L(2, 1)-Labeling of Unigraphs

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Dedicated to Uri N. Peled (1944-2009).

Abstract: The L(2, 1)-labeling problem consists of assigning colors from the integer set 0, . . . , λ to the nodes of a graph G in such a way that nodes at a distance of at most two get different colors, while adjacent nodes get colors which are at least two apart. The aim of this problem is to minimize λ and it is in general NP-complete. In this paper the problem of L(2, 1)-labeling unigraphs, i.e. graphs uniquely determined by their own degree sequence up to isomorphism, is addressed and a 3/2-approximate algorithm for L(2, 1)-labeling unigraphs is designed. This algorithm runs in O(n) time, improving the time of the algorithm based on the greedy technique, requiring O(m) time, that may be near to Θ(n^2) for unigraphs.

Keywords: L(2, 1)-labeling; frequency assignment; unigraphs.

1 Introduction

The L(2, 1)-labeling problem [10] consists in assigning colors from the integer set 0, . . . , λ to the nodes of a graph G in such a way that nodes at a distance of at most two get different colors, while adjacent nodes get colors which are at least two apart. The aim is to minimize λ.

This problem has its roots in mobile computing. The task is to assign radio frequencies to transmitters at different locations without causing interference. This situation can be modelled by a graph, whose nodes are the radio transmitters/receivers, and adjacencies indicate possible communications and, hence, interference. The aim is to minimize the frequency bandwidth, i.e. λ.

In general, both determining the minimum number of necessary colors [10] and deciding if this number is < k for any fixed k ≥ 4 [9] is NP-complete. Therefore, researchers have focused on some special classes of graphs. For some classes – such as paths, cycles, wheels, tilings and k-partite graphs – tight bounds
for the number of colors necessary for an $L(2, 1)$-labeling are well known in the literature and so a coloring can be computed efficiently. For many other classes of graphs—such as chordal graphs [14], interval graphs [8], split graphs [2], outerplanar and planar graphs [2, 7], bipartite permutation graphs [1], and co-comparability graphs [4]—approximate bounds have been looked for. For a complete survey, see [5].

Unigraphs [11, 12] are graphs uniquely determined by their own degree sequence up to isomorphism and are a superclass including matrogenic graphs, matroidal graphs, split matrogenic graphs and threshold graphs. The interested reader can find information related to these classes of graphs in [13].

In [6] all these subclasses are $L(2, 1)$-labeled: threshold graphs can be optimally $L(2, 1)$-labeled in time linear in $\Delta$ with $\lambda \leq 2\Delta$, while for matrogenic graphs the upper bound $\lambda \leq 3\Delta$ holds, where $\Delta$ is the maximum degree of the graph. In the same paper the problem of $L(2, 1)$-labeling the whole superclass of unigraphs is left open.

In this paper, a $3/2$-approximate algorithm for the $L(2, 1)$-labeling of unigraphs is presented. This algorithm runs in $O(n)$ time, which is the best possible. Observe that a naive algorithm, based on the greedy technique, would obtain an $O(m)$ time complexity, that may be near to $\Theta(n^2)$ for unigraphs.

The technique used in the algorithm takes advantage of the degree sequence analysis. In particular, this algorithm exploits the concept of boxes, i.e. the equivalence classes of nodes in a graph under equality of degree.

This paper is organized as follows.

In the next section all the information required for the rest of the paper is summarized. A recognition algorithm for unigraphs and the corresponding characterization theorem on which it is based are outlined in Section 3. The core of the paper comes in the following three sections. Section 4 provides optimal $L(2, 1)$-labeling without repetitions (i.e. $L'(2, 1)$-labeling) for those graphs listed in the characterization of unigraphs, while an $L(2, 1)$-labeling for the same graphs is presented in Section 5. Finally, in Section 6 a linear time (in $n$ and in $\Delta$) $3/2$ approximate algorithm for $L(2, 1)$-labeling of unigraphs is presented. Concluding remarks and open problems complete the paper.

# 2 Preliminaries

In this section all the definitions and known results that will be used in the rest of the paper are summarized.

We consider only finite, simple, loopless graphs $G = (V, E)$, where $V$ and $E$ are the node and edge sets of $G$ with cardinality $n$ and $m$, respectively. Where no confusion arises, $G = (V, E)$ is called simply $G$.

Let $DS(G) = \delta_1, \delta_2, \ldots, \delta_n$ be the degree sequence of a graph $G$ sorted by non-increasing values: $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_n \geq 0$. We call boxes the equivalence classes of nodes in $G$ under equality of degree. In terms of boxes the degree sequence can be compressed as $d_1^{m_1}, d_2^{m_2}, \ldots, d_r^{m_r}, d_1 > d_2 > \ldots > d_r \geq 0$, where
Figure 1: a. A split graph $G$; b. its complement $\overline{G}$; c. its inverse $G'$. 

di is the degree of the $m_i$ nodes contained in box $B_i(G)$, $1 \leq m_i \leq n$; hence 
$\sum_{i=1}^{t} m_i = n$ and $\sum_{i=1}^{t} d(m_i) = 2m$.

We call a box universal (isolated) if it contains only universal (isolated) nodes, where a node $x \in V$ is called universal (isolated) if it is adjacent to all 
other nodes of $V$ (no other node in $V$); if $x$ is a universal (isolated) node, then 
its degree is $d(x) = n - 1$ ($d(x) = 0$).

A graph $I$ induced by subset $V_I \subseteq V$ is called complete or clique if any two 
distinct nodes in $V_I$ are adjacent in $G$, stable or null if no two nodes in $V_I$ are 
adjacent in $G$.

A graph $G$ is said to be split if there is a partition $V = V_K \cup V_S$ of its nodes 
such that the induced subgraphs $K$ and $S$ are complete and stable, respectively 
(see Figure 1.a).

