

Minimum Energy Broadcast and Disk Cover in Grid Wireless Networks ^{*}

(Extended Abstract)

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Abstract. The *Minimum Energy Broadcast* problem consists in finding the minimum-energy range assignment for a given set S of n stations of an ad hoc wireless network that allows a source station to perform broadcast operations over S .

We prove a nearly tight asymptotical bound on the optimal cost for the Minimum Energy Broadcast problem on square grids. We emphasize that finding tight bounds for this problem restriction is far to be easy: it involves the *Gauss’s Circle* problem and the *Apollonian Circle Packing*. We also derive near-tight bounds for the *Bounded-Hop* version of this problem. Our results imply that the best-known heuristic, the MST-based one, for the Minimum Energy Broadcast problem is far to achieve optimal solutions (even) on very regular, well-spread instances: its worst-case approximation ratio is about π and it yields $\Omega(\sqrt{n})$ hops.

As a by product, we get nearly tight bounds for the *Minimum Disk Cover* problem and for its restriction in which the allowed disks must have *non-constant* radius.

Finally, we emphasize that our upper bounds are obtained via polynomial time constructions.

1 Introduction

An *ad-hoc* wireless network consists of a set S of radio stations connected by wireless links. We assume that stations are located on the Euclidean plane. A transmission range is assigned to every station: a *range assignment* $r : S \rightarrow R$ determines a directed *communication graph* $G(S, E)$ where edge $(i, j) \in E$ if and only if $\text{dist}(i, j) \leq r(i)$ where $\text{dist}(i, j)$ is the Euclidean distance between i and j . In other words, $(i, j) \in E$ if and only if j belongs to the *disk* of radius $r(i)$ centered at i . The transmission range of a station depends on the energy power supplied to the station. In particular, the power P_s required by a station s to

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transmit data to another station t must satisfy the inequality

$$\frac{P_s}{\text{dist}(s, t)^\alpha} \geq 1$$

where $\alpha \geq 1$ is the *distance-power gradient*. In the empty space, $\alpha = 2$ (see [20]): this is the case considered in this paper.

Stations of an ad-hoc network cooperate in order to provide specific network connectivity properties by adapting their transmission ranges. A *Broadcast Range Assignment* (for short *Broadcast*) is a range assignment that yields a communication graph G containing a directed spanning tree rooted at a given source station s . A fundamental problem in the design of ad-hoc wireless networks is the *Minimum Energy Broadcast Problem* (for short *Minimum Broadcast*): it consists in finding a Broadcast of minimal *overall energy power* [7, 10, 18]. A range assignment r can be represented by the corresponding family $\mathcal{D} = \{D_1, \dots, D_\ell\}$ of disks, and its overall energy power (i.e. $\text{cost}(\mathcal{D})$) is defined as

$$\text{cost}(\mathcal{D}) = \sum_{i=1}^{\ell} r_i^2 \quad \text{where } r_i \text{ is the radius of } D_i \quad (1)$$

The Minimum Broadcast problem is known to be NP-hard [5] and the best-known approximation algorithm is the MST-based heuristic [1, 10]. The MST-based heuristic computes the minimum spanning tree of the complete graph induced by S , then, it assigns a direction to the edges from the source s to the leaves; finally, it assigns to each node i a range equal to the length of the longest edge outgoing from i . This heuristic is efficient and easy to implement, so, its worst-case approximation analysis has been the subject of several works over the last five years. In particular, the first *constant* upper bound ($\simeq 40$) on the approximation ratio was determined in [5]. A rather sophisticated analysis, recently introduced in [1], yields the *tight* upper bound 6. The tightness follows from the lower bound proved in [4, 10] by considering *unlike* input configurations. The worst-case analysis is often not sufficient to evaluate the practical interest of a heuristic. It might be the case that the MST-based heuristic provides *nearly* optimal solutions *for most* of natural and practically-relevant instances. Recently, experimental studies have been presented on this issue [11, 6, 10].

