Hypernatural numbers in ultra-Ramsey theory

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Overview

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2. Notation for trees

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3. The alpha-Ramsey theorem

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5. Every nonprincipal ultrafilter $\mathcal{U}$ is of the form $\{X \subseteq \mathbb{N} : \beta \in {}^*X\}$ for some $\beta \in {}^*\mathbb{N} \setminus \mathbb{N}$.
6. The framework is convenient but unnecessary. The proofs can be carried out by referring directly to the ultrafilters or the notion of a functional extensions as introduced by Forti.

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Example

$[\mathbb{N}]^{<\infty}$ is an $\vec{\alpha}$-tree.

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2. $[S] \cap \mathcal{X} = \emptyset$.
3. *For all $\vec{\alpha}$-trees S', if $S' \subseteq S$ then $[S'] \not\subseteq \mathcal{X}$ and $[S'] \cap \mathcal{X} \neq \emptyset$.*

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Note that $H = [\mathbb{N}]^{<\infty} \setminus (F \cup G)$.

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For each $n \in A$, let T_n be an $\vec{\alpha}$-tree with stem $s \cup \{n\}$ such that $[T_n] \subseteq \mathcal{X}$.

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$$H = \{s \in [\mathbb{N}]^{<\infty} : \forall \vec{\alpha}\text{-tree } S \subseteq T \text{ with stem } s, [S] \not\subseteq \mathcal{X} \text{ and } [S] \cap \mathcal{X} \neq \emptyset\}.$$

Wlog assume that $st(T) \in H$.

By the previous Lemma and Claim there is an $\vec{\alpha}$-tree $S \subseteq T$ such that $st(S) = st(T)$ and $S/st(S) \subseteq H$.

If $S' \subseteq S$ is an $\vec{\alpha}$-tree then $st(S') \in S/st(S) \subseteq H$.

Since $S' \subseteq T$ and $st(S') \in H$, $[S'] \not\subseteq \mathcal{X}$ and $[S'] \cap \mathcal{X} \neq \emptyset$. □

$\vec{\alpha}$-Ramsey theorem

Theorem (T. [3])

For all $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ and for all $\vec{\alpha}$-trees T there exists an $\vec{\alpha}$-tree $S \subseteq T$ with $st(S) = st(T)$ such that one of the following holds:

1. $[S] \subseteq \mathcal{X}$.
2. $[S] \cap \mathcal{X} = \emptyset$.
3. *For all $\vec{\alpha}$-trees S', if $S' \subseteq S$ then $[S'] \not\subseteq \mathcal{X}$ and $[S'] \cap \mathcal{X} \neq \emptyset$.*

Finite dimensional $\vec{\alpha}$-Ramsey theorem

Theorem (T.)

Suppose that $n \in \mathbb{N}$. For all $A \subseteq [\mathbb{N}]^n$ and for all $\vec{\alpha}$-trees T there exists an $\vec{\alpha}$-tree $S \subseteq T$ with $st(S) = st(T)$ such that either $S(n) \subseteq A$ or $S(n) \cap A = \emptyset$.

- Let $\mathcal{X} = \{Y \in [\mathbb{N}]^\infty : r_n(Y) \in A\}$.

Finite dimensional $\vec{\alpha}$-Ramsey theorem

Theorem (T.)

Suppose that $n \in \mathbb{N}$. For all $A \subseteq [\mathbb{N}]^n$ and for all $\vec{\alpha}$-trees T there exists an $\vec{\alpha}$-tree $S \subseteq T$ with $st(S) = st(T)$ such that either $S(n) \subseteq A$ or $S(n) \cap A = \emptyset$.

- Let $\mathcal{X} = \{Y \in [\mathbb{N}]^\infty : r_n(Y) \in A\}$.
- $\mathcal{X}$ can not satisfy conclusion (3) in the statement of alpha Ramsey theorem because any $\vec{\alpha}$-tree S with $|st(S)| \geq n$ will either have $[S] \subseteq \mathcal{X}$ or $[S] \cap \mathcal{X} = \emptyset$.

Finite dimensional $\vec{\alpha}$-Ramsey theorem

Theorem (T.)