If $G = (V, E)$ is a graph, its complement is $\overline{G} = (V, V \times V - E)$ (see Figure 
1.b). If $G = (V_K \cup V_S, E)$ is a split graph, its inverse $G'$ is obtained from 
$G$ by deleting the set of edges $\{(a_1, a_2) : a_1, a_2 \in V_K\}$ and adding the set of 
edges $\{(b_1, b_2) : b_1, b_2 \in V_S\}$ (see Figure 1.c).

Given a graph $G$, if its node set $V$ can be partitioned into three disjoint sets 
$V_K, V_S$ and $V_C$ such that $K$ is a clique, $S$ is a stable set and every node in $V_C$ is 
adjacent to every node in $V_K$ and to no node in $V_S$, then the subgraph induced 
y $V_C$ is called crown.

In the following the definitions of some special graphs are recalled [15]:

$mK_2$: it is the union of $m$ node-disjoint edges $m \geq 1$, also called perfect 
matching (see Fig. 2.a).

$U_2(m, s)$: it is the disjoint union of a perfect matching $mK_2$ and a star $K_{1,s}$, 
for $m \geq 1, s \geq 2$ (see Fig. 2.b).

$U_3(m)$: for $m \geq 1$, this graph is constructed as follows: fix a node in each 
component of the graph obtained as disjoint union of the chordless cycle $C_4$ and 
m triangles $K_3$, and merge all these nodes in one (see Fig. 2.c).

$S_2 = (p_1, q_1; \ldots; p_t, q_t)$: to obtain this graph, add all the edges connecting 
the centers of $l$ non isomorphic arbitrary stars $K_{1,p_i}, i = 1, \ldots, t$, each one 
occuring $q_i$ times, where $p_i, q_i, t \geq 1, q_1 + \ldots + q_t = l \geq 2$ (see Fig. 3.a). 
Without loss of generality, in the following we assume $p_1 \leq \ldots \leq p_t$. 

\[3\]
Figure 2: a. $mK_2$; b. $U_2(m, s)$; c. $U_3(m)$.

$S_3(p, q_1; q_2)$: take a graph $S_2(p, q_1; p + 1, q_2)$ where $p \geq 1$, $q_1 \geq 2$ and $q_2 \geq 1$; add a new node $v$ to the stable part of the graph and add the set of $q_1$ edges $\{\{v, w\} : w \in V_K \text{ and } \deg_{V_S}(w) = p\}$; the obtained graph is $S_3$ (see Fig. 3.b).

$S_4(p, q)$: it is constructed taking a graph $S_3(p, 2; q)$, $q \geq 1$, adding a new node $u$ to the clique part and connecting it with each node of the stable except $v$ (see Fig. 3.c).

Figure 3: a. $S_2(p_1, q_1; \ldots; p_t, q_t)$; b. $S_3(p, q_1; q_2)$; c. $S_4(p, q)$.

It is easy to see that $S_2$, $S_3$ and $S_4$ are split graphs, where the clique part is constituted by the centers of the stars for $S_2$ and $S_3$, and by the centers of the stars and $u$ for $S_4$.

3 Characterization and Recognition of Unigraphs

In this section we recall a characterization of unigraphs in terms of superposition of a red and a black graph.

Theorem 3.1 [3] A graph $G$ is a unigraph if and only if its node set can be partitioned into three disjoint sets $V_K$, $V_S$ and $V_C$ such that:

(i) $V_K \cup V_S$ induces a split unigraph $F$ in which $K$ is the clique and $S$ is the stable set;
(ii) $V_C$ induces a crown $H$ and either $H$ or $\overline{H}$ is one of the following graphs:

$$C_5, \ mK_2, m \geq 2, \ U_2(m, s), \ U_3(m);$$

(iii) the edges of $G$ can be colored red and black so that:

a. the red partial graph is the union of the crown $H$ and of node-disjoint pieces $P_i, i = 1, \ldots, z$. Each piece $P_i$ (or $P_i^1$ or $P_i^2$) is one of the following graphs:

$$K_1, \ S_2(p_1, q_1; \ldots; p_t, q_t), \ S_3(p, q_1; q_2), \ S_4(p, q),$$

considered without the edges in the clique;

b. the linear ordering $P_1, \ldots, P_z$ is such that each node in $V_K$ belonging to $P_i$ is not linked to any node in $V_S$ belonging to $P_j$, $j = 1, \ldots, i - 1$, but is linked by a black edge to every node in $V_S$ belonging to $P_j$, $j = i + 1, \ldots, z$. Furthermore, any edge connecting either two nodes in $V_K$ or a node in $V_K$ and a node in $V_C$ is black.

In view of the previous lemma, although not explicitly mentioned, when we speak about a unigraph $G$ we mean that its node set is partitioned into the three sets $V_C$, $V_K$, and $V_S$, inducing the crown, the clique and the stable part, respectively.

It is worthy to be noticed that there is a basic difference between a matching inside the red graph of a split unigraph and a matching constituting the crown of a unigraph: the nodes of the first one induce an $S_2(1, q)$ (i.e., the red edges of the matching plus the black edges connecting as a clique the nodes of one partition); the second one corresponds to an $mK_2$. An analogous difference holds between the graph induced by the nodes of an antimatching inside the red graph of a split unigraph ($S_2(1, q)$) and the crown inducing an antimatching ($mK_2$). This difference will be very important when we will $L(2, 1)$-label the pieces of the unigraph, as we underline in Sections 4 and 5.

In Fig. 4 a unigraph is depicted, and its red and black partial graphs are highlighted. The pieces $P_i$ defined by the previous theorem are included in dotted rectangles. Observe that in this figure, and all over the paper, we depict all nodes belonging to $V_K$ above all nodes belonging to $V_s$, that always lie on the bottom part of the drawing; moreover, we avoid to draw all the edges of the clique, but we include the nodes of $V_K$ in a rectangle to underline that they induce a clique.

