Expansion Properties of (Secure) Wireless Networks

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February 6, 2004

Abstract

We show that some topologies arising naturally in the context of wireless networking are low-degree, expander graphs.

1 Introduction

In this paper we show that some network topologies that arise naturally in the context of wireless networking are, with high probability, low-degree expander graphs. The expansion property is highly desirable because it implies useful network properties such as low diameter, high connectivity and resilience against network partitioning that may be caused by node malfunctioning.

To describe our results precisely we make use of the the following widely used and natural setup. A number n of nodes are distributed in some geographical region. Links can be established between any pair of nodes within a given distance r from each other. We shall for simplicity take the geographical region to be the unit square $[0,1]^2$, and also make the probabilistic model assumption that the positions of the n nodes are random: independent and uniformly distributed on $[0,1]^2$.

The first model we analize is the following.

Definition 1 *Fix $r > 0$ and a positive integer c. We take $G^n_{r,c} = (V^n_{r,c}, E^n_{r,c})$ to denote the geometric random graph defined as follows.*

- *The vertex set $V^n_{r,c}$ consists of n points, picked independently according to the uniform distribution on $[0,1]^2$.*
- *Each node $v \in V^n_{r,c}$ connects to c nodes chosen uniformly at random among those within distance r. If the number of such nodes is less than c then v connects to all of them. This is done independently for all nodes v.*

The resulting graph $G^n_{r,c}$ is called the **Bluetooth graph** *with parameters r, c, and n.*

As the name suggests, the motivation here comes from the study of networks adopting the Bluetooth standard, one of the most promising technologies for multihop wireless networking. As shown by Ferraguto et al. [5, 6], $G^n_{r,c}$ approximates very well the graphs generated by certain protocols for the Bluetooth so-called device discovery phase. When a Bluetooth network starts operating each device explores its neighborhood in order to establish reliable communication

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links with devices within transmission range. The goal of device discovery is to set up enough links in order to have a connected network. Since device discovery is both time and energy consuming it is not possible to set up all possible links. The problem then becomes that of finding a local strategy to set up a subset of all possible links in order to have high probability of global connectivity. In [5, 6] a device discovery strategy where each device connects to c neighbors is studied, and the Bluetooth graph of Definition 1 makes this notion mathematically precise. The approach leads to device discovery protocols outperforming all existing solutions (see [5, 6] and references therein).

In [4] the authors confirm the experimental findings of [5, 6] by showing analytically that G^n_{rc} is connected with high probability for all fixed r and $c > 1$. In this paper we continue this line of research showing the stronger property that, with high probability, G^n_{rc} is an expander for all fixed $r > 0$ and $c > c_0$. The expansion property implies connectivity, but since the value of c_0 is considerably higher than 2 the results of [4] are not subsumed. The actual value of c_0 is 10^7 but we emphasize that this choice has been made to keep the proofs as simple as possible. The same arguments allow for better constants, but we have opted for simplicity of presentation.

We remark that the assumption that r is constant is not particularly restrictive from the point of view of the applications.

The expansion property is interesting also in view of the following. A common simulation to be performed when studying wireless networks is to eliminate at random nodes from the network and ask if the network stays connected. It is commonly observed that even low-degree networks such as G^n_{rc} stay connected even when a large fraction of the nodes have been removed. The expansion property explains this.

We should also emphasize that the graphs G^n_{rc} arise naturally in other contexts. Definition 1 is actually a very simple (constant-time!) distributed algorithm that, as shown in [4], generates a very sparse, but connected, overlay network with high probability. From the results of this paper we now know that the overlay network is not only connected, but is an expander. Given the wide applicability of sparse overlay networks, the notion of G^n_{rc} appear to be of interest in a variety of network settings, not limited to wireless communication.

Note that the degree (i.e., the number of links incident to) of a given node v may exceed c, because in addition to the c links created by v, there may be further links resulting from other nodes choosing to connect to v. However, the total number of links cannot exceed cn, and therefore the average degree among the n nodes cannot exceed $2c$. Thus G^n_{rc} is indeed sparse.

The *visibility graph*, denoted as G^n_r, is the graph in which links are established between *all* pairs of nodes within distance r from each other. Since adding edges can only improve expansion, the above result implies that G^n_r too is an expander with high probability. This however can be established directly in a rather straightforward way. And, for trivial reasons, G^n_r is not an expander when $r = o(1)$.

