Containers made easy

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1. **Containers theorem for finite hypergraphs** — joint with Anton Bernshteyn, Michelle Delcourt, and Henry Towsner

2. **Containers theorem for algebraic hypergraphs** — joint with Anton Bernshteyn and Michelle Delcourt

From “dense” to “sparse random” setting

Transference principle in extremal combinatorics

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$$\lim_{n\to\infty} \mathbb{P}\left[[\boldsymbol{n}]_{p_n} \text{ is } (\delta, k)\text{-Szemerédi}\right] = 1,$$

  where $[\boldsymbol{n}]_p$ denotes a randomly chosen subset where each element is included with probability p.

Associated k-hypergraph

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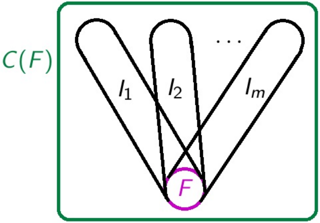
- Each fingerprint F determines a container $C(F)$ of size less than $(1-\alpha)n$, where $\alpha \in (0,1)$, such that if F is a fingerprint of an independent set I, then $I \subseteq C(F)$.

Hypergraph Containers (continued)

- **Containers Theorem:** $I \mapsto F \mapsto C(F)$ such that $F \subseteq I \subseteq C(F)$.



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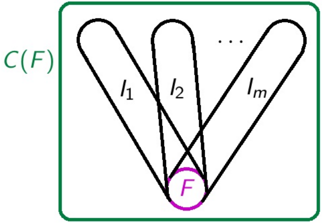

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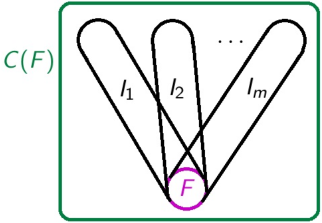

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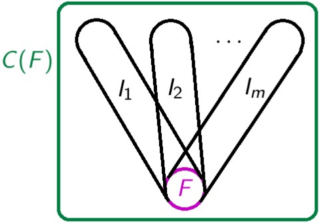

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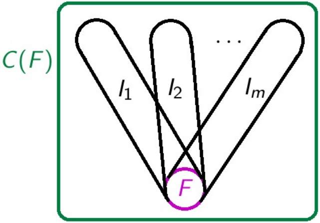

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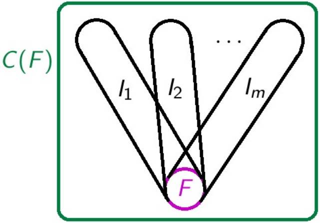

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- Note: in this calculation we used the trivial upper bound $2^{(1-\alpha)n}$ on the number of independent sets inside a given container; in practice, the above machinery is iteratively applied to the container, leading to stronger results, such as the **Sparse Random Szemerédi** theorem.

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- As a result, our proof is nonalgorithmic: it builds H' in one step.

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- We say that H is δ-bounded if $\delta_X(H) \leqslant \delta$.
- For $\delta, \varepsilon \in [0,1]$, we call H (δ, ε)-homogeneous if it is δ-bounded and $\log_{|X|} |H| \geqslant 1 + (k-1)\delta - \varepsilon$

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- We say that H is δ-bounded if $\delta_X(H) \leqslant \delta$.
- For $\delta, \varepsilon \in [0,1]$, we call H (δ, ε)-homogeneous if it is δ-bounded and $\log_{|X|} |H| \geqslant 1 + (k-1)\delta - \varepsilon$, so

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- Note: even when H is δ-bounded, the fiber hypergraph H_U need not be δ-bounded, especially for large $U \subseteq X$.

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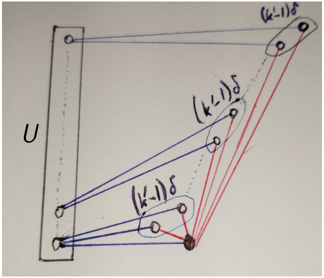

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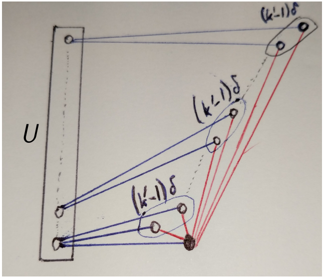

Let $|H|_\delta$ denote the maximum size of a δ-bounded subhypergraph of H:

$$|H|_\delta := \max\left\{|H'| : H' \subseteq H \text{ and } H' \text{ is } \delta\text{-bounded}\right\}.$$

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$$\bigcup \boldsymbol{F} \subseteq I \subseteq \bigcup \boldsymbol{F} \cup C;$$
- each container $C \in \mathrm{im}(\nearrow)$ has a large complement:
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Theorem (Bernshteyn–Delcourt–Towsner–Ts. 2018)

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- *In particular, for a container C, having $\log_{|X|} |X \setminus C| \geqslant 1 - \sigma$ usually implies $|C| \leqslant (1-\alpha)|X|$ for some $\alpha \in (0,1)$.*

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- Need to assign a fingerprint $\boldsymbol{F}$ to I and a container C to $\boldsymbol{F}$.

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- However, as the degrees of the polynomials defining it get higher, it looks more and more random — enough for our arguments to go through.

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Theorem (Bernshteyn–Delcourt–Ts. 2018+)

If a given algebraic hypergraph is "dense", then a Zariski-generic low-dimensional polynomially parameterized subset of the vertices induces a subhypergraph that is also "dense" (in particular, far from independent).

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- Model theory to the rescue: long live saturation and compactness!

An ad that eventually popped up on Henry's twitter

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Thank you