From the characterization stated in Theorem 3.1, and recalling that a unigraph is a graph uniquely determined by its own degree sequence up to isomorphism, it is possible to derive a linear time recognition algorithm for unigraphs that identifies the structure of the graph analyzing only its degree sequence [3]. In particular, this algorithm exploits the concept of boxes. If there is not an isolated or universal box ($K_1$ in item (iii).a of Theorem 3.1), a group of boxes can induce either a crown as specified in item (ii), or one of the graphs $S_2, S_3,$
Figure 4: A unigraph where its crown $C_5$ and its pieces $S_3(1,2;1)$, $K_1$ and $S_2(2,2)$ are highlighted by dotted rectangles. Edges are colored according to Theorem 3.1 (edges completely contained into the dotted rectangles and the edges of the crown $C_5$ are red).

$S_4$ (or their complement, their inverse, or the inverse of their complement) in item (iii).a. This algorithm for recognizing unigraphs works pruning the degree sequence $d_1^{\text{m}}, \ldots, d_r^{\text{m}}$ of a given graph $G$. At each step, the algorithm finds one of the node-disjoint pieces $P_i$ of $G$, checking the first $p$ and the last $q$ boxes, according to part (iii).a. of Theorem 3.1. The algorithm proceeds on the pruned graph $G - P_i$, that represents a unigraph if $G$ is a unigraph (part (iii).b of Theorem 3.1). This step is iterated until either $G$ is recognized to be a unigraph or some contradiction is highlighted because either part (iii).a or part (iii).b are recognized to be not true.

4 L'(2,1)-Labeling of the Crown and of the Pieces

An $L'(2,1)$-labeling (also called $L(2,1)$-labeling without repetitions) [8] is a one-to-one $L(2,1)$-labeling into the set $0, \ldots, \lambda'$, with the aim of minimizing $\lambda'$.

In order to design the $L(2,1)$-labeling algorithm for unigraphs, in this section we will show how to optimally $L'(2,1)$-label the graphs cited in Theorem 3.1 and, for each of them, we provide the number of used colors, taking into account the black connections of Theorem 3.1. In the following it will be clear why we need to $L'(2,1)$-label some pieces of a unigraph in order to get an $L(2,1)$-labeling of it.

We underline that, from now on, in the figures, when we depict complement and inverse graphs, we omit to draw all the edges, except the absent ones, represented by dotted lines. Moreover, the unused colors are highlighted in a queue.

4.1 Crown

In order to $L(2,1)$-label the $V_C$ nodes of the crown of a unigraph $G$, we have to consider whether there are other nodes in the unigraph or it is constituted by
the only crown; in other words, we have to distinguish if the crown is the only piece in the graph or not. If at least another (split) piece exists, all the nodes in $V_C$ are at mutual distance two, since the crown is completely connected to the nodes of $V_K$. When $V_K = \emptyset$ this condition is not required. It follows that in the first case we have to $L'(2,1)$-label the crown, while in the second case we have to $L(2,1)$-label it.

In the following we will show how to optimally $L'(2,1)$-label the crown.

**Lemma 4.1** [3] Let $G$ be a unigraph with $V_K \neq \emptyset$. If its crown $H$ is:
- the cycle $C_5 = C_5$ then it can be optimally $L'(2,1)$-labeled with 5 consecutive colors;
- a matching $mK_2$, then it can be optimally $L'(2,1)$-labeled with $2m$ consecutive colors if $m > 1$ and with $3 = 2m + 1$ colors if $m = 1$; in this latter case one color remains unused.
- a hyperoctaedron $mK_2$, then it can be optimally $L'(2,1)$-labeled with $3m - 1$ consecutive colors and $m - 1$ colors remain unused.

In Figure 5.a, 5.b, 5.c and 5.d $L'(2,1)$-labeling of $C_5$, $K_2$, $mK_2$ and $mK_2$ when $m = 4$ are reported. It is to notice that the $L(2,1)$- and $L'(2,1)$-labelings coincide for $C_5$ and $4K_2$ because no colors can be repeated in this graphs.

**Lemma 4.2** Let $G$ be a unigraph with $V_K \neq \emptyset$. If its crown $H$ is:
- $U_2(m, s)$ then it can be optimally $L'(2,1)$-labeled with $2m + s + 1$ consecutive colors;
- $\overline{U_2(m, s)}$ then it can be optimally $L'(2,1)$-labeled with $3m + 2s - 1$ colors and $m + s - 2$ colors remain unused;
- $U_3(m)$ then it can be optimally $L'(2,1)$-labeled with $2m + 4$ consecutive colors;
- $\overline{U_3(m)}$ then it can be optimally $L'(2,1)$-labeled with $3m + 3$ colors and $m - 1$ colors remain unused.
Proof: As $U_2(m, s)$ is the disjoint union of an $mK_2$ and a star $K_{1,s}$, in view of Lemma 4.1, if $m \geq 2$, we need $2m$ colors for $mK_2$, let them be $1, \ldots, 2m$, while the star can be easily optimally $L'(2,1)$-labeled with $s+1$ colors, assigning 0 to the center of the star and colors $2m+1, \ldots, 2m+s$ to the leaves (see Figure 6.a). If $m = 1$ we need the same number of colors simply arranging them in a different way (e.g. using 1 and 3 for the $K_2$, 0 for the center of the star, and the other ones for its leaves).