We believe that our proofs extend to the case G^n_{rp} (for $p \sim \log n/n$ and possibly smaller values). Here, two vertices within distance r are connected with probability p.

We shall consider one more model, which has been proposed as an approach to generate cryptographically secure wireless networks [8].

Definition 2 *Let K be a finite set of keys, and let $k \leq K$ be a fixed parameter. Let $[K] = \{1, 2, \ldots, K\}$. The graph $G^N_{r,k,K}$ is defined as the geometric random graph obtained by the following procedure.*

- *First, each node u is assigned a subset $S_u \subseteq [K]$ by sampling $[K]$ with replacement k times.*

- *Second, the N nodes are distributed uniformly at random on the unit square.*
- *Third, uv is an edge if (a) the two nodes are within distance r* and *(b) $S_u \cap S_v \neq \emptyset$.*

The resulting graph $G^N_{r,k,K}$ is called a **Kryptograph** *with parameters r, k, K and N.*

The motivation behind the model is that two nodes can communicate securely if and only if they are within distance r and moreover they share a key. Asking for connectivity of the resulting graph is equivalent to asking if any two nodes in the network can communicate securely. In [3] it is shown that these graphs are connected with high probability, provided that $k \geq 4$ and $K = n/\log n$. In setting up these parameters the resource to keep in mind is k, while K can be set quite freely. In this paper we show that these graphs are expanders with high probability, provided that $k > 10\log n$. Like those above, this parameter choice is realistic and quite useful for the applications.

2 Preliminaries

We recall some basic facts and definitions for later use:

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

and

$$\binom{n}{k} \leq 2^n.$$

Proposition 3 [Chernoff-Hoeffding Bounds] *Let $X := \sum_{i=1}^n X_i$ be the sum of identically distributed, binary random variables X_i such that* $\Pr[X_i = 1] = p$ *for all i. Then,*

$$\Pr[|X - pn| \geq \delta n] \leq 2\exp(-2\delta^2 n).$$

Definition 4 *The graph G is an ϵ-*expander *if there exists an $\epsilon > 0$ such that for all $S \subseteq V(G)$, with $|S| \leq \frac{|V(G)}{2}$,*

$$|\Gamma(S) - S| \geq \epsilon|S|.$$

3 Expansion of Bluetooth Graphs

Our goal in this section is to show the following theorem about Bluetooth Graphs. Let N denote the total number of points.

Theorem 5 *If $c \geq 10^7$, then with probability $1 - o(1)$, the graph G^N_{rc} is an ϵ-expander, where $\epsilon = (r^2/(5 \cdot 10^5))$.*

To prove this theorem, we need to show that a subset S of at most $N/2$ nodes has at least $\epsilon|S|$ neighbors outside S. We do this in three steps, each of which helps us establish some local expansion properties for our graph. We then put these together to argue that G^N_{rc} is an ϵ-expander.

Step 1: Suppose N nodes are placed at random in the unit square uniformly and independently. We expect regions with the same area to receive similar number of nodes. To formalize this, we subdivide the unit square into m^2 smaller axis-aligned squares, called *cells,* each of dimension $\frac{1}{m} \times \frac{1}{m}$. We choose m to be the smallest integer so that every two nodes in adjacent (i.e., sharing a common edge) cells are within distance r; that is we set $m = \lceil \sqrt{5}/r \rceil$. Note that since r is a constant, m is also a constant. Let N_C be the number of nodes in cell C. The expectation of N_C is $n := \frac{N}{m^2}$. Lemma 6 below shows that with probability $1 - o(1)$ each N_C is in the range $[\alpha n, \beta n]$, where $\alpha := 0.9$ and $\beta := 1.1$.

Step 2: For a set S of nodes, let $\Gamma(S)$ denote the set of nodes that are connected to some node in S by an edge. We refer to a the set $S \cap C$ as a *pocket* of S contained in the cell C. We will classify pockets into three kinds: small, medium and large. A *small* pocket has less than $10^{-4}\beta n$ nodes; a *medium* pocket has at least $10^{-4}\beta n$ and at most $\frac{\beta n}{2}$ nodes; a *large pocket* has more than $\frac{\beta n}{2}$ nodes.

