Bridges in highly connected graphs

Paul Wollan[*†]
Department of Computer Science
University of Rome, "La Sapienza"
Via Salaria 113
00198 Rome, Italy

Abstract

Let $\mathcal{P} = \{P_1, \ldots, P_l\}$ be a set of internally disjoint paths contained in a graph G and let S be the subgraph defined by $\bigcup_{i=1}^t P_i$. A $\mathcal{P}$-*bridge* is either an edge of $G - E(S)$ with both endpoints in $V(S)$ or a component C of $G - V(S)$ along with all the edges from $V(C)$ to $V(S)$. The attachments of a bridge B are the vertices of $V(B) \cap V(S)$. A bridge B is k-stable if there does not exist a subset of at most $k-1$ paths in $\mathcal{P}$ containing every attachment of B. A classic theorem of Tutte states that if G is a 3-connected graph, there exists a set of internally disjoint paths $\mathcal{P}' = \{P_1', \ldots, P_l'\}$ such that P_i and P_i' have the same endpoints for $1 \leq i \leq t$ and every $\mathcal{P}'$-bridge is 2-stable. We prove that if the graph is sufficiently connected, the paths $P_1', \ldots, P_l'$ may be chosen so that every bridge containing at least two edges is in fact k-stable. We also give several simple applications of this theorem related to a conjecture of Lovász on deleting paths while maintaining high connectivity.

Key Words : graph, connectivity, bridges, non-separating cycles

1 Introduction

The graphs we consider will be simple with no loops or multiple edges. We include in the definition of a path the *trivial path* consisting of a single vertex. An *internal* vertex of a path is a vertex not equal to one of its endpoints. Two paths P and Q are *internally disjoint* if every vertex in $V(P) \cap V(Q)$ is an endpoint of both P and Q. Given two graphs G_1 and G_2, the graph $G_1 \cup G_2$ has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. In general, we will follow the notation of [2].

Let G be a graph. A *path system* $\mathcal{P}$ is a set $\{P_1, P_2, \ldots, P_l\}$ of pairwise internally disjoint paths in G such that if P_i is trivial, then for all j, $j \neq i$, $V(P_i) \nsubseteq V(P_j)$. The *order* of the path system $\mathcal{P}$, denoted $|\mathcal{P}|$, is the number of elements. Let $\mathcal{P} = \{P_1, \ldots, P_l\}$ be a path system of order l and label

[*]This work partially supported by a fellowship from the Alexander von Humboldt Foundation
[†]Email address: `wollan@di.uniroma1.it`

the endpoints of P_i to be s_i and t_i (with $s_i = t_i$ in the case P_i is trivial). A path system $\mathcal{P}'$ of order l is *equivalent* to $\mathcal{P}$ if we can label the elements of $\mathcal{P}'$ as $P_1', \dots, P_l'$ so that the endpoints of P_i' are s_i and t_i for all $1 \leq i \leq l$.

Let $\mathcal{P} = \{P_1, \dots, P_l\}$ be a path system in a graph G and let $H_{\mathcal{P}}$ be the subgraph of G given by $\bigcup_1^l P_i$. A $\mathcal{P}$-*bridge* in G is a connected subgraph B such that one of the following holds.

(*i*) B is a single edge of $E(G) \setminus E(H_{\mathcal{P}})$ with both endpoints contained in $V(H_{\mathcal{P}})$. In this case, B is called a *trivial bridge.*

(*ii*) $B - V(H_{\mathcal{P}})$ is a connected component of $G - V(H_{\mathcal{P}})$ and B contains every edge of G with one end in $V(B) \setminus V(H_{\mathcal{P}})$ and one end in $V(H_{\mathcal{P}})$.

The *attachments* of the bridge B are the vertices of $V(B) \cap V(H_{\mathcal{P}})$.

A classic theorem of Tutte states the following.

Theorem 1.1 ([11]) *Let $l \geq 3$ be a positive integer and let $\mathcal{P}$ be a path system in a graph G of order l. If G is 3-connected, then there exists a path system $\mathcal{P}'$ which is equivalent to $\mathcal{P}$ such that for all $\mathcal{P}'$-bridges B, there does not exist an element $P_i' \in \mathcal{P}'$ which contains every attachment of B.*

The theorem is sometimes known as the stable bridges theorem, where a $\mathcal{P}$-bridge B of a path system $\mathcal{P}$ is *stable* if there does not exist an element $P \in \mathcal{P}$ which contains every attachment of B. We can generalize this notion of stability in the following way. Let $\mathcal{P} = \{P_1, \dots, P_l\}$ be a path system. A $\mathcal{P}$-bridge B is k-*stable* if there does not exist a subset $I \subseteq \{1, 2, \dots, l\}$ with $|I| \leq k-1$ such that every attachment of B is contained in $\bigcup_{i \in I} V(P_i)$. Stability is then equivalent to 2-stability. Obviously, a trivial bridge can never be k-stable for $k \geq 3$. However, one might hope that the following natural generalization of Tutte's theorem holds.