1.1 Our results

Minimum Broadcast Problem. In this paper, we address the above issue by adopting an analytical approach: we consider Minimum Broadcast and some other related problems on *square grids*. Square grids have been often considered in wireless networks since they well-model some *well-spread*, practically relevant ad-hoc network topologies [8, 19, 20]. One can see that the MST-based heuristic, on a square *grid* of n points (without loss of generality, adjacent points are placed at unit distance), returns, *in the worst-case*, a solution of cost $n - 1$. On the other hand, what is the optimal cost on the square grids? One may think that

determining this cost is an easy task for so simple instances. On the contrary, this is far to be true: as we will see later, this analysis involves the well-known mathematical *Gauss' Circle problem* [15, 17] and the *Apollonian Circle Packing* [13, 21]. Our first contribution is the following result.

Theorem [Broadcast]. *If \mathcal{B}^* is any optimal Broadcast for the square grid \mathcal{G} of n points, then*

$$\frac{n}{\pi} - O(\sqrt{n}) \leq \text{cost}(\mathcal{B}^*) \leq 1.01013 \frac{n}{\pi} + O(\sqrt{n})$$

The upper bound is achieved via a polynomial time construction.

The above upper bound implies that the MST-based heuristic yields, in the worst-case, a solution cost which is about π times larger than the optimum.

Minimum Cover Problem. Any Broadcast yields a (*disk*) *cover* of the grid and a communication graph that contains a spanning tree. A cover \mathcal{C} of a set S of points is a set of disks $\mathcal{C} = \{D_1, \dots, D_\ell\}$ of radius at least 1, centered at some points of S , that covers all points in S . The cost of \mathcal{C} is defined as $\text{cost}(\mathcal{C})$ (see Eq. 1). The *Minimum Cover* problem consists in finding a cover for S of minimum cost. Observe that this is a variant of the well-known NP-hard *Minimum Geometric Disk Cover* [9, 16].

In general, a cover does not suffice to provide a feasible solution for the Minimum Broadcast problem. A natural question here is whether (or when) the minimum cover cost is asymptotically equivalent to the minimum broadcast cost. This question is formally addressed by determining the cost of a minimum cover for square grids.

Theorem [Cover]. *If \mathcal{C}^* is any optimal cover of the square grid \mathcal{G} of n points, then*

$$n/5 \leq \text{cost}(\mathcal{C}^*) \leq n/5 + O(\sqrt{n})$$

The upper bound is achieved via a polynomial time construction.

From the above theorems, it turns out that the cost of the cover is significantly lower than the cost of the broadcast. However, next theorem shows that this is not the case when we require that the disks are sufficiently large.

Theorem [Large Disk Cover] *Let $f(n) = \omega(1)$. The cost of any cover of \mathcal{G} with disks of radius at least $f(n)$ is at least $\frac{n}{\pi} - o(n)$. The upper bound is achieved via a polynomial time construction.*

We emphasize that there are important network scenarios in which the *installing* cost (i.e. the cost of installing an omni-directional transmitter at a given location) is rather high and it must be "amortized" by a *relevant* use of the antenna. In such cases, it is convenient to assign positive range to a station only if such a range (so, disk) is large enough.

Bounded-Hop Broadcast. An important version of the Minimum Broadcast problem is the one in which feasible solutions must guarantee a *bounded number*

of hops: The number of links (i.e. *hops*) in the path from the source to *any* other node must be not larger than a fixed bound. This problem version is relevant since the number of hops is closely related to the delay transmission time. The hop restriction finds another application in the context of *reliability*: Assume that, in a communication network, link faults happen with probability p and that all faults occur independently. Then, the probability that a multi-hop transmission fails exponentially increases with the number of hops. For further motivations in studying bounded hops communication see [2, 12, 14, 22].