Suppose that $n \in \mathbb{N}$. For all $A \subseteq [\mathbb{N}]^n$ and for all $\vec{\alpha}$-trees T there exists an $\vec{\alpha}$-tree $S \subseteq T$ with $st(S) = st(T)$ such that either $S(n) \subseteq A$ or $S(n) \cap A = \emptyset$.

- Let $\mathcal{X} = \{Y \in [\mathbb{N}]^\infty : r_n(Y) \in A\}$.
- $\mathcal{X}$ can not satisfy conclusion (3) in the statement of alpha Ramsey theorem because any $\vec{\alpha}$-tree S with $|st(S)| \geq n$ will either have $[S] \subseteq \mathcal{X}$ or $[S] \cap \mathcal{X} = \emptyset$.
- Thus, either $S(n) \subseteq A$ or $S(n) \cap A = \emptyset$ depending on whether $[S] \subseteq \mathcal{X}$ or $[S] \cap \mathcal{X} = \emptyset$, respectively.

Definition

For $s \in [\mathbb{N}]^{<\infty}$ and $X \in [\mathbb{N}]^{\infty}$, let

$$[s, X] = \{Y \in [\mathbb{N}]^{\infty} : s \sqsubseteq Y \subseteq X\}.$$

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Suppose that $\mathcal{C} \subseteq [\mathbb{N}]^{\infty}$.

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Definition

Suppose that $\mathcal{C} \subseteq [\mathbb{N}]^{\infty}$. $\mathcal{X} \subseteq [\mathbb{N}]^{\infty}$ is $\mathcal{C}$-**Ramsey** if for all $[s, X] \neq \emptyset$ with $X \in \mathcal{C}$ there exists $Y \in [s, X] \cap \mathcal{C}$ such that either $[s, Y] \subseteq \mathcal{X}$ or $[s, Y] \cap \mathcal{X} = \emptyset$.

Local Ramsey Theory

Definition

For $s \in [\mathbb{N}]^{<\infty}$ and $X \in [\mathbb{N}]^{\infty}$, let

$$[s, X] = \{Y \in [\mathbb{N}]^{\infty} : s \sqsubseteq Y \subseteq X\}.$$

Definition

Suppose that $\mathcal{C} \subseteq [\mathbb{N}]^{\infty}$. $\mathcal{X} \subseteq [\mathbb{N}]^{\infty}$ is $\mathcal{C}$-**Ramsey** if for all $[s, X] \neq \emptyset$ with $X \in \mathcal{C}$ there exists $Y \in [s, X] \cap \mathcal{C}$ such that either $[s, Y] \subseteq \mathcal{X}$ or $[s, Y] \cap \mathcal{X} = \emptyset$.

Definition

$\mathcal{X} \subseteq [\mathbb{N}]^{\infty}$ is $\mathcal{C}$-**Ramsey null** if for all $[s, X] \neq \emptyset$ with $X \in \mathcal{C}$ there exists $Y \in [s, X] \cap \mathcal{C}$ such that $[s, Y] \cap \mathcal{X} = \emptyset$.

Definition

Suppose that $\mathcal{C} \subseteq [\mathbb{N}]^\infty$. We say that $([\mathbb{N}]^\infty, \mathcal{C}, \subseteq)$ is a **topological Ramsey space** if the following conditions hold:

1. $\{[s, X] : X \in \mathcal{C}\}$ is a neighborhood base for a topology on $[\mathbb{N}]^\infty$.
2. The collection of $\mathcal{C}$-Ramsey sets coincides with the σ-algebra of sets with the Baire property with respect to the topology generated by $\{[s, X] : X \in \mathcal{C}\}$.
3. The collection of $\mathcal{C}$-Ramsey null sets coincides with the σ-ideal of meager sets with respect to the topology generated by $\{[s, X] : X \in \mathcal{C}\}$.

Local Ramsey Theory

Theorem (The Ellentuck Theorem)

$([\mathbb{N}]^\infty, [\mathbb{N}]^\infty, \subseteq)$ *is a topological Ramsey space.*

Local Ramsey Theory

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$([\mathbb{N}]^\infty, [\mathbb{N}]^\infty, \subseteq)$ *is a topological Ramsey space.*

Theorem (Louveau)

If $\mathcal{U}$ *is a selective ultrafilter then* $([\mathbb{N}]^\infty, \mathcal{U}, \subseteq)$ *is a topological Ramsey space.*

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Remark

Local Ramsey theory is concerned with characterizing the conditions on $\mathcal{C}$ which guarantee that $([\mathbb{N}]^\infty, \mathcal{C}, \subseteq)$ forms a Ramsey space.