We will show in Lemma 7 that with probability $1 - o(1)$, every medium size pocket P contained in a single cell C, expands into every adjacent cell and in its own cell. More precisely, for each cell C' that is adjacent to the cell C we have

$$|\Gamma(P) \cap C'| \geq 1.1|P| \tag{3.1}$$

The claim also holds with $C' = C$. This step is enough to establish expansion as soon as S has one medium or one large pocket.

Step 3: Next consider small pockets, that is, sets P of at most $10^{-4}\beta n$ nodes all contained in a single cell. Lemma 8 below shows that with probability $1 - o(1)$

$$|\Gamma(P)| \geq 200|P|. \tag{3.2}$$

We will justify these steps below. First, let us now verify that G_{rc}^N is an expander with high probability, assuming that that the steps above are correct, and the favorable events mentioned there hold with probability $1 - o(1)$.

Proof of Theorem 5. By Step 1, we may assume that each $N_C \in [\alpha n, \beta n]$. Fix a set S of at most $N/2$ nodes. We have three cases.

First, suppose there is a large pocket (that is, there is a cell C such that $|S \cap C| > \beta n/2$). Since $|S| < \frac{N}{2}$, all pockets cannot be large, and there must be two adjacent cells C and C', such that $|S \cap C| > \beta n/2$ and $|S \cap C'| \leq \beta n/2$. Now, consider a subset P of $S \cap C$ of size $\beta n/2$. By Equation 3.1, we see that

$$|\Gamma(P) \cap C'| \geq 1.1\frac{\beta n}{2}.$$

Since $|S \cap C'| \leq \beta n/2$, this implies that

$$|\Gamma(S) - S| \;\geq\; \frac{\beta n}{20} \;\geq\; \frac{\beta}{10m^2}\frac{N}{2} \geq \frac{\beta}{10m^2}|S|$$

since $|S| \leq \frac{N}{2}$.

Next, suppose there is a medium pocket P. Again, by Equation 3.1, P expands in its own cell C. It follows that

$$|\Gamma(S) - S| \;\geq\; |(\Gamma(P) \cap C) - P| \;\geq\; \frac{1}{10}|P| \geq 10^{-5}\beta n \;=\; \left(\frac{\beta}{10^5 m^2}\right) N.$$

It follows that $|\Gamma(S) - S| \geq (2\beta/10^5 m^2)|S|$, since $|S| \leq N/2$.

Finally, assume that all the pockets of S are small. We need some notation. Assume a coordinate system so that a cell can be denoted as C_{ij} $(0 \leq i, j < m)$. For instance, C_{00} is the leftmost, bottom most cell. Let S be any set of cardinality at most $N/2$, and let

$$S_{ij} := S \cap C_{ij}.$$

The *sector* of a given S_{ij}, denoted as $\mathcal{S}_{ij}$ is defined as,

$$\mathcal{S}_{ij} := \bigcup_{\substack{i-6 \leq k \leq i+6 \\ j-6 \leq \ell \leq j+6}} S_{k\ell}.$$

The set S_{ij} is called the *center* of its sector. With this notation, consider the following greedy procedure: Repeatedly select the largest pocket, remove the entire sector around it, and continue until there are no pockets left. Let I denote the index set of the pockets (and sectors) so selected.

By Equation 3.2, each center of every sector expands at least 200 times. The greedy procedure ensures that the expansion regions of the selected centers do not overlap. Each sector contains at most 169 cells. In particular, if $ij \in I$, then total number of nodes in a sector $\mathcal{S}_{ij}$ is at most $169|S_{ij}|$. Thus,

$$\begin{aligned} |\Gamma(S) - S| &\geq \sum_{ij \in I} |\Gamma(S_{ij}) - \mathcal{S}_{ij}| \\ &\geq \sum_{ij \in I} 31|S_{ij}| \\ &\geq \frac{31}{169}|S|. \end{aligned}$$

For the last inequality we used the fact that $|\mathcal{S}_{ij}| \leq 168|S_{ij}|$. Therefore, S expands non-trivially in this case as well. This completes the proof of our theorem. □

We now show the correctness of Steps 1, 2 and 3 described above.

In Step 1, let N_C denote the number of nodes that fall in cell C. N_C is a random variable with expectation. We first observe that with high probability, for all C, N_C is close to its expectation $n := N/m^2$.