Question 1 *Does there exist a constant c such that the following holds? Let G be a ck-connected graph, and let $\mathcal{P} = \{P_1, P_2, \dots, P_l\}$ be a path system of order l for $l \geq k+1$. Then there exists a path system $\mathcal{P}'$ which is equivalent to $\mathcal{P}$ such that every trivial $\mathcal{P}'$-bridge is stable and every non-trivial $\mathcal{P}'$-bridge is k-stable.*

Unfortunately, there is an annoying counter-example to the above question. Let G be a very highly connected graph and let $k \geq 2$ be a positive integer. Let $x_1, \dots, x_k$ be k distinct vertices in G. Let $\mathcal{P} = \{P(i,j) : 1 \leq i < j \leq k\}$ be a path system of order $\binom{k}{2}$ such that the endpoints of $P(i,j)$ are x_i and x_j. In effect, $\mathcal{P}$ forms a subdivision of the complete graph K_k. Assume also that for every $1 \leq i < j \leq k$, the vertices x_i and x_j are adjacent. If the question were true, then there would exist a path system $\mathcal{P}'$ equivalent to $\mathcal{P}$ with $\mathcal{P}' = \{P(i,j)' : 1 \leq i < j \leq k\}$ such that $P(i,j)'$ has endpoints x_i and x_j. Moreover, every trivial $\mathcal{P}'$-bridge is stable and every non-trivial $\mathcal{P}'$-bridge is k-stable. Specifically, this implies that each element $P(i,j)'$ of $\mathcal{P}'$ is an induced path from x_i to x_j and consequently consists of the single edge x_ix_j. We conclude that the subgraph formed by

$\bigcup_{P' \in \mathcal{P}'} P'$ is exactly the K_k subgraph of G induced by $\{x_1, \ldots, x_k\}$. Any non-trivial bridge, even if it has as attachments all the vertices $\{x_1, \ldots, x_k\}$ will have all its attachments contained in $k-1$ elements of $\mathcal{P}'$, namely $P(1,2)', P(1,3)', \ldots, P(1,k)'$. Thus, no non-trivial bridge can be k-stable.

In conclusion, despite the fact that $\mathcal{P}$ is a path system of order $O(k^2)$ and that G could be chosen to be arbitrarily highly connected, the desired equivalent path system cannot exist. However, we will see that modulo this technicality, Question 1 is true.

Fix $\mathcal{P} = \{P_1, \ldots, P_l\}$ to be a path system in a graph G. Let $\mathcal{P}' = \{P_1', \ldots, P_l'\}$ be a path system equivalent to $\mathcal{P}$ such that the endpoints of P_i' are s_i' and t_i'. We say $\mathcal{P}'$ is an *induced simplification of* $\mathcal{P}$ if for all indices i such that P_i' has length at least two and s_i' is adjacent to t_i', then there exists an index j such that P_j' is the edge $s_i't_i'$. In other words, an induced simplification is obtained by repeatedly replacing an element of $\mathcal{P}$ of length at least two by a single edge connecting its endpoints. Note that the induced simplification of $\mathcal{P}$ need not be unique. Assume that there exist elements P_i and P_j which are both paths between the same two vertices u and v, and, moreover, assume there exists the edge uv in G. A induced simplification could be obtained by replacing either P_i or P_j by the edge uv. Finally, for all positive integers k, we say a path system $\mathcal{P}$ is *k-spread* if there does not exist an induced simplification $\mathcal{P}' = \{P_1', P_2', \ldots, P_l'\}$ of $\mathcal{P}$ and a subset $I \subseteq \{1, \ldots, l\}$ with $|I| < k$ such that $\bigcup_{i \in I} V(P_i') = \bigcup_1^l V(P_i')$. Note that if the elements of $\mathcal{P}$ are pairwise disjoint and $l \geq k$, then $\mathcal{P}$ is k-spread. The following theorem is the main goal of this article.

Theorem 1.2 *Let $k \geq 2$ be a positive integer and let G be an $83k$-connected graph. Let $\mathcal{P}$ be a k-spread path system contained in G. Then there exists a path system $\mathcal{P}'$ which is equivalent to $\mathcal{P}$ such that every trivial $\mathcal{P}'$-bridge is stable and every non-trivial $\mathcal{P}'$-bridge is k-stable.*

The following corollary follows immediately from Theorem 1.2 and the definition of k-spread.

Corollary 1.3 *Let $k \geq 2$ be a positive integer and let G be a $83k$-connected graph. Let $\mathcal{P}$ be a path system of order l, $l \geq k$, such that every element of $\mathcal{P}$ is an induced path of length at least two. Then there exists a path system $\mathcal{P}'$ which is equivalent to $\mathcal{P}$ such that every trivial $\mathcal{P}'$-bridge is stable and every non-trivial $\mathcal{P}'$-bridge is k-stable.*

This article is organized as follows. In section 2, we give some necessary notation and auxiliary results. In Section 3, we present two applications of Theorem 1.2 to questions related to a conjecture of Lovász on deleting paths while maintaining high connectivity. In the following sections, we give the proof of the main theorem. We conclude with a brief discussion on the possibility of improving the amount of connectivity necessary in the main theorem as well as a lower bound for the best possible value of the constant.

2 Notation

We first give some further notation which we will use going ahead. A *separation* (A, B) of a graph G is a pair of subsets of $V(G)$ such that $A \cup B = V(G)$ and $G = G[A] \cup G[B]$. The *order* of a separation is $|A \cap B|$. A separation (A, B) is *trivial* if $A \subseteq B$ or $B \subseteq A$. Given a path P in a graph G and two specified vertices x and y in P, we refer to the subpath of P with endpoints x and y by xPy.

While it is most convenient, and perhaps most natural, to phrase Theorem 1.2 in terms of path systems, for the proof and in one of the applications, we will use a slightly different statement focusing on the subgraph formed by the union of the elements of the path system.

Let S be an arbitrary subgraph of a graph G. A set of *branch vertices of* S is any subset $X \subseteq V(S)$ such that X contains every vertex of S of deg_S at least three and every vertex of deg_S at most one. Fix a set X of branch vertices of S. An (S, X)*-segment* is either

(i) a non-trivial path P contained in S with both endpoints contained in X and no internal vertex in X, or

(ii) a trivial path P consisting of a vertex of degree 0 in S.