α-Ramsey Theory

Definition

$\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is said to be $\vec{\alpha}$-**Ramsey** if for all $\vec{\alpha}$-trees T there exists an $\vec{\alpha}$-tree $S \subseteq T$ with $st(S) = st(T)$ such that either $[S] \subseteq \mathcal{X}$ or $[S] \cap \mathcal{X} = \emptyset$.

α-Ramsey Theory

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Definition

$\mathcal{X}$ is said to be $\vec{\alpha}$-**Ramsey null** if for all $\vec{\alpha}$-trees T there exists an $\vec{\alpha}$-tree $S \subseteq T$ with $st(S) = st(T)$ such that $[S] \cap \mathcal{X} = \emptyset$.

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$\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is said to be $\vec{\alpha}$-**Ramsey** if for all $\vec{\alpha}$-trees T there exists an $\vec{\alpha}$-tree $S \subseteq T$ with $st(S) = st(T)$ such that either $[S] \subseteq \mathcal{X}$ or $[S] \cap \mathcal{X} = \emptyset$.

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Definition

The topology on $[\mathbb{N}]^\infty$ generated by $\{[T] : \text{T is an } \vec{\alpha}\text{-tree}\}$ is called **the** $\vec{\alpha}$-**Ellentuck topology**.

Definition

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$\mathcal{X}$ is said to be $\vec{\alpha}$-**Ramsey null** if for all $\vec{\alpha}$-trees T there exists an $\vec{\alpha}$-tree $S \subseteq T$ with $st(S) = st(T)$ such that $[S] \cap \mathcal{X} = \emptyset$.

Definition

The topology on $[\mathbb{N}]^\infty$ generated by $\{[T] : \text{T is an } \vec{\alpha}\text{-tree}\}$ is called **the** $\vec{\alpha}$-**Ellentuck topology**.

Remark

The $\vec{\alpha}$-*Ellentuck space* is a zero-dimensional Baire space on $[\mathbb{N}]^\infty$ with the countable chain condition.

Corollary

Every $\vec{\alpha}$-open set is $\vec{\alpha}$-Ramsey.

Proof.

- Suppose that $\mathcal{X}$ is not $\vec{\alpha}$-Ramsey and then apply the $\vec{\alpha}$-Ramsey theorem.

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Proof.

- Suppose that $\mathcal{X}$ is not $\vec{\alpha}$-Ramsey and then apply the $\vec{\alpha}$-Ramsey theorem.
- Then there exists an $\vec{\alpha}$-tree T and S such that $S \subseteq T$ with $st(S) = st(T)$ and for all $\vec{\alpha}$-trees $S' \subseteq S$, $[S'] \not\subseteq \mathcal{X}$ and $[S'] \cap \mathcal{X} \neq \emptyset$.

Corollary

Every $\vec{\alpha}$-open set is $\vec{\alpha}$-Ramsey.

Proof.

- Suppose that $\mathcal{X}$ is not $\vec{\alpha}$-Ramsey and then apply the $\vec{\alpha}$-Ramsey theorem.
- Then there exists an $\vec{\alpha}$-tree T and S such that $S \subseteq T$ with $st(S) = st(T)$ and for all $\vec{\alpha}$-trees $S' \subseteq S$, $[S'] \not\subseteq \mathcal{X}$ and $[S'] \cap \mathcal{X} \neq \emptyset$.
- Let X be any element of $[S] \cap \mathcal{X}$.

Corollary

Every $\vec{\alpha}$-open set is $\vec{\alpha}$-Ramsey.

Proof (Cont).

- If S' is an $\vec{\alpha}$-tree and $X \in [S']$ then $X \in [S] \cap [S']$.

Corollary

Every $\vec{\alpha}$-open set is $\vec{\alpha}$-Ramsey.

Proof (Cont).

- If S' is an $\vec{\alpha}$-tree and $X \in [S']$ then $X \in [S] \cap [S']$.
- So either $st(S') \sqsubseteq st(S) \sqsubseteq X$ or $st(S) \sqsubseteq st(S') \sqsubseteq X$.