A main question here is the following: Does broadcasting with a bounded number of hops require a *significantly* larger cost than broadcasting with an unbounded number of hops? Intuitively speaking, one might figure out that the right answer is the positive one since the cost is proportional to the area of the solution disks and bounded-hop solutions require larger disks. Observe also that the use of *large* disks yields *large* disk overlapping. Rather surprisingly, this is not the case: we derive a broadcast for grids that uses only a constant (i.e. not depending on n) number of disks and thus yields a *constant* number of hops. This solution has a cost which is very close to that of the unbounded-hops version.

Theorem [Broadcast with few Hops]. *A positive constant c exists such that it is possible to construct in polynomial time a broadcast \mathcal{B} for \mathcal{G} with (only) c disks (of radius $\Omega(\sqrt{n})$) and such that*

$$\text{cost}(\mathcal{B}) < 1.1171 \frac{n}{\pi} + O(\sqrt{n}).$$

By comparing the above theorem with Theorem [Large Disks Cover], we can state that covering and broadcasting over grids have almost asymptotically-equivalent cost when the solution disks have *non-constant* radius (remind that any broadcast is also a cover). We also remark that the MST-based heuristic *always* returns a solution for the grid that has an *unbounded* (i.e. $\Omega(\sqrt{n})$) number of hops. So, our almost optimal polynomial-time construction yields bounded-hop solutions whose structure significantly departs from that of the MST-based solutions.

Square grids are thus the first family of well-spread, natural instances that perfectly capture the "hardness" of solving the Minimum Broadcast problem via the MST-based heuristic. It is our opinion that the set of results presented in this paper provides strong theoretical arguments that open new possibilities in the design of an efficient heuristic that significantly improves over the MST-based one (at least) in the case of *well-spread* and *uniform-random* instances.

1.2 Preliminaries

We consider a Cartesian coordinates system and a square grid \mathcal{G} of side length $m - 1$ with its bottom left vertex in the origin. \mathcal{G} contains $n = m^2$ points at integer coordinates; the coordinates of point P of the grid will be denoted as x_p and y_p . A \mathcal{G} -*disk* D is a disk centered at any point of the grid and having at least one point of the grid on its boundary. We also denote as D the set of points of grid \mathcal{G} covered by D .

2 The Minimum Cover Problem on the Grid

In this section we study two versions of the disk cover problem of the grid \mathcal{G} . In the first version, we consider coverings by disks of arbitrary radius, while, in the second one, disks are required to have a minimal non constant radius. For both versions, we need to evaluate the number $N(r)$ of points of the *infinite* grid covered by a \mathcal{G} -disk of radius r . This problem, known as *Gauss' Circle problem*, has been extensively studied [15, 17] in order to derive the best exponent $\delta < 1$ such that $N(r) \leq \pi r^2 + cr^\delta$ for some constant c . However, all these studies are not useful to provide a good bound on c : instead, we need an upper bound on $N(r)$ with a small constant c while the exponent δ can be 1. The proof of next lemma is given in the full version of the paper [3].

Lemma 1. *For any radius $r \geq 1$, it holds that $N(r) < \pi r^2 + (\pi\sqrt{2} - 2)r + \frac{1}{5}\sqrt{r} + \frac{\pi}{2}$. Moreover, for $r > \sqrt{10}$, it holds that $N(r) < \pi r^2 + 2\sqrt{2}r - 5$.*

The above lemma is now exploited to prove asymptotically tight lower and upper bounds on the minimum cost of a cover of grid \mathcal{G} .

Theorem 1. *If \mathcal{C}^* is any minimum cover of the square grid \mathcal{G} of n points, then*

$$n/5 \leq \text{cost}(\mathcal{C}^*) \leq n/5 + O(\sqrt{n})$$

The upper bound is achieved via a polynomial time construction.