$U_2(m, s)$ is given by a hyperoctahedron $mK_2$ completely connected with the complement of a star $K_{1,s}$. $3m-1$ colors are needed to optimally $L'(2,1)$-label the hyperoctahedron (see Lemma 4.1), and the unused colors cannot be used inside the same $U_2$ in view of the complete connection with $mK_2$. In order to label the complement of the star we need $2s-1$ colors more, whose $s-2$ are unused. Finally, we have to add a further color between the colors of the hyperoctahedron and of $K_{1,s}$ because they are completely connected (see Figure 6.b), so also one more color remains unused. By summing all the contributions, the thesis follows.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{Optimal $L'(2,1)$-labeling of: a. $U_2(4,3)$; b. $U_2(4,3)$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7.png}
\caption{Optimal $L'(2,1)$-labelings of: a. $U_3(3)$; b. $U_3(3)$.}
\end{figure}

In order to label $U_3(m)$, let 0 be the color of the maximum degree node. It is not difficult to give different colors to all the other nodes in order to get an optimal $L'(2,1)$-labeling with a number of colors equal to the number of nodes. (see Figure 7.a).

Observe that $U_3(m)$ is constituted by a hyperoctahedron $mK_2$, completely connected to three nodes, two of which are connected, and the third one is
adjacent to a degree 1 node. Consequently, $3m - 1$ colors are necessary to label the hyperoctahedron (see Lemma 4.1); one of the colors unused by the hyperoctahedron can be used for the degree 1 node. 4 more consecutive colors are necessary for the remaining three nodes, since the first of them cannot be used (see Figure 7.b). The unused colors are hence $m - 1$.

We highlight that the $L(2,1)$- and $L'(2,1)$-labelings coincide on $\overline{U_2}$ and $\overline{U_3}$ graphs as they have diameter 2.

4.2 Split pieces

Each split piece $P_i$ ($S_2$, $S_3$ and $S_4$ of Theorem 3.1) must be colored using colors at mutual distance at least two in the clique part.

For what concerns the stable part, we have to distinguish two cases, according to the fact that $P_i$ is the first piece in the linear ordering of item iii.(b) of Theorem 3.1 (i.e. $i = 1$) or not (i.e. $i > 1$). Indeed, black edges defined in item iii.(b) impose to use different colors for the nodes in the stable part of each $P_i$, $i > 1$, hence for this piece we have to provide an $L'(2,1)$-labeling. Only colors in the stable part of $P_1$ can be eventually repeated.

In this subsection we show how to $L'(2,1)$-label split pieces.

Lemma 4.3 Let $G$ be a unigraph. If one of its pieces $P_i$, $i > 1$, is

- $S_2(p_1, q_1; \ldots; p_t, q_t)$ then it can be optimally $L'(2,1)$-labeled with $\sum_{i=1}^t (p_i + 1)q_i$ consecutive colors;
- $S_2(p_1, q_1; \ldots; p_t, q_t)$ then it can be optimally $L'(2,1)$-labeled with $\sum_{i=1}^t (p_i + 1)q_i$ colors; if $q_1 > 2$ and $p_1 = 1$ then it can be optimally $L'(2,1)$-labeled with $\sum_{i=1}^t (p_i + 1)q_i + \lceil q_1 / 2 \rceil$ colors and $\lfloor q_1 / 2 \rfloor$ of them remain unused;
- $S_2(p_1, q_1; \ldots; p_t, q_t)$ then it can be optimally $L'(2,1)$-labeled with $2 \sum_{i=1}^t p_i q_i - 1$ colors and $\sum_{i=1}^t q_i(p_i - 1) - 1$ of them remain unused; if $p_1 = 1$ then both the number of used and unused colors must be incremented by $\lceil q_1 / 2 \rceil$;
- $S_2(p_1, q_1; \ldots; p_t, q_t)$ then it can be optimally $L'(2,1)$-labeled with $2 \sum_{i=1}^t p_i q_i - 1$ colors and $\sum_{i=1}^t q_i(p_i - 1) - 1$ of them remain unused; if $t = 1$ and $q_1 = 1$ then it can be optimally $L'(2,1)$-labeled with $2p_1 + 1$ colors and $p_1$ of them remain unused.

Proof: For the $\sum_{i=1}^t q_i$ centers of the stars of $S_2$, that are connected in a clique, $2 \sum_{i=1}^t q_i - 1$ colors are necessary, and $\sum_{i=1}^t q_i - 1$ of them are unused. Let $U$ be the set of these unused colors. Colors from $U$ are assigned to the leaves of each star taking into account to avoid those colors at distance one from the color assigned to the center (see Figure 8.a). In order to complete the labeling, further $\sum_{i=1}^t (p_i - 1)q_i + 1$ consecutive colors will be necessary. The number of used colors is hence $\sum_{i=1}^t p_i q_i + \sum_{i=1}^t q_i$, that is exactly the number of nodes of $S_2$. Observe that, if $\sum_{i=1}^t q_i = 2$, in order not to discard any color, the nodes
in the clique must be labeled with a different rule (see Figure 8.b). Indeed, if the clique was labeled with 0 and 2, color 1 would be discarded.

For what concerns $S'_2$, again a number of colors equal to the number of nodes is necessary and sufficient, but the labeling must be performed in the following way: label the first of the $p_i$ leaves of each star with the first available color $c$; label the center of the star with color $c+1$, and the remaining $p_i-1$ leaves with colors $c+2, \ldots, c+p_i$ (see Figure 8.c). This method works if $p_1 \geq 2$.

But, if it holds that $q_1 > 2$ and $p_1 = 1$, then the first $q_1$ stars constitute a matching and more colors are necessary. Namely, for each color $g$ assigned to a node of the matching in the clique, both $g-1$ and $g+1$ cannot be assigned to any node in the clique and to any node in the stable set, except its mate; hence one between $g-1$ and $g+1$ must remain unused (see Figure 8.d).

It is easy to see that for labeling $S_2(p_1, q_1; \ldots; p_t, q_t)$ and $S_2(p_1, q_1; \ldots; p_t, q_t)$, $2 \sum_{i=1}^{t} p_i q_i - 1$ colors are always necessary and sufficient. Indeed, they are necessary for $L'(2,1)$-labeling the clique containing all the leaves of the stars, and each center of a star may be colored with one of the colors unused during the labeling of the leaves opportunely chosen (see Figure 8.e). It follows that $\sum_{i=1}^{t} q_i(p_i-1) - 1$ colors remain unused. Observe that if $p_1 = 1$ in $S'_2$, arguments similar to those explained for $S_2$ can be used, and the thesis follows. Finally, if $t = 1$ and $q_1 = 1$ in $S'_2$, it is easy to see that 2 colors more are needed since $S'_2$ is a clique with $p+1$ nodes.