Corollary

Every $\vec{\alpha}$-open set is $\vec{\alpha}$-Ramsey.

Proof (Cont).

- If S' is an $\vec{\alpha}$-tree and $X \in [S']$ then $X \in [S] \cap [S']$.
- So either $st(S') \sqsubseteq st(S) \sqsubseteq X$ or $st(S) \sqsubseteq st(S') \sqsubseteq X$.
- Thus, $S' \cap S$ is an $\vec{\alpha}$-tree.

Corollary

Every $\vec{\alpha}$-open set is $\vec{\alpha}$-Ramsey.

Proof (Cont).

- If S' is an $\vec{\alpha}$-tree and $X \in [S']$ then $X \in [S] \cap [S']$.
- So either $st(S') \sqsubseteq st(S) \sqsubseteq X$ or $st(S) \sqsubseteq st(S') \sqsubseteq X$.
- Thus, $S' \cap S$ is an $\vec{\alpha}$-tree.
- Since $S' \cap S \subseteq S$, $[S' \cap S] \not\subseteq \mathcal{X}$ and $[S' \cap S] \cap \mathcal{X} \neq \emptyset$.

Corollary

Every $\vec{\alpha}$-open set is $\vec{\alpha}$-Ramsey.

Proof (Cont).

- If S' is an $\vec{\alpha}$-tree and $X \in [S']$ then $X \in [S] \cap [S']$.
- So either $st(S') \sqsubseteq st(S) \sqsubseteq X$ or $st(S) \sqsubseteq st(S') \sqsubseteq X$.
- Thus, $S' \cap S$ is an $\vec{\alpha}$-tree.
- Since $S' \cap S \subseteq S$, $[S' \cap S] \not\subseteq \mathcal{X}$ and $[S' \cap S] \cap \mathcal{X} \neq \emptyset$.
- In particular, $[S'] \not\subseteq \mathcal{X}$ as $[S' \cap S] \subseteq [S']$.
- So X is not an interior point of $\mathcal{X}$.
- Thus X is not $\vec{\alpha}$-open.

Definition

$\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is $\vec{\alpha}$-**nowhere dense/ is** $\vec{\alpha}$-**meager/ has the** $\vec{\alpha}$-**Baire property** if it is nowhere dense/ is meager/ has the Baire property with respect to the $\vec{\alpha}$-Ellentuck topology.

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Theorem (T., The $\vec{\alpha}$-Ellentuck Theorem)

For any sequence of nonstandard hypernatural numbers $\vec{\alpha}$,

Definition

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Theorem (T., The $\vec{\alpha}$-Ellentuck Theorem)

For any sequence of nonstandard hypernatural numbers $\vec{\alpha}$*, the collection of* $\vec{\alpha}$*-Ramsey sets coincides with the* σ*-algebra of sets with the* $\vec{\alpha}$*-Baire property*

Definition

$\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is **$\vec{\alpha}$-nowhere dense/ is $\vec{\alpha}$-meager/ has the $\vec{\alpha}$-Baire property** if it is nowhere dense/ is meager/ has the Baire property with respect to the $\vec{\alpha}$-Ellentuck topology.

Theorem (T., The $\vec{\alpha}$-Ellentuck Theorem)

For any sequence of nonstandard hypernatural numbers $\vec{\alpha}$, the collection of $\vec{\alpha}$-Ramsey sets coincides with the σ-algebra of sets with the $\vec{\alpha}$-Baire property and the collection of $\vec{\alpha}$-Ramsey null sets coincides with the σ-ideal of $\vec{\alpha}$-meager sets.

Theorem (T.)

Suppose that $\mathcal{U} := \{X \subseteq \omega : \beta \in {}^*X\}$ is selective ultrafilter on $\mathbb{N}$. For $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ the following are equivalent:

1. *$\mathcal{X}$ has the β-Baire property.*
2. *$\mathcal{X}$ is β-Ramsey.*
3. *$\mathcal{X}$ has the $\mathcal{U}$-Baire property.*
4. *$\mathcal{X}$ is $\mathcal{U}$-Ramsey.*

Furthermore, the following are equivalent:

1. *$\mathcal{X}$ is β-meager.*
2. *$\mathcal{X}$ is β-Ramsey null.*
3. *$\mathcal{X}$ is $\mathcal{U}$-meager.*
4. *$\mathcal{X}$ is $\mathcal{U}$-Ramsey null.*

Definition (Strong Cauchy Infinitesimal Principle)

Every nonstandard hypernatural number β is the ideal value of an increasing sequence of natural numbers.