Proof. We first observe that, for any $r > 0$, it holds that

$$N(r) \leq 5r^2. \tag{2}$$

Indeed, $N(1) = 5$, $N(\sqrt{2}) = 9$, and Lemma 1 implies that $N(r) \leq 5r^2$, for any $r \geq 2$. Let D_1, D_2, \dots, D_t be the \mathcal{G} -disks of an optimal cover and let cost^* be its cost. Let r_i be the radius of D_i , $1 \leq i \leq t$. Since D_i covers $N(r_i)$ points, Inequality (2) implies that

$$n \leq \sum_{i=1}^t N(r_i) \leq \sum_{i=1}^t 5r_i^2 = 5 \cdot \text{cost}^*$$

and so $\text{cost}^* \geq \frac{n}{5}$.

A cover of \mathcal{G} with cost $\frac{n}{5} + O(\sqrt{n})$ is shown in Figure 1 for $m = 11$. It is easy to see that the number of grey \mathcal{G} -disks (i.e. disks not completely contained in \mathcal{G}) is $O(\sqrt{n})$, and the number of white \mathcal{G} -disks (i.e. disks completely contained in \mathcal{G}) is not greater than $\frac{n}{5}$. Since all \mathcal{G} -disks have unit radius, then the cost $\frac{n}{5} + O(\sqrt{n})$ follows. It is easy to check that the above construction can be computed in time polynomial in n . \square

The cover resulting by the construction in Theorem 1 uses only \mathcal{G} -disks of unit radius. Next theorem investigates the cost of covers using only \mathcal{G} -disks of large, non constant radius.

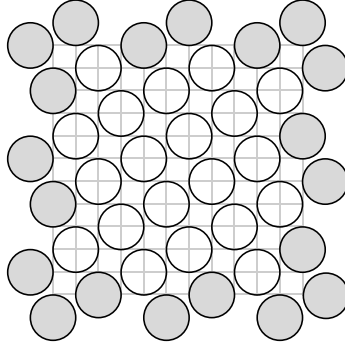


Fig. 1. An asymptotically optimum disk cover for \mathcal{G} with $m = 11$.

Theorem 2. Let $f(n) = \omega(1)$. The cost of any cover of \mathcal{G} with \mathcal{G} -disks of radius at least $f(n)$ is at least $\frac{n}{\pi} - o(n)$.

Proof. Let D_1, D_2, \dots, D_t be the \mathcal{G} -disks of a cover of \mathcal{G} and let cost be its cost. Let r_i be the radius of D_i , $1 \leq i \leq t$. As D_i covers $N(r_i)$ points, Lemma 1 implies that

$$\begin{aligned} n &< \sum_{i=1}^t N(r_i) < \sum_{i=1}^t \left(\pi r_i^2 + (\pi\sqrt{2})r_i + \frac{1}{5}\sqrt{r_i} + \frac{\pi}{2} \right) < \quad (3) \\ &< \sum_{i=1}^t (\pi r_i^2 + 2\pi r_i) = \pi \text{cost} + 2\pi \sum_{i=1}^t r_i \end{aligned}$$

By hypothesis $r_i \geq f(n)$, hence we get

$$\text{cost} = \sum_{i=1}^t r_i^2 \geq f(n) \sum_{i=1}^t r_i$$

and thus

$$\sum_{i=1}^t r_i \leq \frac{\text{cost}}{f(n)}$$

From the above inequality and from Inequality 3, we get $n < \pi \text{cost} + 2\pi \frac{\text{cost}}{f(n)}$ and, finally,

$$\text{cost} > n \left(\frac{f(n)}{\pi f(n) + 2\pi} \right) = \frac{n}{\pi} \left(1 - \frac{2}{f(n) + 2} \right) = \frac{n}{\pi} - o(n).$$

□

As we shall see in the next section, the lower bound of this theorem is almost tight.