Lemma 6

$$\Pr[\exists C \ |N_C - n| \geq 0.1n] \leq 2m^2 \exp\left\{-\frac{N}{50m^4}\right\}.$$

Proof We use Proposition 3. By taking $p := \frac{1}{m^2}$ and $\delta := \frac{1}{10m^2}$, we see that the probability that N_C is not within $\pm(\frac{1}{10m^2})N$ of its expectation is at most $2\exp(-(\frac{1}{50m^4})N)$. Since there are at most m^2 cells in all, the claim follows immediately. □

Let us turn to Step 2.

Lemma 7 *Suppose $N_C \in [\alpha n, \beta n]$, for each cell C. Let P be a medium pocket, that is a set such that $10^{-4}n \leq |P| \leq \beta n/2$, and let $c \geq 10^7$. Then, with probability $1 - o(1)$, for each set P at most $\frac{\beta n}{2}$ in a cell C, and each cell C' adjacent to C (including C),*

$$|\Gamma(P) \cap C'| \geq 1.1|P|.$$

Proof Let $p := |P|$. If P does not have the required number of neighbors then its neighbors are confined to a subset of size less than $1.1p$. A node in P can connect to nodes in at most 24 other cells and thus chooses c neighbors from a pool of at most $24\beta n$ vertices. The probability that no vertex in a given cell is chosen outside the set of size $1.1p$ is at most

$$\begin{aligned} \left(1 - \frac{\alpha n - 1.1p}{24\beta n}\right)^{cp} &\leq \left(1 - \frac{1}{100}\right)^{cp} \\ &\leq \exp(-cs/100). \end{aligned}$$

There are no more than $2^{\beta N}$ many ways of choosing P, and no more than $2^{\beta N}$ many ways of choosing a neighbor set of size $1.1p$. Therefore, recalling that $p \geq 10^{-4}\beta n$, the total probability of failure is at most,

$$2^{2\beta N} \cdot \exp(-10^{-6} cN) = e^{-\Omega(N)},$$

for $c \geq 10^7$. Since, there are at most $5m^2$ possible pairs (C, C'), the total probability of failure goes to 0 as N becomes large. □

Finally, we justify Step 3, by showing that small pockets expand.

Lemma 8 *Suppose that $N_C \in [\alpha n, \beta n]$ for every cell C, and let $c \geq 800$. Then, with probability $1 - o(1)$, every set of P of at most $10^{-4}\beta n$ nodes contained in a single cell has more than $200|P|$ neighbors.*

Proof Fix a cell C and a set of P of at most $10^{-4}\beta n$ nodes in C. Let $p := |P|$.

Fix a set T of $200p$ nodes. Now, the probability that a given neighbor of P falls in T is at most $\frac{200p}{5\alpha n}$, because there are at least 5 other cells (each with at least αn nodes) that each point in S sees. Thus,

$$\Pr[\Gamma(P) \subseteq T] \leq \left(\frac{200p}{5\alpha n}\right)^{pc}.$$

The number of choices for P is at most

$$\binom{\beta n}{p} \leq \binom{2n}{p}$$

and the number of ways to choose T vertices is at most

$$\binom{25\beta n}{200p} \leq \binom{30n}{200p}.$$

Then, the probability that some small pocket P does not expand sufficiently is at most,

$$\begin{aligned} \sum_{p=1}^{10^{-4}n} \binom{2n}{p}\binom{30n}{200p}\left(\frac{200p}{4.5n}\right)^{pc} &\leq \sum_{p=1}^{10^{-4}n} \binom{30n}{200p}^2 \left(\frac{400p}{9n}\right)^{pc} \\ &\leq \sum_{p=1}^{10^{-4}n} \left(\frac{30en}{200p}\right)^{400p} \left(\frac{400p}{9n}\right)^{800p} \\ &\leq \sum_{p=1}^{10^{-4}n} \left(\frac{30e}{200}\right)^{400p} \left(\frac{400}{9}\right)^{800p} \left(\frac{p}{n}\right)^{400p} \\ &\leq \sum_{p=1}^{10^{-4}n} \left(\frac{1000p}{n}\right)^{400p}. \end{aligned}$$

The successive terms of the last summation decrease exponentially. To see this, consider the function $f(p) := \ln(\frac{1000p}{n})^p = p\ln(\frac{1000p}{n})$. Then, the derivative of f with respect to p is

$$\ln(\frac{1000p}{n}) + 1 < -1,$$

for $p \leq 10^{-4}n$. So, $f(p+1) \leq f(p) - 1$ and $e^{200f(p+1)} \leq e^{f(p)-200}$.

