Non-zero disjoint cycles in highly connected group labelled graphs

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Abstract

Let $G = (V, E)$ be an oriented graph whose edges are labelled by the elements of a group Γ. A cycle C in G has non-zero weight if for a given orientation of the cycle, when we add the labels of the forward directed edges and subtract the labels of the reverse directed edges, the total is non-zero. We are specifically interested in the maximum number of vertex disjoint non-zero cycles.

We prove that if G is a Γ-labelled graph and $\overline{G}$ is the corresponding undirected graph, then if $\overline{G}$ is $\frac{31}{2}k$-connected, either G has k disjoint non-zero cycles or it has a vertex set Q of order at most $2k-2$ such that $G - Q$ has no non-zero cycles. The bound "$2k-2$" is best possible.

This generalizes the results due to Thomassen [18], Rautenbach and Reed [13] and Kawarabayashi and Reed [10], respectively.

Key Words : Non-zero disjoint cycles, highly connected graphs, group-labelled graphs,
Mathematics Subject Classification (2000) : 05C40, 05C83.

1 Introduction

A family $\mathcal{F}$ of graphs has the *Erdős-Pósa property*, if for every integer k there exists an integer $f(k, \mathcal{F})$ such that every graph G contains either k vertex-disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$ or a set C of at most $f(k, \mathcal{F})$ vertices such that $G - C$ has no subgraph isomorphic to a graph in $\mathcal{F}$. The term *Erdős-Pósa property* arose because in [5], Erdős and Pósa proved that the family of cycles has this property.

The situation is different when we consider the family of odd cycles. Lovàsz characterizes the graphs having no two disjoint odd cycles, using Seymour's result on regular matroids [15]. No such characterization is known for more than three odd cycles. In fact, the Erdős-Pósa property does not hold for odd cycles. Reed [14] observed that there exists a class of cubic projective planar graphs $\{G_t : t \in \mathbb{N}\}$ such that G_t does not contain two disjoint odd cycles, and yet there do not exist a set A of t vertices or a set B such that $G - A$ and $G - B$ are bipartite. In fact, this example shows that the Erdős-Pósa property does not necessarily hold for any cycle of length $\not\equiv 0$ modulo m, see in [18].

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While the Erdős-Pósa property does not hold for odd cycles in general, Reed [14] proved that the Erdős-Pósa property holds for odd cycles in planar graphs. This result was extended to an orientable fixed surface in [9]. Note that the Erdős-Pósa property does not hold for odd cycles in nonorientable surfaces, even for projective planar graphs as the above examples show. But such an example on the projective plane or on the non-orientable surface is not 5-connected, so one can hope that if a graph is highly connected compared to k, then the Erdős-Pósa property holds for odd cycles. Motivated by this, Thomassen [18] was the first to prove that there exists a function $f(k)$ such that every $f(k)$-connected graph G has either k disjoint odd cycles or a vertex set X of order at most $2k-2$ such that $G-X$ is bipartite. Hence, he showed that the Erdős-Pósa property holds for odd cycles in highly connected graphs. Soon after that, Rautenbach and Reed [13] proved that the function $f(k) = 576k$ suffices. Very recently, Kawarabayashi and Reed [10] further improved the function to $f(k) = 24k$ The bound "$2k-2$" is best possible in a sense since a large bipartite graph with edges of a complete graph on $2k-1$ vertices added to one side of the bipartition set shows that no matter how large the connectivity is, there are no k disjoint odd cycles.

In this paper, we are interested in group labelled graphs. Let Γ be an arbitrary group. We will use additive notation for groups, though they need not be abelian. Let G be an oriented graph. For each edge e in G, we assign a weight γ_e. The weight γ_e is added when the edge is traversed according to the orientation and subtracted when traversed contrary to the orientation. Rigorously, given an oriented graph G and a group Γ, a Γ-*labelling of* G consists of an assignment of a label γ_e to every edge $e \in E$, and function $\gamma : \{(e, v) | e \in E(G), v \text{ an end of } e\} \rightarrow \Gamma$ such that for every edge $e = (u, v)$ in G where u is the tail of e and v is the head, $\gamma(e, u) = -\gamma_e = -\gamma(e, v)$. Let $C = (v_0 e_1 v_1 e_2 \dots e_k v_k = v_0)$ be a (not necessarily directed) cycle in G. Then the weight of C, denoted by $w(C)$, is $\sum_{i=1}^{k} \gamma(e_i, v_i)$. While the weight of a cycle will generally depend upon the orientation in which we traverse the edges and the vertex chosen to be v_0, we will in general only be concerned whether or not the weight of a particular cycle is non-zero. This is independent of the orientation of the cycle or the initial vertex.

