Exact (Exponential) Algorithms for NP-hard Problems

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Outline

- Exact Algorithms
- Independent Set Problem
- A Toy-Algorithm
- Memorization
- Folding
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- Other Results
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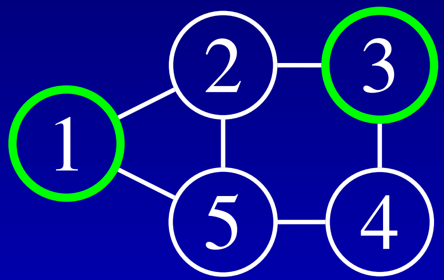
Exact (Exponential) Algorithms

Prb: Designing exact algorithms for NP-hard problems with the smallest possible worst-case running time.

- Need for exact solutions (e.g. decision problems).
- Reducing the running time from, say, 2^n to 1.5^n increases the size of the instances solvable by a constant *multiplicative* factor.
- Classical approaches (heuristics, approximation algorithms, parameterized algorithms...) have limits and drawbacks (no guaranty, hardness of approximation, $W[t]$-completeness...).
- New combinatorial and algorithmic challenges.

Maximum Independent Set

Prb: given an n-node graph $G = (V, E)$, determine the maximum cardinality of a subset of pairwise non-adjacent nodes (independent set).



$$OPT = 2$$

Maximum Independent Set

- NP-hard.
- Hard to approximate within $n^{1-\epsilon}$.
- $W[1]$-complete (no fast parameterized algorithm).
- No exact $c^{o(n)}$ algorithm (unless SNP$\subseteq$SUBEXP).

$\Rightarrow$ The best we can hope for is a 2^{cn} algorithm for some small $c \in (0, 1]$.

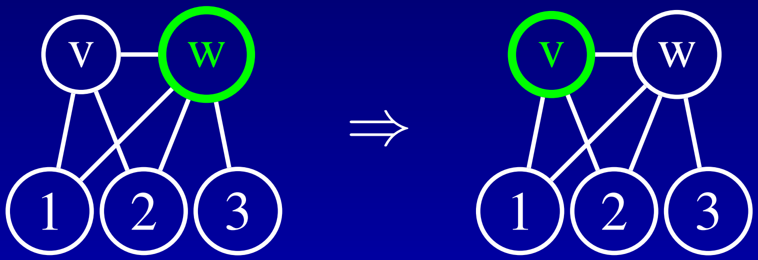
Maximum Independent Set

- [Tarjan&Trojanowski'77]: $\mathbf{O}(2^{0.334n})$ poly-space.
- [Jian'86]: $\mathbf{O}(2^{0.304n})$ poly-space.
- [Robson'86]: $\mathbf{O}(2^{0.296n})$ poly-space, $\mathbf{O}(2^{0.276n})$ exp-space.
- [Fomin,Grandoni&Kratsch'06]: $\mathbf{O}(2^{0.288n})$ poly-space.
- [Beigel'99, Chen,Kanj&Xia'03]: better results for sparse graphs.

Domination

Lem: If there are two nodes v and w such that $N[v] \subseteq N[w]$, there is a maximum independent set which does not contain w ($N[x]{=}N(x){\cup}\{x\}$).

Prf:



A Toy-Algorithm

```
int mis(G) {
    if(|G| ≤ 1) return |G|; //Base case
    if(∃ component C ⊂ G) //Components
        return mis(C)+mis(G − C);
    if(∃ nodes v and w: N[v] ⊆ N[w]) //Domination
        return mis(G − {w});
    //"Greedy" branching
    select a node v of maximum degree; //d(v) ≥ 2
    if(deg(v)=2) return poly-mis(G); //cycles
    return max{mis(G−{v}), 1+mis(G−N[v])};
}
```

A Toy-Algorithm

- The algorithm produces a search tree of exponential size, where branching takes polynomial time.
- Thus the analysis reduces to bounding the number of subproblems generated.
- The bound is obtained by defining a measure of the size of the subproblems. Each branching rule leads to some linear recurrences in the measure, which are used to lower-bound the progress made by the algorithm at each branching step.