![Figure 8: Optimal $L'(2,1)$-labelings of: a. $S_2(2, 2; 4, 1)$; b. $S_2(1, 1; 2, 1)$; c. $S'_2(2, 2; 4, 1)$; d. $S'_2(1, 5; 2, 1)$; e. $S_2(2, 2; 4, 1)$.](image)

**Lemma 4.4** Let $G$ be a unigraph. If one of its pieces $P_i$, $i > 1$, is

- $S_3(p, q_1; q_2)$ then it can be optimally $L'(2, 1)$-labeled with $pq_1 + (p + 1)q_2 + q_1 + q_2 + 1$ consecutive colors;
- $S'_3(p, q_1; q_2)$ then it can be optimally $L'(2, 1)$-labeled with $pq_1 + (p + 1)q_2 + q_1 + q_2 + 1$ consecutive colors; if $q_1 > 2$ and $p_1 = 1$ then it can be optimally $L'(2, 1)$-labeled with $pq_1 + (p + 1)q_2 + q_1 + q_2 + 1 + [q_1/2]$ colors and $[q_1/2]$ of them are unused;
- $S_2(p, q_1; q_2)$ then it can be optimally $L'(2, 1)$-labeled with $2p(q_1 + q_2) + 2q_2 + 1$ colors and $pq_1 + pq_2 - q_1 - 1$ of them remain unused; if $p_1 = 1$ then it can
be optimally $L'(2, 1)$-labeled with $2p(q_1 + q_2) + 2q_2 + 1 + \lfloor q_1/2 \rfloor$ colors and $pq_1 + pq_2 - q_1 - 1 + \lfloor q_1/2 \rfloor$ of them remain unused;

- $S_3(p, q_1; q_2)$ then it can be optimally $L'(2, 1)$-labeled with $2p(q_1 + q_2) + 2q_2 + 1$ colors and $pq_1 + pq_2 - q_1 - 1$ of them remain unused.

**Proof:** Remind that $S_3(p, q_1; q_2)$ is obtained adding a node $v$ to the stable set of a graph $S_2(p, q_1; p + 1, q_2)$, $p \geq 1$, $q_1 \geq 2$, $q_2 \geq 1$ and $v$ is connected to the first $q_1$ centers. Consequently the methods for labeling $S_3$, $S_3'$, $S_4$ and $S_4'$ are the same presented for $S_2$, $S_2'$, $S_4$ and $S_4'$ only taking care of node $v$. In order not to overburden the exposition we omit further details and we present only Figure 9.a in which the labeling of $S_3(2, 1; 1)$ is depicted.

![Figure 9](image_url)

**Figure 9:** Optimal $L'(2, 1)$-labelings of: a. $S_3(2, 2; 1)$; b. $S_4(2, 1)$; c. $S_4(2, 1)$; d. $S_4(1, 2)$.

**Lemma 4.5** Let $G$ be a unigraph. If one of its pieces $P_i$ (cf. Th. 3.1) is

- $S_4(p, q)$ then it can be optimally $L'(2, 1)$-labeled with $2p + qp + 2q + 4$;

- $S_4(p, q)$ then it can be optimally $L'(2, 1)$-labeled with $2pq + 4p + 2q + 1$ colors and $pq + 2p - 3$ of them remain unused; if $p = 1$ then it can be labeled with $2pq + 4p + 2q + 2$ colors and $pq + 2p - 2$ of them remain unused;

- $S_4(p, q)$ then it can be optimally $L'(2, 1)$-labeled with $2pq + 4p + 2q + 2$ colors and $pq + 2p - 2$ of them remain unused;

- $S_4(p, q)$ then it can be optimally $L'(2, 1)$-labeled with $3q + 2p + qp + 4$ colors and $q$ of them remain unused.

**Proof:** The clique part of $S_4(p, q)$ requires $2(q + 3) - 1$ colors, whose $q + 3$ are used. Node $u$ in the clique part is colored with 0. All the nodes in the stable part (except $v$) are connected to $u$ and hence at mutual distance two and require distinct colors. The colors discarded while labeling the clique can be opportune used in the stable part, and in particular node $v$ is labeled with 1.
The centers of the two stars with \( p \) leaves cannot receive color 2. The remaining nodes are labeled with consecutive new colors (see Figure 9.b). Totally, the number of necessary colors is the same as the number of nodes of \( S_4 \).

The clique part of \( S_4 \) can be labeled with all the even colors from 0 to \( e = 2(2p + (p + 1)q) \). The odd colors, opportunely used, are sufficient to label the stable part, and \( pq + 2p - 3 \) colors remain unused (see Figure 9.c). In the special case when \( p = 1 \), one color must be unused, hence one color more is necessary, as shown in Figure 9.d.

Analogous considerations hold for \( S_4^I \). In this case, color \( e \) is assigned to node \( u \) and node \( v \) must have a new odd color \( e + 1 \). The number of unused colors is one more than in the previous case, i.e. \( pq + 2p - 2 \).

Finally, an optimal labeling of \( S_4^I \) is obtained using \( qp + 2p + 3q + 5 \) colors. Indeed \( 2(q + 3) - 1 \) colors are required by the clique. Let 0, 2 and 4 the colors assigned to \( v \) and to the centers of the stars with \( p \) leaves each one. Color 3 is suitable for labelling \( u \). Moreover color 1 can be assigned to one of the leaves of the star with center labeled with 2. No other odd colors from 5 to \( 2(q + 3) - 3 \) can be utilized in the stable part so, \( q \) colors must remain unused. Since nodes in the stable part must have different colors (in view of the fact that each pair is at distance two), we have to add \( 2p + qp + q - 1 \) consecutive different colors for completing the coloring of \( S_4^I \).

