymptotic Result First

Theorem (Asymptotic Version)

For all k and odd n, $R_k(C_n) = 2^{k-1}n + o(n)$ as $n \to \infty$.

An Asymptotic Result First

Theorem (Asymptotic Version)

For all k and $\varepsilon > 0$ there is an n_0 such that, for any odd $n > n_0$, $R_k(C_n) \leq (1+\varepsilon)2^{k-1}n$.

An Asymptotic Result First

Theorem (Asymptotic Version)

For all k and $\varepsilon > 0$ there is an n_0 such that, for any odd $n > n_0$, $R_k(C_n) \leq (1+\varepsilon)2^{k-1}n$.

By Łuczak's method of connected matchings it suffices to prove the following:

An Asymptotic Result First

Theorem (Asymptotic Version)

For all k and $\varepsilon > 0$ there is an n_0 such that, for any odd $n > n_0$, $R_k(C_n) \leq (1+\varepsilon)2^{k-1}n$.

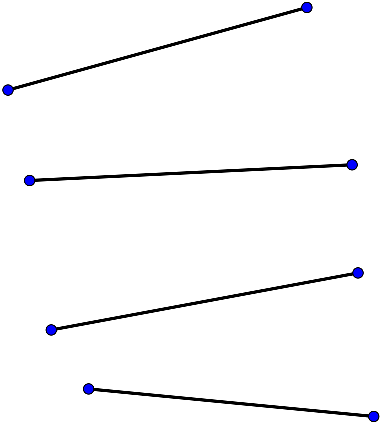
By Łuczak's method of connected matchings it suffices to prove the following:

Proposition

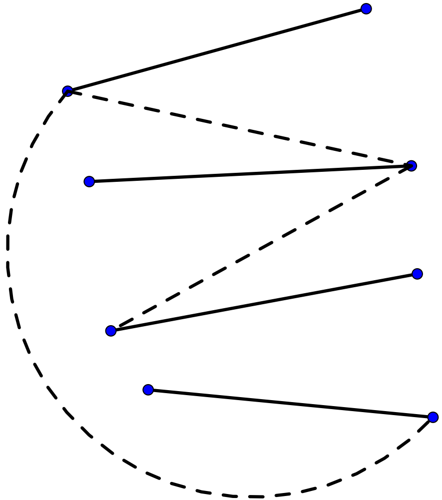
For every $\varepsilon > 0$ there exist a $\delta > 0$ and an n_0 such that the following holds:

If $n \geq n_0$ is odd and G is a graph with $N = (1+\varepsilon)2^{k-1}n$ vertices and at least $(1-\delta)\binom{N}{2}$ edges,
then each k-coloring of G contains a monochromatic odd connected matching of size $(n+1)/2$.

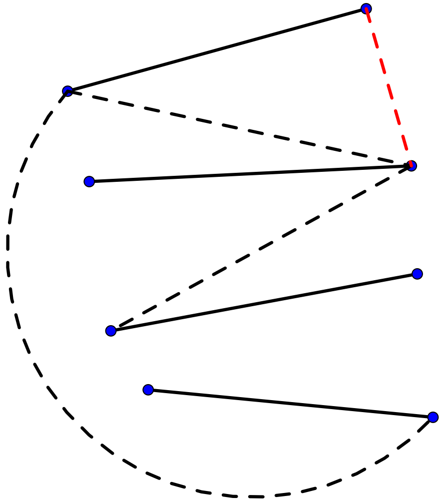
Matching



Connected Matching



Odd Connected Matching



An Asymptotic Result First

Theorem (Asymptotic Version)

For all k and $\varepsilon > 0$ there is an n_0 such that, for any odd $n > n_0$, $R_k(C_n) \leq (1+\varepsilon)2^{k-1}n$.

By Łuczak's method of connected matchings it suffices to prove the following:

Proposition

For every $\varepsilon > 0$ there exist a $\delta > 0$ and an n_0 such that the following holds:

If $n \geq n_0$ is odd and G is a graph with $N = (1+\varepsilon)2^{k-1}n$ vertices and at least $(1-\delta)\binom{N}{2}$ edges,
then each k-coloring of G contains a monochromatic odd connected matching of size $(n+1)/2$.

Graph Decomposition

Let G be a k-colored graph and let $i \in [k]$ be any colour.

Graph Decomposition

Let G be a k-colored graph and let $i \in [k]$ be any colour.

- Let G_i denote the spanning subgraph of G induced by edges of color i.