Our main theorem is the following.

Theorem 1.1 *Let G be an oriented graph and Γ a group. Let the function γ be a Γ labelling of G. Let $\overline{G}$ be the underlying undirected graph. If $\overline{G}$ is $\frac{31}{2}k$-connected, then G has either k disjoint non-zero cycles or it has a vertex set Q of order at most $2k-2$ such that $G-Q$ has no non-zero cycles.*

This generalizes the results due to Thomassen [18], Rautenbach and Reed [13] and Kawarabayashi and Reed [10], respectively. Given a graph G, assign edge directions arbitrarily, and let each edge have weight 1. If Γ is the group on 2 elements, the $\gamma(e, v) = \gamma(e, u)$ for all edges $e = (u, v)$. Thus the non-zero cycles are simply the odd cycles in G. Then the above theorem finds disjoint odd cycles or a set of $2k-2$ vertices intersecting all odd cycles. We state this formally:

Corollary 1.2 *Suppose G is $\frac{31}{2}k$-connected. Then G has either k disjoint odd cycles or it has a vertex set Q of order at most $2k-2$ such that $G-Q$ has no odd cycles.*

The bound "$2k-2$" in the above theorem is best possible as the above example shows.

A conference version of this article outlining the results and proof techniques is accepted to appear in [11].

For graph-theoretic terminology not explained in this paper, we refer the reader to [4]. Given a vertex x of a graph G, $d_G(x)$ denotes the degree of x in G. When G is an oriented graph, $d(x)$ will count all edges incident the vertex x, regardless of orientation. For a subset S of $V(G)$, the subgraph induced by S is denoted by $G[S]$. For a subgraph H of G, $G - H = G[V(G) - V(H)]$, and for a vertex x of $V(G)$ and for an edge e of E(G), $G - x = G[V(G) - \{x\}]$ and $G - e$ is the graph obtained from G by deleting e.

We briefly introduce several necessary results. For a fixed set of vertices A in a graph G, an *A-path* is a nontrivial path P with both ends in A and no other vertices in A. Chudnovsky, Geelen, Gerards, Goddyn, Lohman, and Seymour examined non-zero A-paths in a group labelled graph G, proving that for any set of vertices $A \subset V(G)$, the Erdős-Pósa property holds for non-zero A paths. Notice that in a group labelled graph, as in the case of non-zero cycles, the weight of an A path will depend on the direction in which the path is traversed. However, whether or not the weight is non-zero will not. Specifically, Chudnovsky et al. proved:

Theorem 1.3 ([3]) *Let Γ be a group, and G be an oriented graph. If γ is a Γ labelling of G, then for any set S of vertices of G and any positive integer k, either*

1. *there are k disjoint non-zero S paths, or*
2. *there is a vertex set X of order at most $2k - 2$ that meets each such non-zero S path.*

Following the notation of Chudnovsky et. al. in [3], consider a vertex $x \in V(G)$ and a value $\alpha \in \Gamma$. Then for each edge e with head v and tail u, we consider a new assignment of weights:

$$\gamma'_e = \begin{cases} \gamma_e + \alpha & \text{if } v = x \\ -\alpha + \gamma_e & \text{if } u = x \\ \gamma_e, & \text{otherwise} \end{cases}$$

We say γ' is obtained by *shifting* γ at x by the value α. Notice that if we shift γ at some vertex $x \in V(G) - A$, then the weight of any A-path remains unchanged. Similarly, the weight of a cycle also remains invariant under shifting γ.

