Extremal Functions for Shortening Sets of Paths

Paul Wollan *

School of Mathematics

Georgia Institute of Technology

Atlanta, Georgia 30332

wollan@math.gatech.edu

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Abstract

Let $P_1, \dots, P_k$ be k vertex disjoint paths in a graph G where the ends of P_i are x_i, and y_i. Let H be the subgraph induced by the vertex sets of the paths. We find edge bounds $E_1(n)$, $E_2(n)$ such that,

1. If $e(H) \geq E_1(|V(H)|)$, then there exist disjoint paths $P'_1, \dots, P'_k$ where the ends of P'_i are x_i and y_i such that $|\bigcup_i V(P_i)| > |\bigcup_i V(P'_i)|$.
2. If $e(H) \geq E_2(|V(H)|)$, then there exist disjoint paths $P'_1, \dots, P'_k$ where the ends of P'_i are x'_i and y'_i such that $|\bigcup_i V(P_i)| > |\bigcup_i V(P'_i)|$ and $\{x_1, \cdots, x_k\} = \{x'_1, \cdots, x'_k\}$ and $\{y_1, \cdots, y_k\} = \{y'_1, \cdots, y'_k\}$.

The bounds are the best possible, in that there exist arbitrarily large graphs H' with $e(H') = E_i(H') - 1$ without the properties stipulated in 1 and 2.

1 Introduction and Results

We consider the following problem. Given a graph G containing k vertex disjoint paths $P_1, \dots, P_k$, we ask how many edges can the subgraph induced by the union of the vertex sets of the paths have without ensuring that it is possible to reroute the paths to shorten the sums of their lengths. We find the exact solution to two variants of this problem. In the first case, we would like to ensure that the path ends are individually fixed. We obtain the following bound:

Theorem 1.1 *Let G be a graph and $P_1, \dots, P_k$ be k disjoint paths in G. Assume G has n vertices and $V(G) = \bigcup_i V(P_i)$. Let the ends of P_i be x_i and y_i. Then if $e(G) \geq (2k-1)n - 3\binom{k}{2} - k + 1$, then there exist paths $P'_1, \dots, P'_k$ where the ends of P'_i are x_i and y_i and $\sum_i |V(P_i)| > \sum_i |V(P'_i)|$.*

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Alternatively, what if we do not maintain the specified path ends, but rather, we want to find shorter paths with ends in the same sets. In other words, given $P_1, \ldots, P_k$ are paths with ends in two disjoint sets X and Y, we want to ensure that there exist disjoint paths comprising fewer total vertices, such that each path still has an end in X and an end in Y. We prove the following lower bound.

Theorem 1.2 *Let G be a graph and $P_1, \ldots, P_k$ be k disjoint paths in G. Assume G has n vertices and $V(G) = \bigcup_i V(P_i)$. Let the ends of P_i be x_i and y_i. Then if $e(G) \geq \frac{(3k-1)}{2}n - 2\binom{k}{2} - k + 1$, then there exist paths $P'_1, \ldots, P'_k$ where the ends of P'_i are x'_i and y'_i such that*

1. $\{x_1, \cdots, x_k\} = \{x'_1, \cdots, x'_k\}$ *and* $\{y_1, \cdots, y_k\} = \{y'_1, \cdots, y'_k\}$, *and*
2. $\sum_i |V(P_i)| > \sum_i |V(P'_i)|$.

An application of Theorem 1.1 arises studying the problem of graph linkages. A graph G with at least $2k$ vertices is k-linked if for any set of distinct vertices $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$, there exist vertex disjoint paths $P_1, \ldots, P_k$ where the ends of P_i are s_i and t_i. In [5], it is shown that every $2k$-connected graph G on n vertices with $5kn$ edges is k-linked. In the same source, it is conjectured that every $2k$-connected graph with $(2k-1)|V(G)| - (3k+1)k/2 + 1$ edges is k-linked. A lower bound for this conjecture arises as follows. Let $P_1, \ldots, P_{k-1}$ be $k-1$ disjoint paths with the ends of P_i labeled s_i and t_i. Add to these paths the maximal number of edges such that there do not exist $P'_1, \ldots, P'_{k-1}$ where the endpoints of P'_i are s_i and t_i ,and with the property that the paths P'_i use fewer vertices. Then by adding vertices s_k and t_k adjacent to every other vertex in the graph except each other, we see that this graph cannot be k-linked. The most edges such a graph could have is determined by the optimal bound obtained in Theorem 1.1.