5 \( L(2, 1) \)-Labeling of the Crown and of the Pieces

5.1 Crown

We recall that the \( L(2, 1) \)- and \( L'(2, 1) \)-labelings of \( C_5 \) and \( mK_2 \) coincide (see Figures 5.a and 5.d). Furthermore, in view of their structure, \( U_2 \) and \( U_3 \) are graphs with diameter 2, hence even their \( L(2, 1) \)- and \( L'(2, 1) \)-labelings coincide and they must be labeled with all different colors, independently of the rest of the unigraph. Finally, in \( U_3 \) only one node can be labeled re-using a color (the node labeled by 1 in Figure 7.a) hence the number of colors necessary to \( L'(2, 1) \)- and to \( L(2, 1) \)-label \( U_3 \) is the same, but in the latter case one color remains unused. So, it remains to prove the following lemma.

Lemma 5.1 Let \( G \) be a unigraph constituted only by its crown. If \( G \) is:

- \( mK_2 \), then it can be optimally \( L(2, 1) \)-labeled with 3 colors and one color remains unused;
- \( U_2(m, s) \) then it can be optimally \( L(2, 1) \)-labeled with \( s + 2 \) colors and one color remains unused.

Proof: The \( L(2, 1) \)-labeling of a matching with 3 colors is trivial: 0 and 2 can be used for adjacent nodes, and color 1 remains unused (see Figure 10.a).

As \( U_2 \) is constituted by the disjoint union of a matching and a star, we can optimally label the star with \( s + 2 \) colors, whose one color is unused; for the
matching, it is possible to re-use a couple of not adjacent already used colors (see Figure 10.b). It is to notice that the pair (1, 3) for labeling the matching is also feasible, but the choice of not using a color will be useful later and will be clear during the presentation of the algorithm.

Figure 10: Optimal $L(2,1)$-labelings of: a. 4$K_2$; b. $U_2(4,3)$;

5.2 Split pieces

Observe that $S_2$ is a diameter 2 graph, if $\sum_{i=1}^t q_i > 2$, hence there is no difference between the $L(2,1)$- and $L'(2,1)$-labelings. Furthermore, if $\sum_{i=1}^t q_i = 2$, then the centers of the two stars are at distance three, but there is no way to assign them the same color using the minimum number of colors.

For what concerns $S_4$, the number of used colors is the same as in the case without repetitions, as the maximum number of necessary colors is given by the clique part, but some colors can be replicated in the stable part, hence $\sum_{i=1}^t p_i q_i - 3$ colors remain unused.

$S_2$ is a diameter 2 graph, when $\sum_{i=1}^t q_i > 2$ and hence its $L(2,1)$-labeling coincides with its $L'(2,1)$-labeling. If $\sum_{i=1}^t q_i = 2$, $S_2'$ coincides with $S_2$.

Similar considerations hold for $S_3$, $S_4$, $S_4'$, $S_4'$ and $S_4''$. Finally, in $S_4$ the only node that is at distance three from some leaves is $u$, and hence it is the only node that can receive a repeated color. It follows that the $L(2,1)$-labeling of $S_4$ is identical with respect to the $L'(2,1)$-labeling, except for node $u$. Hence, in order to study the $L(2,1)$-labeling of the split graphs in item iii(a) of Theorem 3.1, it is enough to prove the following result.

**Lemma 5.2** Let $G$ be a unigraph. If its first split piece $P_i$ (cf. Th. 3.1) is

- $S_2(p_1,q_1;\ldots;p_t,q_t)$ then it can be optimally $L(2,1)$-labeled with $(2 \sum_{i=1}^t q_i - 1) + \max\{0,p_t + x - (\sum_{i=1}^t q_i - 1)\}$ colors, where $x = 1$ if $q_t \leq 2$ and $x = 2$ otherwise;

- $S_3(p,q_1;q_2)$ then it can be optimally $L(2,1)$-labeled with $(2 \sum_{i=1}^t q_i - 1) + y + \max\{0,p_t + x - (\sum_{i=1}^t q_i - 1)\}$ colors, where $x = 1$ and $y = 1$ if $q_t \leq 2$ and $x = 2$ and $y = 0$ otherwise.
Proof: $S_2$ is composed by stars whose $\sum_{i=1}^{t} q_i$ centers are connected in a clique. So, at least $2 \sum_{i=1}^{t} q_i - 1$ colors are necessary. The first color must be assigned to one among the $q_i$ centers of the maximum size stars. Each time two distance 2 colors are assigned in the clique, the color in between remains unused. All such colors can be opportune assigned to some nodes in the stable part, possibly many times, paying attention that no leaf of a center of a star labeled $c$ takes label $c - 1$ or $c + 1$. Observe that the $p_i$ leaves of each star must receive all different colors, as they are at mutual distance two. Consider now the $q_i$ stars of maximum size $p_i$. If the unused colors are not enough to label its leaves, some colors must be added. Their number is $p_i - (\sum_{i=1}^{t} q_i - 2)$ if $q_i \leq 2$ (indeed at most one unused color must be discarded, see Figure 11.a) and is one color more if $q_i \geq 3$. Finally, if $p_i$ is sufficiently small, the unused colors are enough to label all the leaves of the maximum size stars and then no other colors must be added.