Definition (Strong Cauchy Infinitesimal Principle)

Every nonstandard hypernatural number β is the ideal value of an increasing sequence of natural numbers.

Theorem (T.)

The following are equivalent:

1. *The strong Cauchy infinitesimal principle.*
2. $\{X \in [\mathbb{N}]^\infty : \alpha \in {}^*X\}$ *is a selective ultrafilter.*
3. *If* T *is an* α*-tree and* $s \in T/st(T)$ *then there exists* $X \in [s, \mathbb{N}]$ *such that* $\alpha \in {}^*X$ *and* $[s, X] \subseteq [T]$.
4. $([\mathbb{N}]^\infty, \{X \in [\mathbb{N}]^\infty : \alpha \in {}^*X\}, \subseteq)$ *is a topological Ramsey space.*

Theorem (T.)

If $\mathcal{X}$ *is* β*-Ramsey for all nonstandard hypernatural numbers* β *then* $\mathcal{X}$ *is Ramsey.*

We extend the main results to the setting of triples

$$(\mathcal{R}, \leq, r)$$

1. $\leq$ is a quasi-order on $\mathcal{R}$,

We extend the main results to the setting of triples

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Abstract α-Ramsey Theory

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1. $\leq$ is a quasi-order on $\mathcal{R}$,
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Example (The Ellentuck Space)

$([\mathbb{N}]^\infty, \subseteq, r)$ where r is the map such that for all $n \in \mathbb{N}$ and for all $X = \{x_0, x_1, x_2, \ldots\}$, listed in increasing order,

$$r(n, X) = \begin{cases} \emptyset & \text{if } n = 0, \\ \{x_0, \ldots, x_{n-1}\} & \text{otherwise.} \end{cases}$$

Abstract α-Ramsey Theory

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Example (The Ellentuck Space)

$([\mathbb{N}]^\infty, \subseteq, r)$ where r is the map such that for all $n \in \mathbb{N}$ and for all $X = \{x_0, x_1, x_2, \dots\}$, listed in increasing order,

$$r(n, X) = \begin{cases} \emptyset & \text{if } n = 0, \\ \{x_0, \dots, x_{n-1}\} & \text{otherwise.} \end{cases}$$

The range of r is $[\mathbb{N}]^{<\infty}$ and for all $s \in [\mathbb{N}]^{<\infty}$ and for all $X \in [\mathbb{N}]^\infty$, $s \sqsubseteq X$ if and only if there exists $n \in \mathbb{N}$ such that $r(n, X) = s$.

The range of r, is denoted by $\mathcal{AR}$.

Abstract α-Ramsey Theory

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For $n \in \mathbb{N}$ and $X \in \mathcal{R}$ we use the following notation

$$\mathcal{AR}_n = \{r(n, X) \in \mathcal{AR} : X \in \mathcal{R}\},$$

$$\mathcal{AR}_n \restriction X = \{r(n, Y) \in \mathcal{AR} : Y \in \mathcal{R} \ \& \ Y \leq X\},$$

$$\mathcal{AR} \restriction X = \bigcup_{n=0}^{\infty} \mathcal{AR}_n \restriction X.$$

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$$\mathcal{AR} \upharpoonright X = \bigcup_{n=0}^{\infty} \mathcal{AR}_n \upharpoonright X.$$

If $s \in \mathcal{AR}$ and $X \in \mathcal{R}$ then we say s **is an initial segment of** X and write $s \sqsubseteq X$, if there exists $n \in \mathbb{N}$ such that $s = r(n,X)$.

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If $s \in \mathcal{AR}$ and $X \in \mathcal{R}$ then we say s **is an initial segment of** X and write $s \sqsubseteq X$, if there exists $n \in \mathbb{N}$ such that $s = r(n, X)$.