The required number of edges for a 4-connected graph to be 2-linked was proven by Jung in [1], and later a complete characterization of two linked graphs was independently found by several authors [2], [3], [6]. Recently, the exact edge bound when $k = 3$ was proven in [4]. In both cases, the conjectured bound is correct. Moreover, for $k = 3$, there exist tight examples consisting of 2 disjoint paths with the endpoints labeled s_1, s_2, t_1, t_2 which cannot be rerouted to shorten their length, and an additional pair of terminals s_3 and t_3 adjacent every other vertex of the graph. This gives some hope that the conjectured bound may in fact be true.

For the purposes of this paper, all graphs will be considered simple graphs. For a graph G and an edge uv of G will be thought of as an unordered pair of vertice, and the graph G/uv will be the graph obtained by identifying the two vertices u and v of G and deleting all parallel edges. If $P = x_1, \ldots, x_l$ is a path in G and x_i and x_j are two vertices on P such that $i \leq j$, then $x_i P x_j$ will denote the subpath of P, $x_i, x_{i+1}, \ldots, x_j$. For any subset X of the vertices of a graph, $G[X]$ will denote the subgraph induced by the vertices X.

To prove each Theorem, we first examine the case when $k = 2$ and we have exactly two paths. Then Theorems 1.1 and 1.2 follow by counting how many edges a system of k disjoint paths must have before this edge bound is violated for some pair of paths. We prove Theorem 1.1 in Section 2 and we prove Theorem

1.2 in section 3. The theorems actually show that the edge bounds ensure that it is possible to reroute some pair of paths to shorten their length. Perhaps fewer edges would suffice to make it possible to simultaneously reroute some larger subset of the paths and decrease the sum of their lengths. However, we show in section 4 that the bounds obtained by Theorems 1.1 and 1.2 are the best bounds possible by constructing arbitrarily large graphs that do not have the desired shorter paths, and yet have only one fewer edge than the bound stipulated in Theorem 1.1 and 1.2.

2 Proof of Theorem 1.1

The proof of Theorem 1.1 hinges upon the following lemma.

Lemma 2.1 *Let G be a graph on n vertices and let P_1 and P_2 be disjoint paths in G with ends x_1, y_1 and x_2, y_2, respectively. Further, assume $V(G) = V(P_1) \cup V(P_2)$. Then if $e(G) \geq 3n - 4$, there exist disjoint paths P_1' and P_2' with ends x_1, y_1 and x_2, y_2 respectively such that $|V(P_1) \cup V(P_2)| > |V(P_1') \cup V(P_2')|$.*

Proof First observe that if $n \leq 5$, then the statement is vacuously true. We proceed by induction on n. The property is preserved under taking minors. If upon contracting an edge $e \subseteq P_i$, we are able to find shorter paths, then in G as well, we could find shorter paths. Thus we may assume that upon contracting each edge $e \subseteq P_i$, we no longer satisfy the edge bound. For that to happen, if $e = uv$, then u and v must have at least 3 common neighbors on the other path.

Now pick some edge $uv \subseteq P_1$ and let w_1, w_2, w_3 be common neighbors on P_2. Assume that one of the pairs (w_1, w_2) and (w_2, w_3) is not an edge of P_2, say (w_2, w_3). Then there exists some $w_4 \in V(P_2)$ adjacent to w_2 with w_4 between w_2 and w_3 on P_2. Then w_2 and w_4 have three common neighbors on P_1 implying w_2 has some neighbor $z \in V(P_1)$, $z \neq u, v$. If $z \in V(vP_1y_1)$, then $x_1P_1uw_2zP_1y_1$ and $x_2P_2w_1vw_3P_2y_2$ are two paths with the desired ends. They are shorter in length than P_1 and P_2 since the new paths do not contain the vertex w_4. The case when $z \in V(x_1P_1u)$ is symmetric. Thus we may assume every edge uv in P_1 has exactly three neighbors in common in P_2 and more over, they are sequential on P_2.

