Ramsey theory in algebraic topological language

Sławomir Solecki

Cornell University

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Unifying efforts in Ramsey theory:

Nešetřil, *Ramsey theory*, 1995

Todorcevic, *Introduction to Ramsey spaces*, 2010

Solecki, *Abstract approach to finite Ramsey theory and a self-dual Ramsey theorem*, 2013

Gromov, *Colorful categories*, 2015

Hubička–Nešetřil, *Ramsey classes with closure operations*, to appear (2017)

Normed induction complexes

A **simplicial complex** $\mathcal{S}$ is a family of non-empty sets closed under taking non-empty subsets.

Sets in $\mathcal{S}$ are called **faces** of $\mathcal{S}$.

$V(\mathcal{S})$ = the set of all vertexes of $\mathcal{S}$.

A **simplicial map** f from $\mathcal{S}_1$ to $\mathcal{S}_2$ is a function $f\colon V(\mathcal{S}_1) \to V(\mathcal{S}_2)$ that preserves faces, that is, for each face $s \in \mathcal{S}_1$ the image of s under f is a face in $\mathcal{S}_2$.

For a face $s \in \mathcal{S}_1$, we write $f(s)$ for the face in $\mathcal{S}_2$ that is the image of s.

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A **partial simplicial map** is a function $f\colon X \to V(\mathcal{S}_2)$, for some $X \subseteq V(\mathcal{S}_1)$, that preserves faces included in X.

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be the **chain simplicial complex of** L whose faces are finite non-empty linearly ordered subsets of L.

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(A) $\delta(f(x)) = f(\delta(x))$ for each $x \in V(\mathcal{S})$, $f \in F$ and $\delta \in \Delta$ for which both sides are defined,

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(B) $|\delta(x)| \leq |x|$ for all $x \in V(\mathcal{S})$ and $\delta \in \Delta$,

(C) if $f(y)$ is defined and $|x| \leq |y|$, for $x, y \in V(\mathcal{S})$ and $f \in F$, then $f(x)$ is defined and $|f(x)| \leq |f(y)|$.

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$\Delta_0 = \{\delta_0\}$, *where for* $x \in V(\mathcal{S}_0)$,

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$L_0 = \mathbb{N}$ *with the natural order and, for* $x \in V(\mathcal{S}_0)$,

$$|x| = \begin{cases} \max x, & \text{if } x \neq \emptyset; \\ 0, & \text{if } x = \emptyset \end{cases}$$

Ramsey degree

$(\mathcal{S}, F, \Delta, |\cdot|)$ a normed induction complex

Let $\mathcal{A}$ be a **family of subsets of** F.

For $a \in \mathcal{A}$ and $s \in \mathcal{S}$, we say $a(s)$ **is defined** if $f(s)$ is defined for each $f \in a$.

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We want $\mathcal{A}$ to act on faces of $\mathcal{S}$.
So we say that $\mathcal{A}$ is **compatible** if, for each face s of $\mathcal{S}$ and each $a \in \mathcal{A}$ with $a(s)$ defined,

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For a compatible $\mathcal{A}$ and $a \in \mathcal{A}$, if $a(s)$ is defined, we write

$$a(s) = \bigcup \{f(s) \colon f \in a\}.$$

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$a_{I,J}$ = *elements of* F_0 *induced by strictly increasing functions whose domains are equal to* I *and whose images are included in* J

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$\mathcal{A}_0$ is compatible.

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For $d \in \mathbb{N}^+$, a d-**coloring** is a simplical map from $\mathcal{S}$ to the $(d-1)$-dimensional simplex, that is, it is an arbitrary function from $V(\mathcal{S})$ to a set with d elements.

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given $d \in \mathbb{N}^+$, there exists $a \in \mathcal{A}$ with $a(s)$ defined and such that,
for each d-coloring, there exists $f \in a$ with $f(s)$ attaining at most r colors.

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The upper bound we are aiming for is

$$\mathrm{rd}(s) \leq \min\{\mathrm{card}(\delta(s)) \mid \delta \in \langle \Delta \rangle\}.$$

Pigeonhole principle

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(PP) Given: a face $s \in \mathcal{S}$ and a number $d \in \mathbb{N}^+$.
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— $f(s_{x,\delta})$ monochromatic and
— $f(y) = y$ for each $y \in \delta(s)$, $|y| \leq |x|$.

The **theorem** will say: (PP) implies

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For this, we need the compatible family $\mathcal{A} \subseteq F$ to **interact more closely** with the normed induction complex $(\mathcal{S}, F, \Delta, |\cdot|)$. We need to impose conditions of **composability** and **extendability** on $\mathcal{A}$.

The general Ramsey theorem

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$$h \restriction s = g \circ (f \restriction s).$$

It follows immediately from (B) and (C) that if $f(s)$ is defined, then so is $f(\delta(s))$ for a face $s \in \mathcal{S}$. The condition of **extendability** below guarantees that we can go the other way in a weak sense.

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$$g \restriction \delta(s) = f \restriction \delta(s).$$

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We write $(\mathcal{S}, \mathcal{A})$ for the Ramsey domain without explicitly mentioning F, Δ and $|\cdot|$.

Theorem (S.)

If (PP) *holds in a Ramsey domain* $(\mathcal{S}, \mathcal{A})$, *then for each face* s *of* $\mathcal{S}$

$$\mathrm{rd}(s) \leq \min\{\mathrm{card}(\delta(s)) \mid \delta \in \langle \Delta \rangle\}.$$

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(2) The pigeonhole principle (PP) finds f ensuring monochromaticity depending on $x \in \delta(s)$. In the theorem, we need an f that is independent of x. This difficulty is handled by another induction argument proceeding on the complexity of x as measured by $|x| \in L$.

Infinite dimensional Ramsey theory

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F is **composable** and **extendable**.

F is topologically **closed**.

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A sequence $\vec{s} = (s_n)$ of faces of $\mathcal{S}$ is **coherent** if

— $s_n = \delta(s_{n+1})$, for all $n \in \mathbb{N}$;

— $\bigcup_n s_n$ is a face in $\mathcal{S}$;

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For a coherent sequence $\vec{s}$, let

$$[\vec{s}] = \{(x_n) \in \varprojlim X \mid x_n \in s_n \text{ for each } n\}.$$

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$(\mathbf{R})_{\rm inf}$ Given $d \in \mathbb{N}$, for each coherent sequence of faces (s_n), and each closed d-coloring of $\varprojlim X$, there exists $f \in F$ such that $f(\bigcup_n s_n)$ is defined and $f([(s_n)])$ is monochromatic.

Theorem (S.)

Let $(\mathcal{S}, F, \Delta, |\cdot|)$ *be a normed induction complex with locally finite faces. Assume* F *is composable, extandable, and closed. Then* $(\mathrm{PP})_{\mathrm{inf}}$ *implies* $(\mathrm{R})_{\mathrm{inf}}$.