3 The Minimum Broadcast Problem on the Grid

The aim of this section is to prove lower and upper bounds on the cost of an optimal broadcast. In particular, in order to prove the lower bound, we introduce the following definitions. A *chain* \mathcal{H} is a sequence of \mathcal{G} -disks D_1, D_2, \dots, D_k , $k \geq 1$, such that D_{i+1} is centered at some point contained in D_i for $1 \leq i < k$. We also say that a chain \mathcal{H} *activates* a disk D if (i) D does not belong to \mathcal{H} , (ii) the center of D is contained in D_k , and (iii) D does not contain the center of D_1 . Furthermore, we define

$$\mathcal{U}(\mathcal{H}) = \bigcup_{i=1}^k D_i$$

where the union refers to points of the infinite grid contained in disks D_i . For any $r \geq 1$, consider any disk D of radius r ; we define

$$M(r) = \min\{|\mathcal{U}(\mathcal{H}) \cap D| \text{ such that } \mathcal{H} \text{ activates } D\}.$$

Notice that $M(r)$ does not depend on the choice of D and that any disk of a broadcast tree not containing the source is activated by a chain of disks belonging to the tree. The cardinality of the intersection between the disk and the chain is at least $M(r)$, where r is the radius of the disk. In order to evaluate the broadcast cost, we need a lower bound on $M(r)$. The proof of next lemma is given in the full version of the paper [3].

Lemma 2. *For any $r \geq 1$, it holds that $M(r) \geq 2\sqrt{2}r - 5$.*

Theorem 3. *The cost of any broadcast of \mathcal{G} is at least $\frac{n}{\pi} - O(\sqrt{n})$.*

Proof. Let D_1, D_2, \dots, D_t be the \mathcal{G} -disks of an optimal broadcast of \mathcal{G} and let cost^* be its cost. Let r_i be the radius of D_i , $1 \leq i \leq t$. If there exists a disk D_i with radius $r_i \geq \sqrt{\frac{n}{\pi}}$, the thesis holds. Hence, we assume that $r_i < \sqrt{\frac{n}{\pi}}$, $1 \leq i \leq t$. In order to exploit Lemma 1, we partition the set $\{D_1, D_2, \dots, D_t\}$ into two sets: X and its complement \bar{X} , where

$$X = \{D_i \mid r_i > \sqrt{10}\}$$

From Lemma 1, it follows that

$$\begin{aligned} \sum_{i=1}^t N(r_i) &= \sum_{D_i \in X} N(r_i) + \sum_{D_i \in \bar{X}} N(r_i) \leq \sum_{D_i \in X} (\pi r_i^2 + 2\sqrt{2}r_i - 5) + \sum_{D_i \in \bar{X}} N(r_i) \\ &= \pi \cdot \text{cost}^* + 2\sqrt{2} \sum_{D_i \in X} r_i - 5|X| + \sum_{D_i \in \bar{X}} (N(r_i) - \pi r_i^2) \end{aligned} \quad (4)$$

As a consequence, we have that

$$\pi \cdot \text{cost}^* \geq \sum_{i=1}^t N(r_i) - 2\sqrt{2} \sum_{D_i \in X} r_i + 5|X| - \sum_{D_i \in \bar{X}} (N(r_i) - \pi r_i^2) \quad (5)$$

Now, we derive a lower bound on $\sum_{i=1}^t N(r_i)$. Observe that the communication graph yielded by the optimal broadcast contains a directed spanning tree T rooted at the source node. We partition $\{D_1, D_2, \dots, D_t\}$ into two sets Y and \bar{Y} , where Y is the set of \mathcal{G} -disks that cover the source point. We observe that every \mathcal{G} -disk $D_i \in \bar{Y}$ is activated by a chain of \mathcal{G} -disks whose centers induce a directed path in T . This implies that the number of intersection points between the activating chain and D_i is at least $M(r_i)$. Now we prove the following inequality:

$$\sum_{i=1}^t N(r_i) \geq n + \sum_{D_i \in \bar{Y}} M(r_i)$$

We consider a numbering of the T disks such that the disks on a root→leaf path have strictly increasing numbers. Let