Graph Decomposition

Let G be a k-colored graph and let $i \in [k]$ be any colour.

- Let G_i denote the spanning subgraph of G induced by edges of color i.
- Write $V(G) = V_0^i \cup V_1^i \cup V_*^i$ where $G_i[V_0^i \cup V_1^i]$ is bipartite and $G_i[V_*^i]$ is a collection of non-bipartite connected components.

Graph Decomposition

Let G be a k-colored graph and let $i \in [k]$ be any colour.

- Let G_i denote the spanning subgraph of G induced by edges of color i.
- Write $V(G) = V_0^i \cup V_1^i \cup V_*^i$ where $G_i[V_0^i \cup V_1^i]$ is bipartite and $G_i[V_*^i]$ is a collection of non-bipartite connected components.
- For $\tau = (\tau_1, \ldots, \tau_k) \in \{0, 1, *\}^k$, let $V_\tau = \bigcap_{j=1}^k V_{\tau_j}^j$.

Graph Decomposition

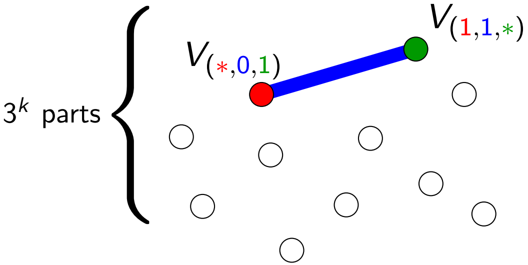
Let G be a k-colored graph and let $i \in [k]$ be any colour.

- Let G_i denote the spanning subgraph of G induced by edges of color i.
- Write $V(G) = V_0^i \cup V_1^i \cup V_*^i$ where $G_i[V_0^i \cup V_1^i]$ is bipartite and $G_i[V_*^i]$ is a collection of non-bipartite connected components.
- For $\tau = (\tau_1, \ldots, \tau_k) \in \{0, 1, *\}^k$, let $V_\tau = \bigcap_{j=1}^k V_{\tau_j}^j$.
- Define the *dimension* of τ as $\omega(\tau) := |\{j \in [k] : \tau_j = *\}|$.

Graph Decomposition

Let G be a k-colored graph and let $i \in [k]$ be any colour.

- Let G_i denote the spanning subgraph of G induced by edges of color i.
- Write $V(G) = V_0^i \cup V_1^i \cup V_*^i$ where $G_i[V_0^i \cup V_1^i]$ is bipartite and $G_i[V_*^i]$ is a collection of non-bipartite connected components.
- For $\tau = (\tau_1, \ldots, \tau_k) \in \{0, 1, *\}^k$, let $V_\tau = \bigcap_{j=1}^k V_{\tau_j}^j$.
- Define the *dimension* of τ as $\omega(\tau) := |\{j \in [k] : \tau_j = *\}|$.
- $\omega(\tau)$ is the maximum number of colors possible on edges in V_τ.



Sketch Proof

Suppose that G is a k-colored graph with $N = (1+\varepsilon)2^{k-1}n$ vertices, $(1-\delta)\binom{N}{2}$ edges such that each monochromatic non-bipartite component has no matching of $(n+1)/2$ edges.

Sketch Proof

Suppose that G is a k-colored graph with $N = (1+\varepsilon)2^{k-1}n$ vertices, $(1-\delta)\binom{N}{2}$ edges such that each monochromatic non-bipartite component has no matching of $(n+1)/2$ edges.

We show that for δ small and n large we reach a contradiction.

Sketch Proof

Suppose that G is a k-colored graph with $N = (1+\varepsilon)2^{k-1}n$ vertices, $(1-\delta)\binom{N}{2}$ edges such that each monochromatic non-bipartite component has no matching of $(n+1)/2$ edges.

We show that for δ small and n large we reach a contradiction.

By the following theorem the monochromatic non-bipartite components of G must be very sparse:

Sketch Proof

Suppose that G is a k-colored graph with $N = (1+\varepsilon)2^{k-1}n$ vertices, $(1-\delta)\binom{N}{2}$ edges such that each monochromatic non-bipartite component has no matching of $(n+1)/2$ edges.

We show that for δ small and n large we reach a contradiction.

By the following theorem the monochromatic non-bipartite components of G must be very sparse:

Theorem (Erdős, Gallai, '59)

Let $m \geq 3$. If G has no cycle of length greater than m (a.k.a. connected matching of $(m+1)/2$ edges), then $e(G) \leq (m-1)(v(G)-1)/2$.