A Toy-Algorithm

Lem: Algorithm `mis` runs in time $O(2^{0.465n})$.

Prf:

- Let $P(n)$ be the number of subproblems solved to solve a problem on n nodes. Then

$$P(n) \leq \begin{cases} 1 & \text{base case/cycles;} \\ 1 + P(|C|) + P(n - |C|) & \text{connected components;} \\ 1 + P(n-1) & \text{domination;} \\ 1 + P(n-1) + P(n-4) & \text{branching } (d(v) \geq 3). \end{cases}$$

- The base of the exponential factor is obtained from

$$c^n \geq c^{n-1} + c^{n-4} \Leftrightarrow c^4 - c^3 - 1 \geq 0.$$

Memorization

- The same subproblem may appear several times.
- **Memorization** consists in storing the solutions of the subproblems solved in an (exponential-size) database, which is queried each time a new subproblem is generated.
- This way, no subproblem is solved twice.

Memorization

Thm: Algorithm `mis`, combined with memorization, has running time $O(2^{0.426\,n})$.

Prf:

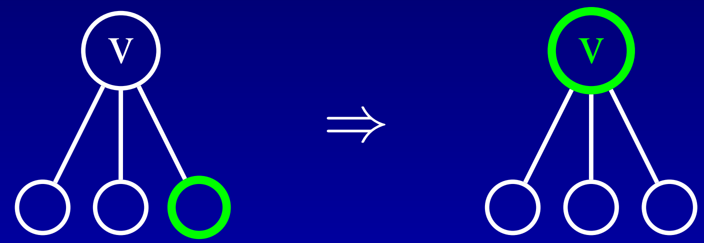
- The subproblems involve induced subgraphs of the original graph. Thus there are at most $\binom{n}{k}$ different subproblems on k nodes.
- From standard analysis, such subproblems are upper bounded also by $2^{0.465(n-k)}$.
- Altogether, using Stirling's formula,

$$P(n) \leq \sum_{k=1}^{n} \min\left\{2^{0.465(n-k)}, \binom{n}{k}\right\} = O(2^{0.426\,n}).$$

Folding

Lem: For every node v, there is a maximum independent set which either contains v or at least two of its neighbors.

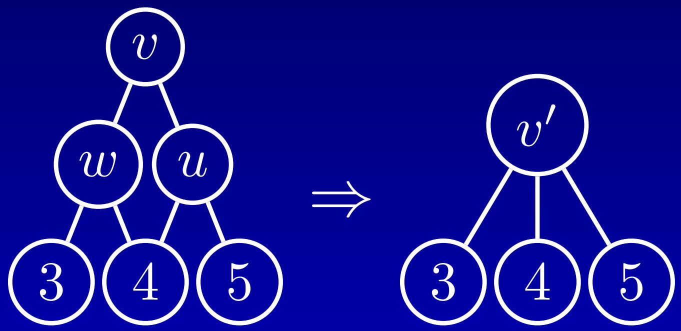
Prf:



Folding

Def: Folding a node v, $N(v) = \{w, u\}$, with w and u not adjacent, means

- replacing v, w, and u with a new node v';
- adding edges between v' and $N(w) \cup N(u) - \{v\}$.



Lem: When we fold a node v, the maximum independent set size decreases exactly by one.

Folding

```
int mis'(G) {
    if(|G| ≤ 1) return |G|; //Base case
    if(∃ component C ⊂ G) //Components
        return mis'(C)+mis'(G − C);
    if(∃ nodes v and w: N[v] ⊆ N[w]) //Domination
        return mis'(G − {w});
    if(∃ v foldable) //Folding
        return 1 + mis'(fold(v, G));
    //"Greedy" branching
    select a node v of maximum degree; //d(v) ≥ 3
    return max{mis'(G−{v}), 1+mis'(G−N[v])};
}
```

Folding

Lem: Algorithm `mis'` runs in time $O(2^{0.406n})$.