If $s \sqsubseteq X$ and $s \neq X$ then we write $s \sqsubset X$. We use the following notation:

$$[s] = \{Y \in \mathcal{R} : s \sqsubseteq Y\},$$

$$[s, X] = \{Y \in \mathcal{R} : s \sqsubseteq Y \leq X\}.$$

A subset T of $\mathcal{AR}$ is called a **tree on** $\mathcal{R}$ if $T \neq \emptyset$ and for all $s, t \in \mathcal{AR}$,

$$s \sqsubseteq t \in T \implies s \in T.$$

For a tree T on $\mathcal{R}$ and $n \in \mathbb{N}$, we use the following notation:

$$[T] = \{X \in \mathcal{R} : \forall s \in \mathcal{AR}(s \sqsubseteq X \implies s \in T)\},$$

$$T(n) = \{s \in T : s \in \mathcal{AR}_n\}.$$

A subset T of $\mathcal{AR}$ is called a **tree on** $\mathcal{R}$ if $T \neq \emptyset$ and for all $s, t \in \mathcal{AR}$,

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For a tree T on $\mathcal{R}$ and $n \in \mathbb{N}$, we use the following notation:

$$[T] = \{X \in \mathcal{R} : \forall s \in \mathcal{AR}(s \sqsubseteq X \implies s \in T)\},$$

$$T(n) = \{s \in T : s \in \mathcal{AR}_n\}.$$

Lemma

If $(\mathcal{R}, \leq, r)$ satisfies A.1(Sequencing), A.2(Finitization) and A.4(Pigeonhole Principle) then for all $s \in \mathcal{AR}$ and for all $X \in \mathcal{R}$ such that $s \sqsubseteq X$, there exists $\alpha_s \in {}^*(\mathcal{AR} \restriction X) \setminus (\mathcal{AR} \restriction X)$ such that

$$s \sqsubseteq \alpha_s \in {}^*\mathcal{AR}_{|s|+1}.$$

Definition

An $\vec{\alpha}$-**tree** is a tree T on $\mathcal{R}$ with stem $st(T)$ such that for all $s \in T/st(T)$,

$$\alpha_s \in {}^*T.$$

Abstract $\vec{\alpha}$-Ramsey Theory

Definition

An $\vec{\alpha}$-**tree** is a tree T on $\mathcal{R}$ with stem $st(T)$ such that for all $s \in T/st(T)$,

$$\alpha_s \in {}^*T.$$

Example

Note that $\mathcal{AR}$ is a tree on $\mathcal{R}$ with stem $\emptyset$. Moreover, for all $s \in \mathcal{AR}$, $\alpha_s \in {}^*\mathcal{AR}$. Thus, $\mathcal{AR}$ is an $\vec{\alpha}$-tree.

Theorem (T.)

Assume that $(\mathcal{R}, \leq, r)$ satisfies A.1, A.2 and A.4 and for all $s \in \mathcal{AR}$, ${}^*s = s$. For all $\mathcal{X} \subseteq \mathcal{R}$ and for all $\vec{\alpha}$-trees T there exists an $\vec{\alpha}$-tree $S \subseteq T$ with $st(S) = st(T)$ such that one of the following holds:

1. $[S] \subseteq \mathcal{X}$.
2. $[S] \cap \mathcal{X} = \emptyset$.
3. *For all $\vec{\alpha}$-trees S', if $S' \subseteq S$ then $[S'] \not\subseteq \mathcal{X}$ and $[S'] \cap \mathcal{X} \neq \emptyset$.*

Definition

Assume that $(\mathcal{R}, \leq, r)$ satisfies A.1, A.2 and A.4 and for all $s \in \mathcal{AR}$, ${}^*s = s$. The topology on $\mathcal{R}$ generated by $\{[T] : T \text{ is an } \vec{\alpha}\text{-tree}\}$ is called **the $\vec{\alpha}$-Ellentuck topology**.

Definition

Assume that $(\mathcal{R}, \leq, r)$ satisfies A.1, A.2 and A.4 and for all $s \in \mathcal{AR}$, ${}^*s = s$. The topology on $\mathcal{R}$ generated by $\{[T] : T \text{ is an } \vec{\alpha}\text{-tree}\}$ is called **the $\vec{\alpha}$-Ellentuck topology**.

Theorem (T.)

Assume $(\mathcal{R}, \leq, r)$ satisfies A.1, A.2 and A.4 and for all $s \in \mathcal{AR}$, ${}^*s = s$.