Observe that by definition the set of (S, X)-segments forms a path system.

Let S be a subgraph of a graph G and X a set of branch vertices of S. Let $\mathcal{P}$ be the path system of (S, X)-segments. An (S, X)*-bridge* will simply be a $\mathcal{P}$-bridge, and an (S, X)-bridge is k-stable if it is k-stable with respect to the path system $\mathcal{P}$. The reader might note that the definition of $\mathcal{P}$-bridge only relies on the subgraph formed by $\bigcup_{P \in \mathcal{P}} P$, i.e. the subgraph S, and therefore the reference to the set X of branch vertices might be superfluous. However, the notion of k-stability only makes sense for a specific set X of branch. Since we will be focusing on stability here, we will only refer to (S, X)-bridges including the reference to the branch set.

We now restate Theorem 1.2 in these new terms.

Theorem 2.1 *Let $k \geq 1$ be given and let G be a $83k$-connected graph. Let S be a subgraph of G, and let X be a set of branch vertices of S. Assume that the set of (S, X)-segments is k-spread. Then there exists a subgraph S' of G such that X is a set of branch vertices of S' with the following properties. The set of (S', X)-segments is equivalent to the set of (S, X)-segments, every trivial (S', X)-bridge is stable, and every non-trivial (S', X)-bridge is k-stable. Moreover, if there exists a non-trivial bridge of (S, X), then there exists a non-trivial bridge of (S', X).*

Theorem 1.2 is an immediate consequence of Theorem 2.1.

The proof of Theorem 2.1 as well as the applications of Theorem 2.1 will make use of the theory of graph linkages. A *linkage problem of size* k in a graph G is a multiset of k subsets of $V(G)$ of size two $\mathcal{L} = \{\{s_i, t_i\} : s_i, t_i \in V(G), 1 \leq i \leq k\}$. A *solution* to a given linkage problem $\mathcal{L} = \{\{s_i, t_i\} : s_i, t_i \in V(G), 1 \leq i \leq k\}$ is a set of k internally disjoint paths $P_1, P_2, \ldots, P_k$ where the endpoints of P_i are s_i and t_i for all $1 \leq i \leq k$. A graph G is k*-linked* if it has at least $2k$ vertices and if there exists a solution to every linkage problem of size k with pairwise disjoint subsets of size two. By assuming a sufficient amount of connectivity, we can assume a given graph is k-linked.

Theorem 2.2 ([9]) *Every $10k$-connected graph is k-linked.*

A graph G is *strongly k-linked* if every linkage problem of size k has a solution. It has been independently shown by Mader [8] as well as by Liu, West, and Yu [5] that every k-linked graph on at least $2k$ vertices is also strongly k-linked. This immediately implies following corollary to Theorem 2.2.

Corollary 2.3 *Every $10k$-connected graph is strongly k-linked.*

Corollary 2.3 can also be directly proven from Theorem 2.2 by a simple construction duplicating vertices contained in multiple pairs of a given linkage problem.

3 Applications to removable path questions

Before proceeding with the proof of Theorem 2.1, we first examine several applications of the theorem to problems arising from a collection of questions we will generally refer to as removable paths conjectures. The following conjecture is due to Lovász:

Conjecture 3.1 (Lovász) *There exists a function $f(k)$ such that for every $f(k)$-connected graph G and every pair of vertices s and t of G, there exists an s-t path P such that $G - V(P)$ is k-connected.*

Progress on the conjecture so far has been limited, leading to the study of a variety of weaker versions of the conjecture.

Conjecture 3.1 has been shown to be true for small values of k. The case when $k = 1$ is an immediate consequence of Theorem 1.1. To see this, let G be a 3-connected graph and let u and v be any pair of vertices of G. If we add the edge uv to the graph in the case u and v are not adjacent, Theorem 1.1 implies that there exist paths P_1 and P_2, each of length at least two, such that every bridge of the path system $\{P_1, P_2, uv\}$ is stable. Either of the paths P_1 or P_2 can be deleted and leave the remaining graph connected. A path P in a graph G where $G - V(P)$ is connected is called a *non-separating* path. The $k = 1$ case of Conjecture 3.1 can be rephrased to state that there exists a non-separating path connecting any pair of vertices, assuming the graph satisfies some connectivity bound. Chen, Gould, and Yu [1] show in fact that a highly connected graph contains many internally disjoint non-separating paths linking any pair of vertices.

Theorem 3.2 ([1]) *Let k be a positive integer and let G be a $(22k + 2)$-connected graph. Then for any pair of vertices u and v of G there exist k internally disjoint non-separating paths $P_1, P_2, \ldots, P_k$ such that the endpoints of P_i are u and v for every $1 \leq i \leq k$.*

We will see that by combining the proof of the $k = 1$ case of Conjecture 3.1 above with Theorem 1.2, we get a slight strengthening of Theorem 3.2 (albeit with a worse constant) as an easy corollary to Theorem 1.2.

A path system $\{P_1, P_2, \ldots, P_l\}$ contained in a graph G is *batch non-separating* if for any subset $I \subseteq \{1, 2, \ldots, l\}$ the graph $G - \left(\bigcup_{i \in I} V(P_i)\right)$ is connected.