$S_3$ is obtained from an $S_2$ by adding a node to the stable part. It is easy to see that the number of colors necessary to label $S_2$ are enough for $S_3$, as the added node $v$ can receive either an already used color or one among the colors unused during the labeling of the clique part (see Figure 11.b). Only if $q_i \leq 2$ then one color more is necessary for $v$, and it must be labeled 1, as shown in Figure 11.c. This is the meaning of $y$ in the formula of the number of colors

\[ 2 \ 4 \ 6 \ 9 \ 0 \ 5 \ 7 \ 1 \ 3 \ 8 \ 5 \ 7 \ 3 \ 5 \ 7 \ 1 \ 3 \ 5 \ 7 \ 8 \]

\[ 2 \ 4 \ 6 \ 8 \ 10 \ 0 \ 3 \ 5 \ 7 \ 9 \ 1 \ 3 \ 5 \ 1 \ 3 \ 5 \ 3 \ 5 \ 7 \]

\[ 3 \ 5 \ 7 \ 9 \ 0 \ 6 \ 8 \ 2 \ 4 \ 1 \ 2 \ 4 \ 6 \ 2 \ 4 \ 6 \]

Figure 11: Optimal $L(2,1)$-labelings of: a. $S_2(2,3;4,2)$; b. $S_3(2,3;3)$; c. $S_3(2,3;2)$.

6 An Algorithm for $L(2,1)$-labeling Unigraphs

The labelings presented in the previous two sections will be used for the linear time algorithm for labeling the whole unigraph detailed in this section.

In Section 3, we have claimed that it is possible to identify the structure of a connected unigraph analyzing only its degree sequence, so the following $L(2,1)$-labeling algorithm will deal with the representation of a graph $G = (V,E)$ in terms of boxes with degree sequence $d_1^{m_1}, d_2^{m_2}, \ldots, d_r^{m_r}, d_1 > d_2 > \ldots > d_r$.

Let us call $k_i$ the largest color used for labeling the clique part of $P_i$ separately, considering that each split piece $P_i$ must be colored using colors at mutual distance at least two in the clique part.
The algorithm labels each piece in two phases. In the first phase, only $k_i + 1$ colors are considered, and in the second phase the labeling is completed. In particular, the algorithm first puts in a queue $S$ the pieces $P_i$, with clique part $K_i$ and stable part $S_i$ described in Theorem 3.1, that it recognizes according to the algorithm in [3], and the crown $H$, if it exists. Then, the algorithm partially labels each piece $P_i$ dequeued from the queue according to its own structure. In order to explain the partial labeling of piece $P_i$, let $c_{i-1} - 1$ be the last color used for the partial labeling of pieces $P_1, \ldots, P_{i-1}$. We label with colors from $c_i - 1$ to $c_i - 1 = c_{i-1} + k_i + 1$ all nodes in the clique and possibly some nodes in the stable set according to the rules of the previous section. In general, some nodes in the stable set remain unlabeled.

Not used colors from $c_i - 1$ to $c_i - 1$ will be inserted into a queue $Q$ together with the information that they have been enqueued by $P_i$.

If some nodes in $S_i$ remain uncolored, $P_i$ is again queued in $S$ together with the information of the number of its uncolored nodes $u_i$. The labeling of the partially labeled pieces will be completed by the last part of the algorithm. Only the crown and the first piece are immediately completely labeled.

The crown, if it is not the unique piece of $G$, is completely $L'(2,1)$-labeled while the first piece, independently from which piece it is, is completely $L(2,1)$-labeled since the nodes in its stable part are not extremes of any black edge and so repetitions of colors are possible. Notice that, if the unigraph is constituted by the only crown, it is the first (and unique) piece, and hence it is correctly $L(2,1)$-labeled.

Observe that a disconnected unigraph consists in a connected one and an isolated box. Hence, if the unigraph is not connected, we can assign the same color to all nodes of the isolated box and run the algorithm for $L(2,1)$-labeling the non trivial connected component. For this reason, as input of the algorithm only connected unigraphs are considered.

Finally, we say that color $k$ is thrown out if we decide not to use it; after $k$ has been thrown out it is not available anymore. Procedure $\text{Recognize-Pieces}(G, S, \text{num})$ takes in input unigraph $G$, recognizes its $\text{num}$ pieces $P_i$ and put them in $S$.

The $L(2,1)$-labeling algorithm is the following:

\begin{verbatim}
ALGORITHM L(2,1)-Label-Unigraphs
INPUT: a connected unigraph $G$ by means of its degree sequence $d^{(1)}_1, \ldots, d^{(r)}_r$
OUTPUT: an $L(2,1)$-labeling for $G$. 
Initialize-QueueColors $Q = \emptyset$;
Recognize-Pieces($G, S, \text{num}$);
PHASE 1.
REPEAT
   DequeuePiece $P_i$ from $S$;
   Step 1 // $P_i$ is completely $L(2,1)$-labeled;
   IF $i = 1$
      THEN completely $L(2,1)$-label $P_1$
      (details in Subsec. 5.1 and 5.2);
   ELSE
      Step 2
\end{verbatim}
IF $P_i$ split component
THEN Partially $L'(2, 1)$-label $P_i$ appropriately with new colors from $c_{i-1}$ to $c_{i-1} + k_i + 1$
(details in Subsec. 4.2);
FOR EACH unused color $d$ between $c_{i-1}$ and $c_{i-1} + k_i + 1$
    EnqueueColor$(d, P_i)$ in $Q$;
$c_i \leftarrow c_{i-1} + k_i + 2$;
IF $P_i$ is partially $L'(2, 1)$-labeled and $u_i$ among its nodes are not labeled
THEN EnqueuePiece$(P_i, u_i)$ in $S$;

**Step 3**
IF $P_i$ crown
    $L'(2, 1)$-label $P_i$ appropriately with new colors starting from $c_{i-1}$
    (details in Subsec. 4.1);
    FOR EACH unused color $u$ in the $L'(2, 1)$-labeling of the crown
        EnqueueColor$(d, P_j)$ in $Q$;
UNTIL$(i = \text{num})$;

**PHASE 2.**
REPEAT
    DequeuePiece$(P_i, u_i)$ from $S$;
    WHILE $(u_i > 0 \text{ AND } Q \neq \emptyset)$ DO
        DequeueColor$(d, P_j)$ from $Q$;
        IF $(j \leq i)$
            THEN throw $d$ out;
        ELSE use $d$ to $L'(2, 1)$-label one uncolored node in $P_i$;
            decrease $u_i$ by 1;
        IF $Q = \emptyset$
            THEN $L'(2, 1)$-label the $u_i$ uncolored nodes of $P_i$ with $m_i$ consecutive new colors from $c_{i-1}$ to $c_{i-1} + m_i - 1$;
UNTIL $(S = \emptyset)$.