The Abstract $\vec{\alpha}$-Ellentuck Theorem

Definition

Assume that $(\mathcal{R}, \leq, r)$ satisfies A.1, A.2 and A.4 and for all $s \in \mathcal{AR}$, ${}^*s = s$. The topology on $\mathcal{R}$ generated by $\{[T] : T \text{ is an } \vec{\alpha}\text{-tree}\}$ is called **the $\vec{\alpha}$-Ellentuck topology**.

Theorem (T.)

Assume $(\mathcal{R}, \leq, r)$ satisfies A.1, A.2 and A.4 and for all $s \in \mathcal{AR}$, ${}^*s = s$. If $\vec{\alpha}$ is a sequence of nonstandard hyperapproximations,

Definition

Assume that $(\mathcal{R}, \leq, r)$ satisfies A.1, A.2 and A.4 and for all $s \in \mathcal{AR}$, ${}^*s = s$. The topology on $\mathcal{R}$ generated by $\{[T] : T \text{ is an } \vec{\alpha}\text{-tree}\}$ is called **the $\vec{\alpha}$-Ellentuck topology**.

Theorem (T.)

Assume $(\mathcal{R}, \leq, r)$ satisfies A.1, A.2 and A.4 and for all $s \in \mathcal{AR}$, ${}^*s = s$. If $\vec{\alpha}$ is a sequence of nonstandard hyperapproximations, then the collection of $\vec{\alpha}$-Ramsey sets coincides with the σ-algebra of sets with the $\vec{\alpha}$-Baire property

The Abstract $\vec{\alpha}$-Ellentuck Theorem

Definition

Assume that $(\mathcal{R}, \leq, r)$ satisfies A.1, A.2 and A.4 and for all $s \in \mathcal{AR}$, ${}^*s = s$. The topology on $\mathcal{R}$ generated by $\{[T] : T \text{ is an } \vec{\alpha}\text{-tree}\}$ is called **the $\vec{\alpha}$-Ellentuck topology**.

Theorem (T.)

Assume $(\mathcal{R}, \leq, r)$ satisfies A.1, A.2 and A.4 and for all $s \in \mathcal{AR}$, ${}^*s = s$. If $\vec{\alpha}$ is a sequence of nonstandard hyperapproximations, then the collection of $\vec{\alpha}$-Ramsey sets coincides with the σ-algebra of sets with the $\vec{\alpha}$-Baire property and the collection of $\vec{\alpha}$-Ramsey null sets coincides with the σ-ideal of $\vec{\alpha}$-meager sets.

Theorem (T.)

Assume that $(\mathcal{R}, \leq, r)$ satisfies A.1, A.2 and A.4 and for all $s \in \mathcal{AR}$, ${}^*s = s$. Let

$$\mathcal{R}_{\vec{\alpha}} = \{X \in \mathcal{R} : \forall s \in \mathcal{AR} \restriction X,\ \alpha_s \in {}^*r_{|s|+1}[s, X]\}.$$

If for all $\vec{\alpha}$-trees T there exists $X \in \mathcal{R}_{\vec{\alpha}}$ such that $\emptyset \neq [st(T), X] \subseteq [T]$, then $(\mathcal{R}, \mathcal{R}_{\vec{\alpha}}, \leq, r)$ is a topological Ramsey space.

Question

Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space satisfying A.1-A.4. Suppose that $\mathcal{U} \subseteq \mathcal{R}$ a selective ultrafilter with respect to $\mathcal{R}$ as defined by Di Prisco, Mijares and Nieto. For each $s \in \mathcal{AR}$, let $\mathcal{U}_s$ be the ultrafilter on $\{t \in \mathcal{AR}_{|s|+1} : s \sqsubseteq t\}$ generated by $\{r_{|s|+1}[s, X] : X \in \mathcal{U}\}$ and $\vec{\mathcal{U}} = \langle \mathcal{U}_s : s \in \mathcal{AR} \rangle$. Is it the case that for all $\vec{\mathcal{U}}$-trees T there exists $X \in \mathcal{R}_{\vec{\mathcal{U}}}$ such that $\emptyset \neq [st(T), X] \subseteq [T]$?

Thank you for your attention.

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[2] Todorcevic, Introduction to Ramsey spaces, 2010.

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