Observation 1.4 *If a subgraph H of G contains no non-zero cycles, then there exists a weight function γ' obtained from γ by shifting at various vertices such that every edge e with both ends in $V(H)$ has $\gamma'_e = 0$.*

Proof. Clearly, it suffices to consider each connected component of H separately. Take a spanning tree T of H. We can ensure that each edge of the spanning tree has weight zero by performing a series of shifts. Then every other edge e of H must also have weight 0, since otherwise $e \cup T$ would contain a non-zero cycle. □

Also note that if for any edge e in G, we flip the orientation of e and also set $\gamma'_e = -\gamma_e$, we do not change the weight of any cycle or A-path.

A graph G is *k-linked* if G has at least $2k$ vertices, and for any $2k$ vertices $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$, G contains k pairwise disjoint paths $P_1, \ldots, P_k$ such that P_i joins x_i and y_i for $i = 1, 2, \ldots, k$.

The study of k-linked graphs has long history. Jung [6] and Larman and Mani [12], independently, proved the existence of a function $f(k)$ such that every $f(k)$-connected graph is k-linked. Bollobás and Thomason [2] were the first to prove that the linear connectivity is enough, i.e., they proved that every $22k$-connected graph is k-linked. Very recently, Kawarabayashi, Kostochka and Yu [8] proved that every $12k$-connected graph is k-linked, and finally, Thomas and Wollan [16] proved that every $10k$-connected graph is k-linked. Actually, they proved the following stronger statement.

Theorem 1.5 ([16]) *Every $2k$-connected graph with at least $5k|V(G)|$ edges is k-linked.*

We will utilize Theorem 1.5 in the proof of Theorem 1.1.

2 Proof of the main result

Assume Theorem 1.1 is false, and let G be a counterexample with γ a labelling from the group Γ such that there do not exist k disjoint non-zero cycles, nor does there exist a set of $2k-2$ vertices intersecting every non-zero cycle. Moreover, assume that G is a counterexample on a minimal number of vertices.

Take disjoint non-zero cycles $C_1, \ldots, C_l$ such that l is as large as possible (but $G-(C_1 \cup C_2 \cup \cdots \cup C_l)$ is non-empty), and subject to that, $|V(C_1) \cup V(C_2) \cup \cdots \cup V(C_l)|$ is as small as possible. Clearly $l < k$. Let W be the induced subgraph on $C_1 \cup C_2 \cup \cdots \cup C_l$. We proceed with several intermediate claims.

Claim 2.1 *For any vertex v in $G - W$, $d_{C_i}(v) \leq 3$ for any i with $1 \leq i \leq l$.*

Proof. Suppose for a contradiction that v has four neighbors $v_1, \ldots, v_4$ in C_i. Let P_j be the directed path of C_i with endpoints v_j and v_{j+1} not containing any other vertices among $v_1, \ldots, v_4$ except for v_j and v_{j+1}, where the addition $j+1$ is taken modulo 4. We may assume that each edge (v, v_j) is directed from v to v_j. Let a_j be the weight of the edge (v, v_j) and let b_j be the weight of P_j.

Define T_j to be the cycle defined by $vv_jP_jv_{j+1}v$. The weight of T_j is $a_j + b_j - a_{j+1}$. Then

$$\sum_{j=1,\ldots,4} w(T_j) = (a_4 + b_4 - a_1) + (a_1 + b_2 - a_2) + \cdots + (a_3 + b_3 - a_4)$$
$$= b_4 + \cdots + b_1.$$

Then since the weight of C_i is non-zero, some T_j must also have non-zero weight. But this contradicts the minimality of the size of C_i, proving the claim.

□

Claim 2.1 implies that the minimum degree of $G - W$ is at least $\frac{31}{2}k - 3(k-1) > \frac{25}{2}k$. Also by the definition of W, $G - W$ has no non-zero cycles. The following result was originally proved in [1]. For the completeness, we shall give a proof here.

Lemma 2.2 ([1]) *Let G be a graph and k an integer such that*

(a) $|V(G)| \geq \frac{5}{2}k$ *and*

(b) $|E(G)| \geq \frac{25}{4}k|V(G)| - \frac{25}{2}k^2$.

Then $|V(G)| \geq 10k + 2$ and G contains a $2k$-connected subgraph H with at least $5k|V(H)|$ edges.