Sketch Proof

Suppose that G is a k-colored graph with $N = (1+\varepsilon)2^{k-1}n$ vertices, $(1-\delta)\binom{N}{2}$ edges such that each monochromatic non-bipartite component has no matching of $(n+1)/2$ edges.

We show that for δ small and n large we reach a contradiction.

By the following theorem the monochromatic non-bipartite components of G must be very sparse:

Theorem (Erdős, Gallai, '59)

Let $m \geq 3$. If G has no cycle of length greater than m (a.k.a. connected matching of $(m+1)/2$ edges), then $e(G) \leq (m-1)(v(G)-1)/2$.

Decomposing G as before, it follows that $|V_\tau| \leq \omega(\tau)n + o(n)$ for all $\tau \in \{0, 1, *\}^k$.

A Quadratic Constraint

For $\tau \in \{0, 1, *\}^k$ let $v_\tau = |V_\tau|/n$ and let $v = (v_\tau : \tau \in \{0, 1, *\}^k)$.

A Quadratic Constraint

For $\tau \in \{0,1,*\}^k$ let $v_\tau = |V_\tau|/n$ and let $v = (v_\tau : \tau \in \{0,1,*\}^k)$.
Counting non-bipartite edges in two different ways we obtain

A Quadratic Constraint

For $\tau \in \{0,1,*\}^k$ let $v_\tau = |V_\tau|/n$ and let $v = (v_\tau : \tau \in \{0,1,*\}^k)$.
Counting non-bipartite edges in two different ways we obtain

$$\left(\sum_{\tau \in \{0,1,*\}^k} v_\tau\right)^2 - 2 \sum_{Q_\sigma \cap Q_\tau = \emptyset} v_\sigma v_\tau - \sum_{\tau \in \{0,1,*\}^k} \omega(\tau) v_\tau \leq 2^{3k} \delta (1+\varepsilon)^2.$$

A Quadratic Constraint

For $\tau \in \{0,1,*\}^k$ let $v_\tau = |V_\tau|/n$ and let $v = (v_\tau : \tau \in \{0,1,*\}^k)$. Counting non-bipartite edges in two different ways we obtain

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k} \omega(\tau) v_\tau \le 2^{3k}\delta(1+\varepsilon)^2.$$

Consider this inequality as a constraint on the non-negative reals v_τ and try to maximize

$$\|v\|_1 = \sum_{\tau\in\{0,1,*\}^k} v_\tau$$

A Quadratic Constraint

For $\tau \in \{0,1,*\}^k$ let $v_\tau = |V_\tau|/n$ and let $v = (v_\tau : \tau \in \{0,1,*\}^k)$. Counting non-bipartite edges in two different ways we obtain

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k} \omega(\tau) v_\tau \le 2^{3k}\delta(1+\varepsilon)^2.$$

Consider this inequality as a constraint on the non-negative reals v_τ and try to maximize

$$\|v\|_1 = \sum_{\tau\in\{0,1,*\}^k} v_\tau = \frac{N}{n}.$$

A Quadratic Constraint

For $\tau \in \{0,1,*\}^k$ let $v_\tau = |V_\tau|/n$ and let $v = (v_\tau : \tau \in \{0,1,*\}^k)$.
Counting non-bipartite edges in two different ways we obtain

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k} \omega(\tau) v_\tau \le 2^{3k}\delta(1+\varepsilon)^2.$$

Consider this inequality as a constraint on the non-negative reals v_τ and try to maximize

$$\|v\|_1 = \sum_{\tau\in\{0,1,*\}^k} v_\tau = \frac{N}{n}.$$

We get the desired contradiction by showing that

$$\|v\|_1 < (1+\epsilon)2^{k-1} = \frac{N}{n}.$$

A Compression Argument

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k} \omega(\tau) v_\tau \leq 0.$$

A Compression Argument

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k} \omega(\tau) v_\tau \leq 0.$$

Suppose that $Q_\rho \cap Q_\pi \neq \emptyset$ for some $\rho, \pi \in \{0,1,*\}^k$, then we may write the above as

$$(A + v_\rho + v_\pi)^2 - Bv_\rho - Cv_\pi - D \leq 0,$$

where $A, B, C, D \geq 0$ do not depend on v_ρ or v_π.