Theorem 3.3 *Let G be a $83(l+1)$-connected graph, and let x and y be two vertices in G. Then there exists a path system $\mathcal{P} = \{P_1, P_2, \ldots, P_l\}$ of x-y paths such that $\mathcal{P}$ is batch non-separating.*

Proof. [Assuming Theorem 1.2]

Let G be a $83(l+1)$-connected graph and let x and y be two vertices of G. If we add the edge xy in the case where x and y are not adjacent, we may assume that there exists a path system $\mathcal{P} = \{P_1, P_2, \ldots, P_{l+1}, xy\}$ of order $l + 2$ such that P_i has length at least two for $1 \leq i \leq l + 1$. Moreover, by the connectivity of G, we may assume that $V(G) \neq \bigcup_1^{l+1} V(P_i)$. The path system $\mathcal{P}$ is $(l+1)$-spread, and so Theorem 1.2 implies that there exists a path system $\mathcal{P}' = \{P'_1, \ldots, P'_{l+1}, xy\}$ consisting of $l + 2$ disjoint x-y paths such that every non-trivial $\mathcal{P}'$-bridge is $l+1$-stable. It follows that every non-trivial $\mathcal{P}'$-bridge must have at least one neighbor which is an internal vertex of P'_i for all $1 \leq i \leq l + 1$.

Given that $V(G) \neq \bigcup_1^{l+1} V(P_i)$, we see that there exists at least one non-trivial $\mathcal{P}'$-bridge. We claim that the path system $\{P'_1, P'_2, \ldots, P'_l\}$ is batch non-separating. Consider for any subset $I \subseteq \{1, \ldots, l\}$, for every $j \notin I$, the path $P'_j - \{x, y\}$ must be in the same component of $G - \bigcup_{i \in I} V(P_i)$ as the subpath $P_{l+1} - \{x, y\}$ since there is a $\mathcal{P}'$-bridge with attachments in both. Thus $G - \bigcup_{i \in I} V(P_i)$ is connected, as desired. □

Returning our attention to Conjecture 3.1, we have seen that the case $k = 1$ is true. The case when $k = 2$ has been shown to be true as well by Kriesell [4] and independently by Chen et al [1] where they show that every 5-connected graph contains a path linking any two vertices such that deleting the path leaves the remaining graph 2-connected. The first open case is when $k = 3$. The following theorem is due to Kawarabayashi, Reed, and Thomassen [3]. We recall that to *subdivide* an edge e of a graph G, we simply replace the edge e by a path of length two. A *subdivision* of a graph H is any graph which can be obtained from H by repeatedly subdividing edges.

Theorem 3.4 ([3]) *There exists a constant c such that for every c-connected graph G and every pair of vertices s and t of G, there exists an s-t path P and a 3-connected graph H such that such that $G - V(P)$ is isomorphic to a subdivision of H.*

Kawarabayashi et al prove Theorem 3.4 from first principles. We give the following short proof. In this case, it will be more convenient to use the notation of Theorem 2.1.

Proof. [Assuming Theorem 2.1]

Let $c = 249$, and let G be a c-connected graph. Fix the vertices s and t in G. Let H be a 3-connected graph and P an s-t path in G such that $G - P$ contains a subgraph isomorphic to a subdivision of H. Such a path P and graph H exist, since deleting a shortest s-t path results in a subgraph with minimum degree larger than 3 which consequently contains a subdivision of K_4 [2].

Assume such a subgraph H and path P are chosen to maximize $|E(H)|$. Let S_H be the subgraph of $G - P$ isomorphic to a subdivision of H, and let $X := \{v \in V(S_H) : deg_{S_H}(v) \geq 3\} \cup \{s, t\}$ be the set of vertices of S_H of degree at least three and the vertices s and t. We may assume that every $(S_H \cup P, X)$-segment contained in S_H is an induced path. The set X is a branch set for the subgraph $S_H \cup P$. Consider the $(S_H \cup P, X)$-segments. Given that H is 3-connected, there exist two disjoint $(S_H \cup P, X)$-segments in S_H. Along with the third $(S_H \cup P, X)$-segment P, we see that there exist three disjoint $(S_H \cup P, X)$-segments and that consequently the $(S_H \cup P, X)$-segments are 3-spread. Thus by applying Theorem 2.1, we may assume that the trivial $(S_H \cup P, X)$-bridges are stable and that the non-trivial $(S_H \cup P, X)$-bridges are 3-stable.

Define a non-trivial path R to be *violating* if the following conditions hold.

1. The endpoints x and y of R are contained in $V(S_H)$ and R has no internal vertex in $V(S_H) \cup V(P)$.

2. Every $(S_H \cup P, X)$-segment contains at most one of the vertices x and y.

We claim that no violating path exists. Assume, to reach a contradiction, that R is such a violating path. The subgraph $S_H \cup R$ forms a subdivision of a graph H' which is disjoint from P. If both x and y are contained in X, then H' is obtained from H by adding an edge. It follows that H' is 3-connected and that $|E(H')| > |E(H)|$. This contradicts our choice of H. Alternatively, if one (or both) of x and y is a vertex of degree two in S_H, then H' is obtained from H by subdividing one (or two) edges and adding an edge. Note that this operation preserves 3-connectivity, guaranteeing that the resulting graph H' is in fact 3-connected. Again, $|E(H')| > |E(H)|$, contradicting our choice of H. We conclude that no such violating path R exists.