Proof. Clearly, if G is a graph on n vertices with at least $\frac{25}{4}kn - \frac{25}{2}k^2$ edges, then $\frac{25}{4}kn - \frac{25}{2}k^2 \leq \binom{n}{2}$. Hence, either $n \leq \frac{25}{4}k + \frac{1}{2} - \frac{1}{4}\sqrt{(25k+2)^2 - 400k^2} < \frac{5}{2}k$ or $n \geq \frac{25}{4}k + \frac{1}{2} + \frac{1}{4}\sqrt{(25k+2)^2 - 400k^2} > 10k + 1$. Since $|V(G)| \geq \frac{5}{2}k$, we get the following:

Claim 1. $|V(G)| \geq 10k + 2$.

Suppose now that the theorem is false. Let G be a graph with n vertices and m edges, and let k be an integer such that (a) and (b) are satisfied. Suppose, moreover, that

(c) G contains no $2k$-connected subgraph H with at least $5k|V(H)|$ edges, and

(d) n is minimal subject to (a), (b) and (c).

Claim 2. *The minimum degree of G is more than $\frac{25}{4}k$.*

Suppose that G has a vertex v with degree at most $\frac{25}{4}k$, and let G' be the graph obtained from G by deleting v. By (c), G' does not contain a $2k$-connected subgraph H with at least $5k|V(H)|$ edges. Claim 1 implies that $|V(G')| = n-1 \geq \frac{5}{2}k$. Finally, $|E(G')| \geq m - \frac{25}{4}k \geq \frac{25}{4}k|V(G')| - \frac{25}{2}k^2$. Since $|V(G')| < n$, this contradicts (d) and the claim follows.

Claim 3. *$m \geq 5kn$.*

The claim follows easily from (b) by using Claim 1.

By Claim 3 and (c), G is not $2k$-connected. Since $n > 2k$, this implies that G has a separation (A_1, A_2) such that $A_1 \setminus A_2 \neq \emptyset \neq A_2 \setminus A_1$ and $|A_1 \cap A_2| \leq 2k-1$. By Claim 2, $|A_i| \geq \frac{25}{4}k+1$. For $i \in \{1,2\}$, let G_i be a subgraph of G with vertex set A_i such that $G = G_1 \cup G_2$ and $E(G_1 \cap G_2) = \emptyset$. Suppose that $|E(G_i)| < \frac{25}{4}k|V(G_i)| - \frac{25}{2}k^2$ for $i = 1,2$. Then

$$\begin{aligned} \frac{25}{4}kn - \frac{25}{2}k^2 &\leq m = |E(G_1)| + |E(G_2)| \\ &< \frac{25}{4}k(n + |A_1 \cap A_2|) - 25k^2 \\ &\leq \frac{25}{4}kn - \frac{25}{2}k^2, \end{aligned}$$

a contradiction. Hence, we may assume that $|E(G_1)| \geq \frac{25}{4}k|V(G_1)| - \frac{25}{2}k^2$. Since $n > |V(G_1)| \geq \frac{25}{4}k+1$ and G_1 contains no $2k$-connected subgraph H with at least $5k|V(H)|$ edges, this contradicts (d), and Lemma 2.2 is proved. □

Lemma 2.2 and Theorem 1.5 imply that $G - W$ has a k-linked subgraph H. Note that H has minimum degree at least $2k$. As we observed in the previous section, by taking an equivalent weight function, we may assume every edge of H has weight 0.

Utilizing Theorem 1.3, we prove the following.

Claim 2.3 *There exist k vertex disjoint non-zero H-paths in G.*

Proof. Assume not. Then by Theorem 1.3, there exists a set X of at most $2k-2$ vertices eliminating all the non-zero H-paths. If $G - X$ contains a non-zero cycle C, then because $G - X$ is still at least 2-connected, there exist two disjoint paths from $V(H) - X$ to C. By routing one way or the other around C, we obtain a non-zero path starting and ending in H. Then there exists a non-zero subpath intersecting H exactly at it's endpoints, contradicting our choice of X. □

Now we have proven that there exist k vertex disjoint non-zero H-paths. Clearly these paths can be completed into cycles by linking up their ends in H with paths of weight zero, contradicting our choice of G as a counterexample to Theorem 1.1. This completes the proof of Theorem 1.1.

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