Let u_1u_2 and u_2u_3 be two edges on P_1 such that the common neighbors of u_1 and u_2 are w_1, w_2, w_3 and the common neighbors of u_2 and u_3 are z_1, z_2, z_3 and assume $\{w_1, w_2, w_3\} \neq \{z_1, z_2, z_3\}$. Then there are two very similar cases: w_1 occurs before z_1 on P_2 and z_1 occurs before w_1. In the first case, $x_1P_1u_1w_2P_2z_1u_3P_1y_1$ and $x_2P_2w_1u_2z_3P_2y_2$ are two paths with the desired ends using fewer vertices. We know the sum of the lengths is less because the paths do not include the vertex z_2 since $w_2 \sim w_1$, and w_1 occurs before z_1. In the second case $x_1P_1u_1w_3P_2z_3u_3P_1y_1$ and $x_2P_2z_1u_2w_3P_2y_2$ are paths with the appropiate ends not utilizing the vertex z_2 in either path.

Thus we may assume that no two consecutive edges on P_1 have a different set of 3 common neighbors. By beginning with the first edge of P_1, it follows that we may assume all edges of P_1 have the same three common neighbors in P_2. Then P_2 can have no other vertex with a neighbor on P_1, and so in fact $|V(P_2)| = 3$. But the argument symetrically shows $|V(P_1)| = 3$. Given the edge bound, it is impossible that P_1 and P_2 are induced paths, i.e. it must be the case that $x_1 \sim y_1$ or $x_2 \sim y_2$, and the statement of the Lemma holds. □

We use the above lemma to prove Theorem 1.1

Assume the theorem is false, and let G and $P_1, \ldots, P_k$ be a counter example. Then clearly by Lemma 2.1 for all i and j, we have $e(G[V(P_i) \cup V(P_j)]) \leq 3(|V(P_i)| + |V(P_j)|) - 5$. Then there are at most $2(|V(P_i)| + |V(P_j)|) - 3$ edges with one end in P_i and the other in P_j. Thus there are at most

$$\sum_{(i,j),i<j} (2\left(|V(P_i)| + |V(P_j)|\right) - 3)$$

edges with ends on distinct paths. Adding the edges contained in the paths, we see that

$$\begin{aligned} e(G) &\leq \sum_{(i,j),i<j} (2\left(|V(P_i)| + |V(P_j)|\right) - 3) + \sum_{i=1}^{k}(|V(P_i)| - 1) \\ &= (2k-1)n - 3\binom{k}{2} - k \end{aligned}$$

contradicting our choice of G and proving the theorem.

3 Proof of Theorem 1.2

As in the previous section, we first prove the bound when we have exactly two paths.

Lemma 3.1 *Let G be a graph on n vertices and let P_1 and P_2 be disjoint paths in G with ends x_1, y_1 and x_2, y_2 respectively. Further, assume $V(G) = V(P_1) \cup V(P_2)$. Then if $e(G) \geq \frac{5}{2}n - 3$, there exist disjoint paths P_1' and P_2' with ends x_1', y_1' and x_2', y_2', respectively, such that $|V(P_1) \cup V(P_2)| > |V(P_1') \cup V(P_2')|$ and $\{x_1, x_2\} = \{x_1', x_2'\}$, $\{y_1, y_2\} = \{y_1', y_2'\}$.*

Proof We proceed by induction on the number of vertices. Notice that the edge bound ensures that both P_1 and P_2 must have at least two vertices, so x_1, x_2, y_1, and y_2 are all distinct vertices. Also, the edge bound implies that we may assume that $n \geq 5$. The property is preserved under taking minors, so if we could contract an edge $e \subseteq V(P_i)$ and satisfy the edge bound, then by induction there would be shorter paths in G/e with the desired property. These paths would extend to paths in G satisfying the claim. Thus we may assume that upon contracting uv, the graph no longer satisfies the edge bound. This implies u and v must have at least 2 common neighbors on the other path.

