Distinct Volume Subsets: The Uncountable Case

Douglas Ulrich [1] William Gasarch [2]

[1]University of California, Irvine
ds_ulrich@hotmail.com

[2]University of Maryland at College Park
gasarch@cs.umd.edu

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3. If we get a rainbow set, then we are done. Otherwise, get a contradiction to Hilbert's basis theorem.
   For example, if f is min-homogeneous, look at

$$V_n := \{x \in \mathbb{C}^d : \sum_{i<d}(x(i) - a_m(i))^2 = \sum_{i<d}(a_n(i) - a_m(i))^2 \text{ for all } m < n\}.$$

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$2^{\aleph_0}$, the cardinality of the continuum, fits somewhere among the $\aleph$'s.

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Suppose $\kappa \leq 2^{\aleph_0}$ is singular. Then there is $X \subseteq \mathbb{R}^d$ of size κ, such that whenever $Y \subseteq X$ has size κ, then Y is not 3-rainbow.

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But note that our coloring f is definable in $(\mathbb{C}, +, \cdot, 0, 1)$, a *stable structure*. This gives us added leverage.

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Note: the volume of a simplex in $\mathbb{R}^d$ is the square root of a certain polynomial (the Cayley-Menger determinant) in the coefficients of the vertices. Thus the coloring f on the previous slide really is definable in $(\mathbb{C}, +, \cdot, 0, 1)$.

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Theorem (Shelah)

Suppose M is a stable structure, $X \subseteq M$ has regular cardinality, and $f : M^n \to c$ is a definable coloring. Then there is $Y \subseteq X$ with $|Y| = |X|$ such that f is constant on tuples of distinct elements from Y.

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Theorem (U., G.)

Suppose $X \subseteq \mathbb{R}^d$ is infinite, with $|X|$ regular. Then there is $Y \subseteq X$ with $|Y| = |X|$, such that for some $2 \leq a \leq d+1$, Y is strictly a-rainbow.

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Suppose M is a stable structure, and $X \subseteq M$ has singular cardinality κ, and $f : M^n \to c$ is definable. Let E be any equivalence relation on X with fewer than κ classes, each of size less than κ.

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Suppose $\kappa \leq 2^{\aleph_0}$ is infinite. Then there is a finite list $(C_i : i < i_*)$ of subsets of $\mathbb{R}^d$ of size κ, such that whenever $X \subseteq \mathbb{R}^d$ has size κ, there is some $i < i_*$ and some injection $F : C_i \to X$, which preserves the relations "u is a degenerate simplex" and "u, v have the same volume," for all tuples u, v of length $2 \leq a \leq d + 1$.

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Theorem (Blass)

Suppose $f : \binom{2^{\mathbb{N}}}{n} \to c$ partitions $\binom{2^{\mathbb{N}}}{n}$ into finitely many pieces, each with the property of Baire. Then there is some perfect $P \subseteq 2^{\mathbb{N}}$ on which f takes on one of $(n-1)!$ canonical behaviors.

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Theorem (U., G.)

Suppose $P \subseteq \mathbb{R}^d$ is perfect. Then there is $Q \subseteq P$ perfect and some $2 \leq a \leq d+1$, such that Q is strictly a-rainbow.

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It is consistent with ZF that there is some uncountable $X \subseteq \mathbb{R}^d$ which has no uncountable 2-rainbow subset.

Thank you