$$E = \{(p, i) \mid \exists i : 1 \leq i \leq t \wedge p \in D_i\} \text{ and}$$

$$F = \{(p, j) \mid (p, j) \in E \wedge j = \min\{k \mid (p, k) \in E\}\}$$

In other words, $(p, j) \in F$ if and only if D_j is the "first" disk that covers p . Clearly, it holds that $|E| = \sum_i N(r_i)$, $F \subseteq E$, and $|F| \geq n$. Now, for every $i \in \bar{Y}$, let \mathcal{H}_i be the chain that activates D_i . Define $E_i = \{(p, i) \mid p \in \mathcal{U}(\mathcal{H}_i) \cap D_i\}$. The following properties hold: (a) $E_i \subseteq E - F$; (b) if $i \neq j$ then $E_i \cap E_j = \emptyset$; (c) $|E_i| \geq M(r_i)$. As for (a), clearly $E_i \subseteq E$. Furthermore, if $(p, i) \in E_i$ then $p \in \mathcal{U}(\mathcal{H}_i) \cap D_i$; thus, there exists a disk $D_j \in \mathcal{H}_i$ such that $p \in D_j$ and $j < i$. This implies that $\min\{k \mid (p, k) \in E\} \leq j < i$ and so $(p, i) \notin F$. The proofs of (b) and (c) are immediate from the definitions of E_i and $M(\cdot)$. Finally, it holds that

$$\sum_{i=1}^t N(r_i) = |E| = |F| + (|E| - |F|) \geq n + \sum_{i \in \bar{Y}} |E_i| \geq n + \sum_{i \in \bar{Y}} M(r_i).$$

Lemma 2 implies that

$$\begin{aligned} \sum_{D_i \in \bar{Y}} M(r_i) &= \sum_{D_i \in \bar{Y} \cap X} M(r_i) + \sum_{D_i \in \bar{Y} \cap \bar{X}} M(r_i) \geq \\ &\geq 2\sqrt{2} \sum_{D_i \in \bar{Y} \cap X} r_i - 5|\bar{Y} \cap X| + \sum_{D_i \in \bar{Y} \cap \bar{X}} M(r_i) \end{aligned}$$

From the above inequality, Inequality (5), and simple calculations, we get:

$$\pi \cdot \text{cost}^* \geq n - 2\sqrt{2} \sum_{D_i \in Y \cap X} r_i + 5|X| - 5|\bar{Y} \cap X| + \sum_{D_i \in \bar{Y} \cap \bar{X}} M(r_i) - \sum_{D_i \in \bar{X}} (N(r_i) - \pi r_i^2)$$

and

$$\begin{aligned} \text{cost}^* &> \frac{n}{\pi} - \frac{2\sqrt{2}}{\pi} \sum_{D_i \in Y \cap X} r_i + \\ &+ \frac{1}{\pi} \sum_{D_i \in \bar{Y} \cap \bar{X}} (M(r_i) - N(r_i) + \pi r_i^2) - \frac{1}{\pi} \sum_{D_i \in Y \cap \bar{X}} (N(r_i) - \pi r_i^2) \end{aligned} \quad (6)$$

Now we bound $\sum_{D_i \in Y \cap X} r_i$. Consider the sets

$$B_k = \{D_j \in Y \mid 2^{k-1} \leq r_j < 2^k\}, \quad 1 \leq k \leq l$$

where $l = \lceil \log r_{max} \rceil + 1$ and $r_{max} = \max\{r_j \mid D_j \in Y\}$. It holds that

$$\sum_{D_i \in Y \cap X} r_i \leq \sum_{D_i \in Y} r_i = \sum_{k=1}^l \sum_{D_i \in B_k} r_i \leq \sum_{k=1}^l \frac{1}{2^{k-1}} \sum_{D_i \in B_k} r_i^2 \quad (7)$$