Thus,

$$\sum_{p=1}^{10^{-4}n} \left(\frac{1000p}{n}\right)^{400p} \leq 2\left(\frac{1000}{n}\right)^{400}.$$

Since there are at most m^2 choices for the cell C, our claim follows from this. □

4 Expansion in Kryptographs

In this section we show that the graphs of Definition 2 are expanders, for an appropriate choice of the parameters. Recall, that in the graph $G^N_{r,k,K}$ each node picks a random subset of $[K]$ of size k, and two vertices are connected by an edge if they are within distance r of each other and their sets intersect. We will be interested in the situation when k is much smaller than K; then, the probabilty that two vertices within distance r are connected by an edge is approximately $\frac{k^2}{K}$.

Theorem 9 *If $k \geq 10\log n$ and $\frac{k^2}{K} \geq 1000\frac{\log n}{n}$, then with probability $1 - o(1)$, $G^N_{r,k,K}$ is an ϵ-expander with $\epsilon = \frac{r^2}{100}$.*

As in the previous section, we subdivide the unit square into m^2 cells where $m = \lceil\sqrt{5}/r\rceil$. This ensures that two nodes in adjacent cells are at most a distance r apart. Recall, that $n = N/m^2$ is the expected number of nodes in each cell. Furthermore, by Lemma 6 with very high probability (exponentially close to 1), each cell has between $0.9n$ and $1.1n$ vertices.

Lemma 10 *With probability $1 - \frac{1}{n}$ for each set S of at most $1.1n/2$ nodes in a cell C and for all neigboring cells $C'(\neq C)$, we have the following:*

- *the number of neighbors of S in C that lie outside S is at least $0.1 \cdot |S|$;*
- *the number of neighbors of S in C' is at least $1.1 \cdot |S|$.*

Proof For a node v, let K_v denote the set of keys selected for v, and for a set S, let

$$K_S := \bigcup_{v \in S} K_v.$$

.

Consider a set S of size at most $1.1n/2$. We have two cases, roughly corresponding to having small S and large S. Let $s := |S|$. First, assume that $sk \leq K$. In this case, one expects that K_S will not be much smaller than sk. Indeed,

$$\begin{aligned} \Pr[|K_S| < sk/10] &\leq \binom{K}{sk/10}\left(\frac{sk}{10K}\right)^{sk} \\ &\leq \left(\frac{eK10}{sk}\right)^{sk/10}\left(\frac{sk}{10K}\right)^{sk} \end{aligned}$$

$$
\begin{aligned}
&\le e^{sk/10}\left(\frac{sk}{10K}\right)^{9sk/10} \\
&\le \left(\frac{sk}{10K}\right)^{4sk/5} \\
&\le n^{-5s}.
\end{aligned}
$$

There are at most m^2 cells, and at most $(1.1n)^s$ such sets S in each. We conclude that with probability at least $1-\frac{1}{n^2}$ for all sets of size $s \le K/k$, $|K_S|$ is at least $sk/10$. Now we will use this to show that S has at least $1.1s$ neighbors in a neighboring cell C' and at least $0.1s$ neighbors outside S in C. Now, if S does not expand sufficiently in C', then all its neighbors are confined to a subset of size $1.1s/2 \le 0.605n$. Since C' has at least $0.9n$ nodes, there is a set R' of $n/4$ nodes in C' that has no neighbor in S. Similarly, if S fails to have $0.1n$ neighbors in C that lie outside S, then there is a subset R of $n/4$ nodes in C that has no neighbor in S.

We now compute the probability that nodes in R fail to have a neighbor in S, by computing the probability of the event $K_S \cap K_R = \emptyset$.

$$
\begin{aligned}
\Pr[K_S \cap K_R = \emptyset] &\le \left(1-\frac{sk}{10K}\right)^{kn/4} \\
&\le \exp(-10s\log n) \\
&\le n^{-10s}.
\end{aligned}
$$

There are at most $(1.1n)^{(2+\epsilon)s}$ choices for (S,R), at most 5 cells neighboring C, and at most m^2 cells C. So, the probability that some S does not expand sufficiently is at most $\frac{1}{n^2}$.