A Compression Argument

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k}\omega(\tau)v_\tau \leq 0.$$

Suppose that $Q_\rho \cap Q_\pi \neq \emptyset$ for some $\rho, \pi \in \{0,1,*\}^k$, then we may write the above as

$$(A + v_\rho + v_\pi)^2 - Bv_\rho - Cv_\pi - D \leq 0,$$

where $A, B, C, D \geq 0$ do not depend on v_ρ or v_π.

Wlog suppose $B \geq C$ and set $v'_\rho = v_\rho + v_\pi$ and $v'_\pi = 0$.

A Compression Argument

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k} \omega(\tau)v_\tau \le 0.$$

Suppose that $Q_\rho \cap Q_\pi \neq \emptyset$ for some $\rho, \pi \in \{0,1,*\}^k$, then we may write the above as

$$(A + v_\rho + v_\pi)^2 - Bv_\rho - Cv_\pi - D \le 0,$$

where $A, B, C, D \ge 0$ do not depend on v_ρ or v_π.

Wlog suppose $B \ge C$ and set $v'_\rho = v_\rho + v_\pi$ and $v'_\pi = 0$.
Call this the (π, ρ)-compression of $v = (v_\tau : \tau \in \{0,1,*\}^k)$.

A Compression Argument

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k} \omega(\tau)v_\tau \leq 0.$$

Suppose that $Q_\rho \cap Q_\pi \neq \emptyset$ for some $\rho, \pi \in \{0,1,*\}^k$, then we may write the above as

$$(A + v_\rho + v_\pi)^2 - Bv_\rho - Cv_\pi - D \leq 0,$$

where $A, B, C, D \geq 0$ do not depend on v_ρ or v_π.

Wlog suppose $B \geq C$ and set $v'_\rho = v_\rho + v_\pi$ and $v'_\pi = 0$.
Call this the (π, ρ)-compression of $v = (v_\tau : \tau \in \{0,1,*\}^k)$.
Compressions preserve $\|v\|_1$ and keep v in the feasible region.

Compressing to a Spherical Constraint

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k}\omega(\tau)v_\tau \le 0.$$

Compressing to a Spherical Constraint

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k} \omega(\tau) v_\tau \leq 0.$$

Applying a compression argument we may assume that $v = (v_\tau : \tau \in \{0,1,*\}^k)$ is supported on a collection $\mathcal{D}$ of disjoint subcubes of Q_k.

Compressing to a Spherical Constraint

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k}\omega(\tau)v_\tau \leq 0.$$

Applying a compression argument we may assume that $v = (v_\tau : \tau \in \{0,1,*\}^k)$ is supported on a collection $\mathcal{D}$ of disjoint subcubes of Q_k.This transforms the constraint equation to

$$\sum_{\tau\in\mathcal{D}}(v_\tau^2 - \omega(\tau)v_\tau) \leq 0.$$

Compressing to a Spherical Constraint

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k} \omega(\tau)v_\tau \leq 0.$$

Applying a compression argument we may assume that $v = (v_\tau : \tau \in \{0,1,*\}^k)$ is supported on a collection $\mathcal{D}$ of disjoint subcubes of Q_k.This transforms the constraint equation to

$$\sum_{\tau\in\mathcal{D}} (v_\tau^2 - \omega(\tau)v_\tau) \leq 0.$$

- Much more amenable to analysis: the constraint is convex (Karush-Kuhn-Tucker conditions, Slater condition for strong duality).

Compressing to a Spherical Constraint

$$\left(\sum_{\tau \in \{0,1,*\}^k} v_\tau \right)^2 - 2 \sum_{Q_\sigma \cap Q_\tau = \emptyset} v_\sigma v_\tau - \sum_{\tau \in \{0,1,*\}^k} \omega(\tau) v_\tau \leq 0.$$

Applying a compression argument we may assume that $v = (v_\tau : \tau \in \{0,1,*\}^k)$ is supported on a collection $\mathcal{D}$ of disjoint subcubes of Q_k.This transforms the constraint equation to

$$\sum_{\tau \in \mathcal{D}} (v_\tau^2 - \omega(\tau) v_\tau) \leq 0.$$

- Much more amenable to analysis: the constraint is convex (Karush-Kuhn-Tucker conditions, Slater condition for strong duality).
- It turns out that an optimal point v must be a 0-1 vector supported on a perfect matching of Q_k.