Replace the \mathcal{G} -disks in $B_1 \cup B_2 \cup \dots \cup B_k$ by a \mathcal{G} -disk with radius (2^{k+1}) and centered in the source point. This operation produces a new broadcast with cost

$$\text{cost}^* - \sum_{D_i \in B_1 \cup B_2 \cup \dots \cup B_k} r_i^2 + (2 \cdot 2^k)^2$$

Hence, from the optimality of the previous broadcast it must be

$$\sum_{D_i \in B_1 \cup B_2 \cup \dots \cup B_k} r_i^2 \leq (2 \cdot 2^k)^2$$

From the above inequality and from Inequality (7) we have

$$\sum_{D_i \in Y \cap X} r_i \leq \sum_{k=1}^l \frac{2^{2k+2}}{2^{k-1}} = \sum_{k=1}^l 2^{k+3} < 2^{l+4} < 2^6 r_{max} = O(\sqrt{n}) \quad (8)$$

where the last step follows from the initial assumption that broadcast \mathcal{G} -disks have radii less than $\sqrt{\frac{n}{\pi}}$. It is possible to exhaustively prove that $M(r) - N(r) + \pi r^2 > 0$ when $r \leq \sqrt{10}$, i.e., $r \in \{1, \sqrt{2}, 2, \sqrt{5}, \sqrt{8}, 3, \sqrt{10}\}$. Hence,

$$\sum_{D_i \in \bar{Y} \cap \bar{X}} (M(r_i) - N(r_i) + \pi r_i^2) > 0 \quad (9)$$

Moreover, the number of \mathcal{G} -disks in $Y \cap \bar{X}$ is bounded by constant $N(\sqrt{10})$. Thus,

$$\sum_{D_i \in Y \cap \bar{X}} (N(r_i) - \pi r_i^2) = O(1) \quad (10)$$

Finally, by combining Inequality (6) with bounds (8), (9) and (10) we get the thesis. \square

The construction of optimal Broadcasts for the grid is somewhat connected with the famous problem known as *Apollonian Circle Packing* [13, 21]. More precisely,

we observe that if it were possible to evaluate the cost of the Apollonian Circle Packing of the grid then it would be possible to obtain the optimal bound on the Broadcast cost. We strongly believe that this is the *only* way to obtain such an optimal bound. The former problem is known to be a hard mathematical problem. In order to get a near-tight bound, we here adopt a simpler construction.

Theorem 4. *Given any source $s \in \mathcal{G}$, it is possible to construct, in polynomial time, a Broadcast for \mathcal{G} of cost $1.01013\frac{n}{\pi} + O(\sqrt{n})$.*

Proof. In order to provide a Broadcast of cost $1.01013\frac{n}{\pi} + O(\sqrt{n})$, we assume that $m - 1$ is a multiple of 6. If this is not the case, we can add $O(m)$ new unit radius \mathcal{G} -disks to our construction in order to broadcast to the remaining points.

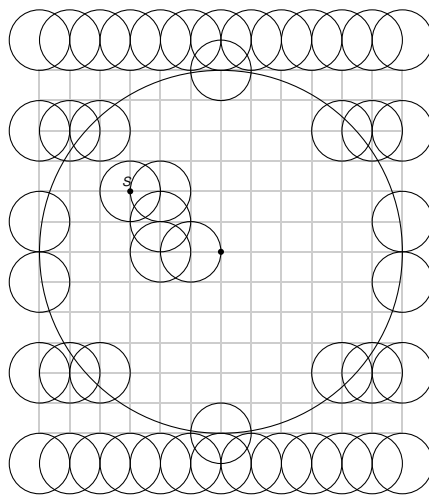


Fig. 2. An almost optimal Broadcast for the grid where $m = 19$.

Consider the Broadcast shown in figure 2. Its cost can be computed by summing up the following three contributions.