Next, consider the case when $s \ge K/k$. First observe that with high probability K_S contains a large fraction of $[K]$:

$$
\Pr[|K_S| \le \frac{K}{10}] \le \left(\frac{1}{10}\right)^{sk} \le n^{-10s}.
$$

Again, there are only m^2 cells and at most $(1.1n)^s$ set S in each cell.So, with probability $1-\frac{1}{n^2}$, we have that $|K_S| \ge \frac{K}{10}$ for all S of size at least K/k.

As in the case above, we now compute the probability that some set R of size at least $n/4$ fails to have a neighbor in S:

$$
\left(\frac{1}{10}\right)^{kn/4} \le n^{-n}.
$$

Since there are at most $4^{1.1n}$ pairs (S,R) and at most five cells adjacent to C and m^2 cells in all, the probability that some S does not expand as claimed in the lemma is at most $\frac{1}{n^2}$.

□

We can now turn to the proof of Theorem 9. Lemma 10 implies that $G^N_{r,k,K}$ is an expander. For, if $|S \cap C| \le \beta n/2$ for all cells C, then by the first part of Lemma 10, $|(\Gamma(S)-S)\cap C| \ge 0.1|S \cap C|$ and clearly $G^N_{r,k,K}$ is an (0.1)-expander. Otherwise, there must be two adjacent cells C and C' such that $|S\cap C| \ge 1.1n/2$, $|S \cap C'| \le 1.1n/2$. But then, $S \cap C$ has at least $1.1n/2$ neighbors in C'. It is then easy to see that $G^N_{r,k,K}$ is a $(0.1 \times 1.1n/(2m^2))$-expander. This establishes our theorem.

References

[1] S. Asmussen and H. Hering, *Branching Processes*, Birkhäuser, Boston, 1983.

[2] S. Banerjee, C. Kommareddy, K. Kar, B. Bhattacharjee and S. Khuller, *Construction of an efficient overlay multicast infrastructure for real-time applications*, Proceedings of Infocom 2003

[3] R. Di Pietro, A. Mei, A. Panconesi and J. Radhakrishnan, *Connectivity properties of secure wireless networks.* Manuscript.

[4] D. Dubhashi, O. Häggström and A. Panconesi, *Connectivity properties of Bluetooth wireless networks.* Manuscript, available at `www.dsi.uniroma1.it/ ale`

[5] F. Ferraguto, G. Mambrini, A. Panconesi and C. Petrioli, *Blue Pleiades, a new solution for device discovery and scatternet formation in multi-hop Bluetooth networks*, submitted to ACM Winet. Available at: `www.di.uniroma1.it/ale`

[6] F. Ferraguto, G. Mambrini, A. Panconesi and C. Petrioli, F. Ferraguto, G. Mambrini, A. Panconesi, C. Petrioli *A new approach to device discovery and scatternet formation in Bluetooth Networks.* To appear in the Proceedings of the International Workshop on Wireless, Mobile, and Ad Hoc Networks (WMAN04, an IEEE IPDPS 2004 workshop), Santa Fe, aprile 2003.

[7] Feng Xue and P. R. Kumar, *The number of neighbors needed for connectivity of wireless networks.* To appear in Wireless Networks.

[8] Laurent Eschenauer, and Virgil D. Gligor, *A key-management scheme for distributed sensor networks.* ACM Conference on Computer and Communications Security 2002: 41-47

[9] O. Häggström and R. Meester, Nearest neighbour and hard sphere models in continuum percolation, *Random Structures and Algorithms*, 9:3, pp. 295–315, 1996.

[10] T.E. Harris, *The Theory of Branching Processes*, Springer, Berlin, 1963.

[11] J.Könemann *Approximation Algorithms for Minimum-Cost Low-Degree Subgraphs.* PhD Thesis in Algorithms, Combinatorics, and Optimization, Carnegie Mellon University, 2003.

[12] R. Meester, M. Penrose and A. Sarkar, "The random connection model in high dimensions", *Statistics and Probability Letters*, 35:2, pp. 145–153, 1997.

[13] R. Meester and R. Roy, *Continuum Percolation*, Cambridge University Press, 1996.

[14] M. Penrose, "On the spread-out limit for bond and continuum percolation", *Annals of Applied Probability* 3:1, pp. 253–276, 1993.