Consider an $(S_H \cup P, X)$-bridge B with at least one attachment in $V(S_H)$. If B is non-trivial, then by the 3-stability of B, there must exist a violating path contained in B, a contradiction. If B is trivial and has both endpoints contained in $V(S_H)$, again by the stability of B, we see that B itself would form a violating path, again a contradiction. We conclude that B is a trivial $(S_H \cup P, X)$-bridge with exactly one endpoint in $V(S_H)$ and the other endpoint in P. Thus $G - P$ is exactly the subgraph S_H, proving the theorem. □

4 Finding a Comb

A *linkage* is a path system such that the elements are pairwise disjoint. Given two sets X and Y in a graph G, a linkage $\mathcal{Q}$ of paths in G is a linkage *from X to Y* if every element of $\mathcal{Q}$ has one endpoint in

X and one endpoint in Y and is otherwise disjoint from $X \cup Y$. We will need to refer several times in this section to the set of all endpoints of components of a given linkage. Given a linkage $\mathcal{P}$ contained in a graph G, we define $Ends(\mathcal{P}) = \{v \in V(G) :$ there exists $P \in \mathcal{P}$ such that v is an endpoint of $P\}$. Note that if $\mathcal{P}$ is a linkage of order k, then $k \leq |Ends(\mathcal{P})| \leq 2k$ with the value depending on the number of trivial elements of $\mathcal{P}$. For the linkage $\mathcal{P}$, we let $Int(\mathcal{P}) = \left(\bigcup_{P \in \mathcal{P}} V(P)\right) \setminus Ends(\mathcal{P})$ be the set of interior vertices.

Definition Let G be a graph and let $\mathcal{P} = \{P_1, \ldots, P_l\}$ be a path system of order l for some positive integer l. Let the endpoints of P_i be labeled s_i and t_i for all $1 \leq i \leq l$. Note that s_i and t_i will not necessarily be distinct from either s_j or t_j. Let H be a subgraph of G such that $V(H) \cap \left(\bigcup_1^l V(P_i)\right) = \emptyset$. An $(H, \mathcal{P})$-*comb of order* k consists of two linkages $\mathcal{Q}$ and $\mathcal{R}$ which satisfy the following.

- (i) $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ is a linkage of order k and $\mathcal{R} = \{R_1, \ldots, R_{l'}\}$ is a linkage of order l' for some $l' \leq l$. Furthermore, $|Ends(\mathcal{R})| = k$ and $\mathcal{Q}$ is an $V(H)$-$Ends(\mathcal{R})$ linkage.
- (ii) After possibly re-indexing the elements of $\mathcal{P}$, we have the property that R_i is a subpath of P_i for all $1 \leq i \leq l'$. Moreover, no $\mathcal{P}$-segment contains two distinct elements of $\mathcal{R}$.
- (iii) The linkage $\mathcal{Q}$ intersects $\mathcal{P}$ only in the vertices of $\mathcal{R}$, i.e. $\left(\bigcup_1^k V(Q_i)\right) \cap \left(\bigcup_1^l V(P_i)\right) \subseteq \bigcup_1^{l'} V(R_i)$.

Combs were first introduced in [10], although with different notation and in a slightly different context where the path system $\mathcal{P}$ is assumed to be a linkage and the subgraph H is allowed to intersect the linkage $\mathcal{P}$.

The main result of this section will be to provide a necessary and sufficient condition for the existence of an $(H, \mathcal{P})$-comb in a given graph G. We first prove the following lemma.

Lemma 4.1 *Let $k, l \geq 1$ be positive integers and let $\mathcal{P} = \{P_1, \ldots, P_l\}$ be a path system of order l contained in a graph G. Let H be a subgraph of $G - \left(\bigcup_1^l V(P_i)\right)$. Let $\mathcal{Q}$ and $\mathcal{R}$ form an $(H, \mathcal{P})$-comb of order k. Then there exists an $(H, \mathcal{P})$-comb given by $\mathcal{Q}'$ and $\mathcal{R}'$ such that either*

- (i) *$\mathcal{Q}'$ and $\mathcal{R}'$ form an $(H, \mathcal{P})$-comb of order $k+1$, for all $R \in R$ there exists an $R' \in \mathcal{R}'$ such that $V(R) \subseteq V(R')$, and furthermore, $||\mathcal{R}| - |\mathcal{R}'|| \leq 1$. Or, alternatively,*
- (ii) *$\mathcal{Q}'$ and $\mathcal{R}'$ form an $(H, \mathcal{P})$-comb of order k, there exists a separation (A, B) of order k such that $V(H) \subseteq B$, $\left(\bigcup_1^l V(P_i)\right) \setminus Int(\mathcal{R}') \subseteq A$.*

Proof. Let $\mathcal{P} = \{P_1, \ldots, P_l\}$ and H be given. Let the linkages $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ and $\mathcal{R} = \{R_1, \ldots, R_{l'}\}$ form an $(H, \mathcal{P})$-comb of order k. We pick an $(H, \mathcal{P})$-comb given by $\mathcal{Q}' = \{Q'_1, \ldots, Q'_k\}$ and $\mathcal{R}' = \{R'_1, \ldots, R'_{l'}\}$ of order k such that for all $1 \leq i \leq l'$, we have that R_i is a subpath of R'_i. Moreover, assume we have selected $\mathcal{Q}'$ and $\mathcal{R}'$ over all such $(H, \mathcal{P})$-combs to maximize $\sum_{R \in \mathcal{R}'} |V(R)|$. Assume that R'_i is a subpath of P_i for $1 \leq i \leq l'$. Let $Z = \left(\bigcup_1^l V(P_i)\right) \setminus Int(\mathcal{R}')$. Let Y be the set of endpoints of $\mathcal{Q}'$ in $V(H)$. Note that by definition, the set of endpoints of $\mathcal{Q}'$ in Z is exactly the set $Ends(\mathcal{R}')$. If there exists a separation (A, B) of order exactly k with $Z \subseteq A$ and $V(H) \subseteq B$, the separation (A, B) satisfies (ii) and the lemma is proven. Alternatively, there exist $k+1$ vertex disjoint $V(H)$-$\bigcup_1^l V(P_i)$ paths $\bar{Q}_1, \dots, \bar{Q}_{k+1}$ and a vertex $z \in Z \setminus Ends(\mathcal{R}')$ and a vertex $u \in V(H) \setminus Y$ such that each $\bar{Q}_i$ has one endpoint in $Ends(\mathcal{R}') \cup \{z\}$ and one endpoint in $Y \cup \{u\}$.