- A chain of \mathcal{G} -disks of radius 1 from the source point to the middle point of \mathcal{G} . The cost of this chain is $O(m)$.
- A big \mathcal{G} -disk of radius $r = \frac{m-1}{2}$ centered in the middle point of \mathcal{G} . This disk has cost $r^2 = \frac{n}{4} - \Theta(m)$.
- A set of \mathcal{G} -disks of radius 1 that broadcast to all nodes of \mathcal{G} out of the big \mathcal{G} -disk. In order to compute the cost of this set, assume that the origin of the Cartesian plane lies in the middle point of \mathcal{G} and compute only the cost of the \mathcal{G} -disks in the first quadrant, multiplied by 4. Furthermore, observe that the contribution of the first quadrant consists of $\frac{m-1}{6}$ horizontal chains of unit-radius \mathcal{G} -disks whose length depends on their y -coordinates. So the cost of this contribution is:

$$\begin{aligned}
C &= 4 \sum_{i=0}^{\frac{r}{3}} \left(r - \left\lfloor \sqrt{r^2 - (3i)^2} \right\rfloor \right) < \frac{4}{3}r^2 - 4 \sum_{i=0}^{\frac{r}{3}-1} \left(\sqrt{r^2 - (3i)^2} - 1 \right) < \\
&< \frac{4}{3}r^2 + \frac{4}{3}r - 4 - 4 \int_0^{\frac{r}{3}} \sqrt{r^2 - (3x)^2} dx < \frac{4}{3}r^2 + \frac{4}{3}r - 4 - \frac{4}{3} \int_0^r \sqrt{r^2 - x^2} dx < \\
&< \frac{4}{3}r^2 + \frac{4}{3}r - 4 - \frac{4}{3} \left[\frac{r^2}{2} \arcsin \frac{x}{r} + \frac{x}{2} \sqrt{r^2 - x^2} \right]_0^r = \\
&= \left(\frac{4 - \pi}{3} \right) \frac{n}{4} + O(m)
\end{aligned}$$

Finally, the cost of this Broadcast is $\frac{n}{4} + \left(\frac{4-\pi}{3}\right) \frac{n}{4} + O(m) = 1.01013 \frac{n}{\pi} + O(\sqrt{n})$. The construction of this solution can be clearly performed in time polynomial in n . □

Even when the \mathcal{G} -disks must be very large, we are able to provide a Broadcast whose cost is very close to the lower bound, as shown in the following result. We remark that its proof makes use of a construction that approximates the Apollonian Circle Packing of the grid. The proof of next lemma is given in the full version of the paper [3].

Lemma 3. *Let $0 < c < 1$ be a constant. For any source $s \in \mathcal{G}$, it is possible to construct, in polynomial time, a broadcast \mathcal{B} for \mathcal{G} with disks of radius at least $c\sqrt{n}$ and such that*

$$\text{cost}(\mathcal{B}) = f(c) \frac{n}{\pi} + O(\sqrt{n})$$

where

$$f(c) < \pi \left(0.35483 + 24.6814c^{2-\log_{1+\sqrt{2}}3} - 0.5551c + 0.5c^2 \right)$$

The following upper bound is an easy consequence of the previous lemma.

Theorem 5. *For any source point, there exists a (polynomial-time computable) Broadcast \mathcal{B} for \mathcal{G} that uses disks with radius at least $\frac{\sqrt{n}}{10^6}$ and such that*

$$\text{cost}(\mathcal{B}) < 1.1171 \frac{n}{\pi} + O(\sqrt{n})$$

As a consequence, \mathcal{B} consists of a constant number of disks.

Observe that Theorem 3 implies that the upper bound of Theorem 5 is almost tight.

4 Future research

Our asymptotical bounds on the Broadcast Problem on grids are not tight: achieving tight bounds here is an interesting theoretical open problem. However, as mentioned in the Introduction, we believe that our results open new promising directions in the design of new, good heuristics for a wide and practically relevant class of input configurations: well-spread, regular instances and uniform random instances [8, 19]. This is, in our opinion, the most relevant challenge in this topic. Efficient implementation, performance analysis and tests of some heuristics inspired by our constructive upper bounds are the subject of our present research activity.

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