Prf: If the algorithm branches at a node of degree 3, then a node of degree 2 is left

$$P(n) \leq \begin{cases} 1 & \text{base case/poly-case;} \\ 1 + P(|C|) + P(n - |C|) & \text{connected components;} \\ 1 + P(n-1) & \text{domination;} \\ 1 + P(n-1) + P(n-5) & \text{branching } (d(v) \geq 4); \\ 1 + P(n-3) + P(n-4) & \text{branching } (d(v) = 3). \end{cases}$$

- Folding is not compatible with memorization.

Measure & Conquer

- Exact recursive algorithms are often very complicated (tedious case analysis).
- But the measure used in their analysis is usually trivial (e.g. number of nodes in IS, as before).

$\Rightarrow$ **Measure & Conquer** approach consists in focusing on the choice of the measure.

Measure & Conquer

- Removing nodes of high degree reduces the degree of many other nodes. This pays of on long term since nodes of degree ≤ 2 can be filtered out without branching.
- This phenomenon is not taken into account with standard analysis/measure.

$\Rightarrow$ The idea is to give a different (smaller) weight to nodes of different (smaller) degree:

$$W(v) = \begin{cases} 0 & \text{if } d(v) \leq 2; \\ \alpha \in (0,1] & \text{if } d(v) = 3; \\ 1 & \text{if } d(v) \geq 4. \end{cases}$$

Measure & Conquer

Thm: Algorithm `mis'` has running time $O(2^{362\,n})$.

Prf:

- First we need to enforce that folding does not increase the size of the problem: $\alpha \geq 0.5$.
- When we branch by discarding a node v, the size of the problem decreases because of: (1) the removal of v, and (2) the decrease of the degree of the neighbors of v.
- When we branch by selecting a node v, the size of the problem decreases because of: (1) the removal of v, and (2) the removal of the neighbors of v.

Measure & Conquer

Prf: Let $P(k)$ be the number of subproblems solved to solve a problem of size $k \leq n$. Then

$$P(k) \leq \begin{cases} 1 + P(k-1) + P(k-6); \\ 1 + P(k-1-\alpha) + P(k-5-\alpha); \\ 1 + P(k-1-2\alpha) + P(k-4-2\alpha); \\ 1 + P(k-1-3\alpha) + P(k-3-3\alpha); \\ 1 + P(k-1-4\alpha) + P(k-2-4\alpha); \\ 1 + P(k-5+4\alpha) + P(k-5); \\ 1 + P(k-4+2\alpha) + P(k-4-\alpha); \\ 1 + P(k-3) + P(k-3-2\alpha); \\ 1 + P(k-2-2\alpha) + P(k-2-3\alpha); \\ 1 + P(k-1-3\alpha) + P(k-1-3\alpha). \end{cases}$$

Measure & Conquer

Prf:

- By solving the recurrences, $P(k)$=$O(c^k)$=$O(c^n)$, where $c = c(\alpha)$ is a quasi-convex function of α [Eppstein'04].
- Imposing $\alpha = 0.6$, one obtains $c < 2^{0.362}$.

Other Results

- Minimum Dominating Set in time $O(2^{0.598\,n})$.
- Maximum Cut in time $O(2^{0.792\,n})$.
- Steiner Tree in time $O(2^{0.773\,n})$.
- Cubic TSP in time $O(2^{0.334\,n})$.
- Chromatic Number in time $O(2.415^n)$.
- 3-Colorability in time $O(1.3289^n)$.
- 3-Satisfiability in time $O(1.4802^n)$.
- Knapsack in time $O(2^{0.5\,n})$.
- ....

Open Problems

- Current best for Hamiltonian Path is poly-space $\Omega(2^n)$.
- Same for TSP, but exp-space.
- Current best for SAT is trivial $\Omega(2^n)$.
- Current best for Feedback Vertex Set is trivial $\Omega(2^n)$.
- ....