As a case, assume there exists an index $i \leq l'$ such that $z \in V(P_i)$. Note that R'_i is a subpath of P_i, and moreover, R'_i is the unique component of $\mathcal{R}'$ completely contained in P_i by the definition of an $(H, \mathcal{P})$-comb. Let $\bar{R}$ be the minimal subpath of P_i containing both the vertex z and the subpath R'_i. Let i' be the index such that $\bar{Q}_{i'}$ has an endpoint contained as an internal vertex of the path $\bar{R}$, if such an index exists. Note that since $\bar{R}$ contains at most one vertex of $Ends(\mathcal{R}')$ as an internal vertex, we have that if i' is defined, then there is a unique such index. If the index i' is not defined, then $\{\bar{Q}_1, \dots, \bar{Q}_{k+1}\}$ and $(\mathcal{R}' \setminus \{R'_i\}) \cup \{\bar{R}\}$ yields an $(H, \mathcal{P})$ comb of order $k+1$ satisfying (i). If instead i' is defined, then $\{\bar{Q}_1, \dots, \bar{Q}_{k+1}\} \setminus \{\bar{Q}_{i'}\}$ and $(\mathcal{R}' \setminus \{R'_i\}) \cup \{\bar{R}\}$ forms an $(H, \mathcal{P})$-comb of order k violating our choice $\mathcal{Q}'$ and $\mathcal{R}'$ to maximize $\sum_{R \in \mathcal{R}'} |V(R)|$.

We conclude that $z \in P_i$ for some index $i > l'$. In this case, we see that $\{\bar{Q}_1, \dots, \bar{Q}_{k+1}\}$ and $\mathcal{R}' \cup \{z\}$ form an $(H, \mathcal{P})$-comb of order $k+1$ satisfying (i). This completes the proof of the lemma. □

We now give the characterization of when a given path system $\mathcal{P}$ and subgraph H admit a comb.

Lemma 4.2 *Let $\mathcal{P} = \{P_1, \dots, P_l\}$ be a path system of order l for some positive integer l contained in a graph G. Let H be a subgraph of $G - \left(\bigcup_1^l V(P_i)\right)$. For all $k \geq 1$, either there exists an $(H, \mathcal{P})$-comb of order k, or there exists a subset $I \subseteq \{1, \dots, l\}$ and a linkage $\mathcal{R} = \{R_i : i \in I\}$ with $|Ends(\mathcal{R})| < k$ which satisfy the following.*

- (i) *For all $i \in I$, R_i is a subpath of P_i.*
- (ii) *There exists a separation (A, B) such that $V(H) \subseteq B$ and $\left(\bigcup_1^l V(P_i)\right) \setminus Int(\mathcal{R}) \subseteq A$.*

Proof. The proof proceeds by repeatedly applying Lemma 4.1. To see that there exists an $(H, \mathcal{P})$-comb, first note that a single $V(H) - \bigcup_1^l V(P_i)$ path Q with R equal to the single vertex $V(Q) \cap \left(\bigcup_1^l V(P_i)\right)$ forms an $(H, \mathcal{P})$-comb of order 1. If no such comb of order one exists, then there exists a separation (A, B) of order 0 with $\bigcup_1^l V(P_i) \subseteq A$ and $V(H) \subseteq B$ satisfying the statement of the lemma.

Thus, by inductively applying Lemma 4.1, either we find an $(H, \mathcal{P})$-comb of order k, or the process terminates for some $k' < k$, and outcome (ii) of Lemma 4.1 ensures we satisfy (ii) above. □

5 Proof of Theorem 2.1 and a lower bound

The proof of Theorem 2.1 will require the following classic theorem of Mader.

Theorem 5.1 ([7]) *Every graph with minimum degree at least $4k$ contains a k-connected subgraph.*

We will also require some basic observations about k-spread path systems. The proof follows immediately from the definition of k-spread, and we omit it here.

Observation 5.2 *Let S and S' be subgraphs of a graph G, and let X be a set of branch vertices for both S and S'. Let $k \geq 1$ be a positive integer. Assume that the (S, X)-segments are equivalent to the (S', X)-segments. Then (S, X)-segments are k-spread if and only if the (S', X)-segments are k-spread. Let X' be a set of branch vertices of S with $X \subseteq X'$. If the (S, X)-segments are k-spread, then the (S, X')-segments are k-spread as well.*

We now proceed with the proof of Theorem 2.1.

Proof. [**Theorem 2.1**]

Let k be a positive integer and let S be a subgraph of G. Let X be a branch set of S, and let the (S, X)-segments be given by the path system $\mathcal{P} = \{P_1, \dots, P_t\}$. Assume that $\mathcal{P}$ is k-spread.

Let S' be a subgraph of G such that X is a branch set of S and if we let $\mathcal{P}' = \{P'_1, \dots, P'_t\}$ be the path system of (S', X)-segments, then we have the property that $\mathcal{P}$ and $\mathcal{P}'$ are equivalent. Let the endpoints of P'_i be s'_i and t'_i. Furthermore, assume we pick S' such that

(a) for all i, if P_i has length at least two and s'_i is adjacent to t'_i, then there exists j such that P_j is equal to the edge $s'_i t'_i$;

(b) subject to (a), the number of vertices contained in k-stable (S', X)-bridges is maximized, and

(c) subject to (a) and (b), the number of vertices in $|V(S')|$ is minimized.

We begin by observing some immediate implications of our choice of graph S' to satisfy (a)-(c). First, property (a) ensures that we begin by selecting an induced simplification of $\mathcal{P}$.