Compressing to a Spherical Constraint

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k}\omega(\tau)v_\tau \le 0.$$

Applying a compression argument we may assume that $v = (v_\tau : \tau \in \{0,1,*\}^k)$ is supported on a collection $\mathcal{D}$ of disjoint subcubes of Q_k.This transforms the constraint equation to

$$\sum_{\tau\in\mathcal{D}}(v_\tau^2 - \omega(\tau)v_\tau) \le 0.$$

- Much more amenable to analysis: the constraint is convex (Karush-Kuhn-Tucker conditions, Slater condition for strong duality).
- It turns out that an optimal point v must be a 0-1 vector supported on a perfect matching of Q_k.
- For such a v we have $\|v\|_1 = 2^{k-1}$ and a compactness argument gives the desired contradiction.

Compressing to a Spherical Constraint

$$\left(\sum_{\tau\in\{0,1,*\}^k} v_\tau\right)^2 - 2\sum_{Q_\sigma\cap Q_\tau=\emptyset} v_\sigma v_\tau - \sum_{\tau\in\{0,1,*\}^k}\omega(\tau)v_\tau \le o_\delta(1).$$

Applying a compression argument we may assume that $v = (v_\tau : \tau \in \{0,1,*\}^k)$ is supported on a collection $\mathcal{D}$ of disjoint subcubes of Q_k. This transforms the constraint equation to

$$\sum_{\tau\in\mathcal{D}}(v_\tau^2 - \omega(\tau)v_\tau) \le o_\delta(1).$$

- Much more amenable to analysis, the constraint is convex (Karush-Kuhn-Tucker conditions, Slater condition for strong duality).
- It turns out that an optimal point v must be a 0-1 vector supported on a perfect matching of Q_k.
- For such a v we have $\|v\|_1 = 2^{k-1} + o_\delta(1)$, however $\|v\|_1 = 2^{k-1} + \varepsilon$, a contradiction.

Towards the Exact Result

Let $\mathcal{S}$ be the set of complete graphs admitting a k-coloring with no monochromatic C_n.

Ramsey Theory	Analysis
Maximize $v(G)$ subject to $$G \in \mathcal{S}.$$	Maximize $\|x\|_1$ subject to $$x \in S \subseteq \mathbb{R}^{3^k}.$$
Extremal construction	Optimal point
An almost extremal construction must be 'close' to an extremal construction (hypercube coloring).	An almost optimal point must be close (in ℓ_1-norm) to an optimal point.

Towards the Exact Result

Let $\mathcal{S}$ be the set of complete graphs admitting a k-coloring with no monochromatic C_n.

Ramsey Theory	Analysis
Maximize $v(G)$ subject to $$G \in \mathcal{S}.$$	Maximize $\|x\|_1$ subject to $$x \in S \subseteq \mathbb{R}^{3^k}.$$
Extremal construction	Optimal point
An almost extremal construction must be 'close' to an extremal construction (hypercube coloring).	An almost optimal point must be close (in ℓ_1-norm) to an optimal point.

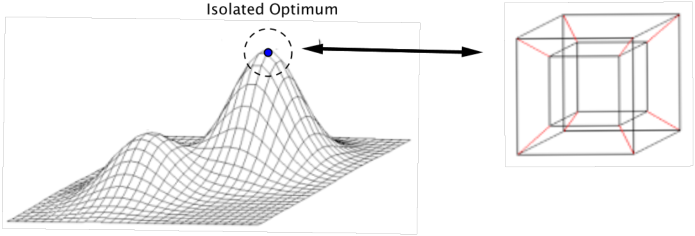
607158046495120886820621 extremal constructions for $k = 7$

Analytic and Combinatorial Stability

The key ingredient for analytic stability is that the optimal points are *isolated*.

Analytic and Combinatorial Stability

The key ingredient for analytic stability is that the optimal points are *isolated*.



Epilogue

Theorem (Jenssen, S. '16+)

For all k, if n is odd and sufficiently large, then

$$R_k(C_n) = 2^{k-1}(n-1) + 1.$$

'sufficiently large' means **huge**

Theorem (Day, Johnson '17)

For all odd n, if k is sufficiently large, then

$$R_k(C_n) > (2 + c(n))^{k-1}(n-1).$$

Final remarks

- The same proof gives $R(C_{n_1}, C_{n_1}, \ldots, C_{n_k}) = 2^{k-1}(\max n_i - 1) + 1$, where n_i's are large and odd.
- How about mixed parities?
- Possible application: finding monochromatic circumference of a k-edge coloured dense graphs.
- Does the analytic approach have wider applications in Ramsey theory?

Thank you for listening!