Define $d^*(v)$ to be the number of neighbors the vertex v has on the other path. The edge bound implies that there are at least $\frac{3}{2}n - 1$ edges with one endpoint in one path and the other endpoint in the second path. Thus,

$$\sum_{v \in V(G)} d^*(v) \geq 3n - 2.$$

First assume that there is no vertex on one path with four or more neighbors on the other path. Then

$$3(n-4) + d^*(x_1) + d^*(x_2) + d^*(y_1) + d^*(y_2) \geq \sum_{v \in V(G)} d^*(v) \geq 3n - 2.$$

Thus we see either $d^*(x_1)+d^*(x_2) \geq 5$ or $d^*(y_1)+d^*(y_2) \geq 5$. Without loss of generality, assume the former. Then we may assume that x_1 has exactly 3 neighbors on P_2, and x_2 has at least 2 neighbors on P_1. Let w_1, w_2, w_3 be the neighbors of x_1 in P_2 and let z be a neighbor of x_2 distinct from x_1. Then if w_3 is the neighbor of x_1 closest to y_2, the paths $x_1w_3P_2y_2$ and $x_2zP_1y_1$ are two paths satisfying the claims of the theorem, but utilizing fewer vertices that P_1 and P_2, since w_2 is not included in the new paths.

Thus we may assume that some vertex has at least 4 neighbors on the other path. Let v be such a vertex and assume that v is in P_1. Then let w_1, w_2, w_3, w_4 be neighbors of v on P_2, and without loss of generality, assume they occur on P_2 in numerical order with w_1 the closest vertex to x_2. The vertex w_2 has some neighbor w_3' on $w_2P_2w_4$ that is distinct from w_4. Then as we have seen above, w_2 and w_3' have at least 2 common neighbors in P_1, specifically, they have a common neighbor distinct from v. Let z be a common neighbor of w_2 and w_3' distinct from v, and without loss of generality, assume z lies in x_1P_1v. Then the paths $x_1zw_3'P_2y_2$ and $x_2w_1vP_1y_1$ are two paths satisfying the claim of the Lemma, since they do not include the vertex w_2. This completes the proof of the Lemma. □

Now we prove Theorem 2.1.

Assume the theorem is false, and let G be a graph on n vertices with $P_1, \ldots, P_k$ be disjoint paths comprising the vertex set of G, and assume the ends of P_i be x_i and y_i. Then by Lemma 3.1, we see that there are strictly less than $\frac{3}{2}|V(P_i) \cup V(P_j)| - 1$ edges with one end in P_i and the other end in P_j for any pair of indices i and j. This implies that

$$\begin{aligned} e(G) &\leq \sum_{(i,j), i<j} \left(\frac{3}{2} \left(|V(P_i)| + |V(P_j)| \right) - 1 \right) + \sum_{i=1}^{k} (|V(P_i)| - 1) \\ &= \frac{(3k-1)}{2} n - \binom{k}{2} - k \end{aligned}$$

contradicting our assumptions on G.

4 Lower Bounds and Tight Examples

Notice that in both the proofs of the above theorems, the bounds we derive are the minimum edge bounds to ensure that pair-wise, we are unable to reroute any two paths to shorten their length, either maintaining the endpoints or swapping the endpoints. The following tight examples show that if we do not satisfy the edge bounds given in the theorems, then there still may be no more complex rerouting of any number of the paths that shortens the sum of the lengths.

First we show the bound of Theorem 1.1 is optimal. Let $V(G)$ consist of k paths. Let P_1 have $n-3(k-1)$ vertices and let v_1, v_2, v_3 be three fixed, sequential vertices of P_1. In other words, the edges v_1v_2 and v_2v_3 are edges of the path P_1. Let $P_2, \ldots, P_k$ be paths on 3 vertices. Let the ends of P_i be x_i and y_i. Then add all edges to $\bigcup_{j, j\neq 1} V(P_j)$ except the edges x_jy_j. Now for all $v \in V(P_1)$, $v \neq v_1, v_2, v_3$, add the edges vx_j and vy_j. For the vertices v_1, v_2, v_3, connect them to all vertices of $\bigcup_{j, j\neq 1} V(P_j)$. Then this graph has $e(G) = (2k-1)n - \binom{k}{2} - k$ and it is impossible to reroute the paths to not use all the vertices of G and still preserve the endpoints of the paths.

To see that the bound for Theorem 1.2 is optimal, let $H_1, \ldots, H_t$ be t disjoint copies of K_k, the complete graph on k vertices. Let every vertex in H_i be adjacent every vertex in H_{i+1}. Then this graph contains k disjoint paths of length $t-1$, each with one end in H_1 and one end in H_t, but there clearly do not exist k paths using fewer vertices where each path still has one end in H_1 and the other end in H_t. This graph has $n = kt$ vertices and $\frac{3k-1}{2}n - \binom{k}{2} - k$ edges, implying that the bound for Theorem 2.1 is the optimal bound.

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