We claim that no trivial (S', X)-bridge has both endpoints contained in a single (S', X)-segment. Assume otherwise and let xy be an edge of $E(G) \setminus E(S')$ such that both x and y are contained in P'_j for a fixed index j. If we label the endpoints of P'_j as s'_j and t'_j, then we can replace P'_j in S' with the segment $s'_j P'_j x y P'_j t'_j$ to find a subgraph S'' such that the set of (S'', X)-segments is equivalent to the set of (S', X)-segments. By (a), it follows that at least one of x and y is not in $\{s'_j, t'_j\}$. Thus any k-stable (S', X)-bridge that has an attachment as an internal vertex of $x P'_j y$ is contained in an (S'', X)-bridge with an internal vertex of $s'_j P'_j x y P'_j t'_j$ as an attachment. Consequently, any vertex contained in a k-stable (S', X)-bridge is also contained in a k-stable (S'', X)-bridge. Thus, S'' has fewer vertices than S' and satisfies (a) and (b). We violate our choice of (c), proving the claim.

Property (c) implies that if a vertex $v \in V(G) \setminus V(S')$ is not contained in a k-stable (S', X)-bridge, then v has at most $3(k-1)$ neighbors in $V(S')$. Otherwise, v would have at least four neighbors in a single segment and it would be possible to shorten the segment by routing through the vertex v while at the same time not decreasing the number of vertices contained in k-stable bridges.

Lest the theorem be proven, we may assume that there exists some non-trivial (S', X)-bridge which is not k-stable. We will define a larger set of branch vertices for S'. Let

$$Y = \{v \in V(S') : \ v \text{ is an attachment of a } k\text{-stable } (S', X)\text{-bridge}\}.$$

We let

$$\overline{X} = X \cup Y.$$

Let s be a positive integer and let $\overline{\mathcal{P}} = \{\overline{P_1}, \overline{P_2}, \dots, \overline{P_s}\}$ be the path system of $(S', \overline{X})$-segments. By Observation 5.2, the path system $\overline{\mathcal{P}}$ is k-spread. Moreover, by our above claim that no edge of $E(G) \setminus E(S')$ has both endpoints in a single (S', X)-segment, we see that $(S', \overline{X})$ is its own induced simplification, and consequently, there do not exist $k-1$ $(S', \overline{X})$-segments which cover all the vertices of S'.

Let G' be the subgraph of G induced by $V(S')$ as well as the vertices of any (S', X)-bridge that is not k-stable. Let B be a non-trivial (S', X)-bridge in G'. Since B is also a (S', X)-bridge in G, we see that for any vertex v of $V(B) \setminus V(S')$, v has at most $3(k-1)$ neighbors in $V(S')$. It follows from Theorem 5.1 that $G[V(B) \setminus V(S')]$ contains a $20k$-connected subgraph H.

We select an $(H, \overline{P})$-comb given by linkages $\mathcal{Q}$ and $\mathcal{R}$ of order at most $2k$ such that $\mathcal{R}$ is a linkage of order exactly k. We can obtain such a comb by repeatedly applying Lemma 4.1. To see this, first observe that if we were to find the separation (A, B) in outcome (ii) of the lemma, it would follow the separation is trivial by the connectivity of G. Since H has strictly more than $2k$ vertices, we see that $\bigcup_1^s V(P_i)$ is contained in the linkage $\mathcal{R}'$ of outcome (ii). Consequently, there exists a subset $I \subseteq \{1, \dots, s\}$, $|I| < k$ such that $\bigcup_1^s V(P_i) \subseteq \bigcup_{R \in \mathcal{R}'} V(R) \subseteq \bigcup_{i \in I} V(P_i)$, contrary to the fact that $\mathcal{P}$ is k-spread. Thus, in each application of Lemma 4.1, we grow the comb until we find an $(H, \mathcal{P})$-comb $\mathcal{Q}$ and $\mathcal{R}$ with $\mathcal{R}$ of order exactly k.

Let $R = \{R_1, \dots, R_k\}$. We fix values k_1 and k_2 with $0 \leq k_1 \leq k_2 \leq k$ such that after possibly re-indexing the components of $\mathcal{R}$, we have that for $1 \leq i \leq k_1$, R_i is a path of length at least two. For $k_1 < i \leq k_2$, R_i is a path of length one, and for $k_2 < i \leq k$, R_i is a trivial path consisting of a single vertex. Then $\mathcal{Q}$ is a linkage of order $k + k_2$. Let $\mathcal{Q} = \{Q_1, \dots, Q_{k+k_2}\}$. We assume that the endpoints of Q_{2i-1} and Q_{2i} are equal to the ends of R_i for $1 \leq i \leq k_2$, and that the trivial path R_i is the end of Q_i for $2k_2 < i \leq k + k_2$. We let the end of Q_i in $V(H)$ be labeled q_i for $1 \leq i \leq k + k_2$.

By the fact that H is $20k$-connected, we see that H is strongly $2k$-linked by Corollary 2.3. Thus there exists a vertex v disjoint from $Ends(\mathcal{Q}) \cap V(H)$ and path systems $\mathcal{Q}' = \{Q_1', \dots, Q_k\}$ and $\mathcal{R}' = \{R_1', \dots, R_{k_1}\}$ such that the elements of $\mathcal{R}'$ and $\mathcal{Q}'$ are pairwise internally disjoint and satisfy the following. For $1 \leq i \leq k_2$, Q_i' has one endpoint equal to v and one endpoint equal to q_{2i-1}, and for $k_2 < i \leq k$, Q_i' has one endpoint equal to q_{2k_2+1} and one endpoint equal to v. For $1 \leq i \leq k + k_2$ and R_i' has endpoints q_{2i-1} and q_{2i} for $1 \leq i \leq k_1$.

We let $\overline{S}$ be the subgraph of G obtained from S' by deleting the interior vertices of the paths R_i for $1 \leq i \leq k_1$ and adding the paths $Q_{2i-1} \cup q_{2i-1} R_i' q_{2i} \cup Q_{2i}$ for $1 \leq i \leq k_1$. By construction, there exists an $(\overline{S}, \overline{X})$-bridge in G' which has as attachments the vertices q_i for $1 \leq i \leq 2k_1$ as well as the vertices $\bigcup_{k_1 < i \leq k} V(R_i)$, namely the bridge which contains the vertex v we selected in H. Label this bridge B. The bridge B is an $(\overline{S}, X)$-bridge both in the subgraph G' as well as in G. If we look at the branch set X, then the set of $(\overline{S}, X)$-segments are equivalent to the set of (S', X)-segments. We satisfy (a) by the fact that we only re-routed segments of length at least two. Moreover, any vertex of G which is contained in a k-stable (S', X)-bridge is also contained in a k-stable $(\overline{S}, X)$-bridge by the fact that we rerouted the segments to preserve the relative positions of the vertices $\overline{X} \setminus X$.

If B is a k-stable bridge in $(\overline{S}, X)$, then we contradict our choice of S' to satisfy (b). Thus, there exist $k-1$ $(\overline{S}, X)$-segments which contain all the attachments of B. This implies as well that there exist $k-1$ (S', X)-segments which contain every element of the path system $\mathcal{R}$, and consequently, some (S', X)-segment contains two distinct R_i and R_j. Given that R_i and R_j are not contained in a single $(S', \overline{X})$-segment, they must be separated by a vertex of $\overline{X} \setminus X$ on the (S', X)-segment. We conclude that there exists a $(\overline{S}, X)$-segment T and vertex $x \in \overline{X} \setminus X$ contained in T such that B has attachments in both components of the $T - x$. By deleting a subpath T which contains x and adding a subpath through the bridge B, we find a new subgraph S'' such that the set of (S'', X)-segments is equivalent to the set of (S', X)-segments which satisfies (a), satisfies the property that every vertex in a k-stable (S', X)-bridge is in a k-stable (S'', X)-bridge, and in addition, the vertex x is also contained in a k-stable (S'', X)-bridge (specifically the k-stable (S', X)-bridge which attached to x "grows" to include the vertex x). Thus S'' violates our choice of S', a contradiction.

In conclusion, we note that as a byproduct of the proof, if there exists at least one (S, X)-bridge, then the there will exist at least one non-trivial (S', X)-bridge as well. This completes the proof of the theorem. □

It would be interesting to know the best possible connectivity function in Theorem 2.1. The large connectivity function obtained in the proof Theorem 2.1 is a consequence of two factors: first, the linkage property used to analyze the highly connected subgraph H, and second, the large number of paths contained in the comb. It is certainly possible that more careful analysis in the proof or an improved connectivity bound for graph linkages would allow one to improve the overall connectivity function in the statement of Theorem 2.1. However, this approach would still likely fail to approach the best known lower bound for the connectivity function of Theorem 2.1.

Consider two large complete graphs G_1 and G_2. Pick two subsets of $2k-2$ vertices in each and identify them to create a graph G. Let our system of paths in G consist of $k-1$ disjoint edges in the intersection of the two cliques as well as an additional l disjoint edges from G_1 disjoint from the vertices of G_2. It is impossible to reroute the paths so that every non-trivial bridge is k-stable, as the bridge containing the vertices unique to G_2 will only have attachments in the $k-1$ paths of the intersection. This example shows that the best possible connectivity function that could be hoped for in Theorem 2.1 would be $2k-1$. Interestingly, this is the value obtained in the $k=2$ case in Theorem 1.1 of Tutte.

ACKNOWLEDGEMENTS

The author would like to thank Bojan Mohar, Robin Thomas, and Sergey Norine for their helpful discussions, as well as special thanks to Matthias Kriesell for his helpful comments on an early draft of this article. The author would also like to thank the anonymous referees whose feedback greatly improved the article.

References

[1] G. Chen, R. Gould, X. Yu, Graph Connectivity after path removal, *Combinatorica*, **23** (2003), 185 - 203.

[2] R. Diestel, Graph Theory, 3rd ed. Springer-Verlag, 2005.

[3] K. Kawarabayashi, *personal communication.*

[4] M. Kriesell, Induced paths in 5-connected graphs, *J.* of Graph Theory, **36** (2001), 52 - 58.

[5] Q. Liu, D. West, and G. Yu, Implications among linkage properties in graphs, *J. of Graph Theory*, **60** (2009) 327-337.

[6] L. Lovász, Problems in Graph Theory, Recent Advances in Graph Theory, ed. M. Fielder, Acadamia Prague, 1975.

[7] W. Mader, Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend großer Kantendichte, *Abh. Math. Sem. Univ. Hamburg* **37** (1972), 86–97.

[8] W. Mader, Topological minors in graphs of minimum degree n, *DIMACS Series in Discrete Math. and Theoretical Comp. Sci.* **49** (1999) 199 - 211.

[9] R. Thomas and P. Wollan, An improved linear edge bound for graph linkages, *Europ. J. of Combinatorics*, **26** (2005), 309 - 324.

[10] R. Thomas and P. Wollan, The extremal function for 3-linked graphs, *J. Combin. Theory, ser. B* **98** (2008), 939-971.

[11] W. T. Tutte, Graph Theory, Addison-Wesley, Menlo Park, CA, 1984.