**Theorem 6.1** Algorithm $L(2, 1)$-Label-Unigraphs correctly $L(2, 1)$-labels a unigraph $G$ in $O(n)$ time.

**Proof:** The correctness of procedure Recognize-Pieces follows from [3]. We will prove that the labeling found by the algorithm is feasible. Indeed, nodes in $V_K$ are labeled with colors at mutual distance at least two. Moreover, each node in $V_S$ cannot be colored with a color at distance $\leq 1$ to the colors of all its adjacent nodes (in $V_K$) in view of the following three facts:

1. Each piece $P_i$ is feasibly labeled according to Sections 4 and 5;
2. The only $L(2, 1)$-labeled piece is the first one, since its nodes in the stable part are not extreme of any black edge;
3. Each dequeued color $d$ (enqueued by $P_j$) is used only for labeling nodes in the stable part of piece $P_i$ with $i < j$, so that black edges cannot join the node labeled $w$ with nodes labeled either $w + 1$ or $w - 1$.

In order to compute the time complexity, we have to add the contribution of the following four actions: the recognition procedure – requiring $O(n)$ time [3], the labeling of $P_1$, the partial labeling of each piece and the completion of the labeling. In order to label each piece $P_i$ with $n_i$ nodes we need $O(n_i)$ time.
Each piece $P_i$ is enqueued in $S$ at most twice, once when it is recognized and possibly a second time if it is only partially labeled. It follows that the algorithm, without the recognition part, requires no more than $\sum_{i=1}^{n} O(n_i) = O(n)$ time; consequently, the whole algorithm needs $O(n)$ time.

Theorem 6.2 Algorithm $L(2,1)$–Label-Unigraphs has a performance ratio of $3/2$.

Proof: The nodes of a unigraph are partitioned into three classes, $V_K$, $V_S$ and $V_C$.

Nodes of the clique induced by $V_K$ must be labeled with colors at mutual distance at least two. Hence, $2|V_K| - 1$ colors are necessary in any labeling for these nodes, but only $|V_K|$ of them are used to label $V_K$. Due to the unigraph structure, the $V_K - 1$ remaining colors could be used for some nodes in $V_S$ but not for the nodes in the crown, as each of them is connected to every node in $V_K$. For this reason, the nodes in $V_C$ must be at distance of at least two from the colors used for $V_K$. Hence the color successive to the maximum used for the clique cannot be used for the crown, so one more color must be added.

Moreover, nodes in the crown induced by $V_C$ must all be different from each other (except for the special case when the unigraph coincides with its crown). Let $|V_C| + \alpha$, where $0 \leq \alpha \leq |V_C|/2 - 1$, be the optimum number of colors necessary for labeling these nodes. Among the $|V_C| + \alpha$ colors, only $|V_C|$ are really used, while $\alpha$ colors could be used for other nodes in $V_S$.

For nodes in $V_S$, we have to distinguish whether they belong to $P_1$ or not, as only in the first case some colors can be repeated (cf. Section 5). Let us call $\beta$, $\beta \leq |P_1 \cap S|$ the optimum number of colors necessary to label nodes of $P_1 \cap V_S$ and $S'$ the set of nodes in $S$ not belonging to $P_1$, i.e. $S' = S - (P_1 \cap S)$.

In the worst case, algorithm $L(2,1)$–Label-Unigraphs is not able to use colors that remain unused after the coloring of $V_K$ and $V_C$. So, the number of used colors is upper bounded by $2|V_K| - 1 + |V_C| + \alpha + 1 + \beta + |S'|$.

Let us now consider the optimum solution. We have to distinguish two cases according to the fact that the number of colors not used in $V_K \cup V_C$ is sufficient for labeling $V_S$ or not:

- If $\beta + |S'| \leq |V_K| + \alpha$, the number of colors used by the optimum solution is lower bounded simply by $2|V_K| - 1 + |V_C| + \alpha + 1$.
- If, on the contrary, $\beta + |S'| > |V_K| + \alpha$, we have to add $|S'| + \beta - |V_K| - \alpha$ colors in order to obtain a lower bound for the optimum solution of $2|V_K| - 1 + |V_C| + \alpha + 1 + (|S'| + \beta - |V_K| - \alpha) = |V_K| + |V_C| + |S'| + \beta$.

Now we compute the approximation ratio in the two cases, using as measure the ratio between the number of colors used by our algorithm and the number of colors used by the optimum solution, i.e. $\frac{\lambda + 1}{\lambda^* + 1}$. By exploiting that $\alpha \leq |V_C|/2 - 1$ and hence $|V_C| \geq 2\alpha + 2 > 2\alpha$, that $\alpha \geq 0$ and the relationships between $\beta + |S'|$ and $|V_K| + \alpha$ we have:

- If $\beta + |S'| \leq |V_K| + \alpha$ then
  $$\frac{\lambda + 1}{\lambda^* + 1} \leq \frac{2|V_K| + |V_C| + |S'| + \alpha + \beta}{2|V_K| + |V_C| + \alpha} \leq 1 + \frac{|S'| + \beta}{2|V_K| + |V_C| + \alpha} < \frac{3}{2}$$
• If $\beta + |S'| > |V_K| + \alpha$ then
  \[
  \frac{\lambda + 1}{\lambda^* + 1} \leq \frac{2|V_K| + |V_C| + |S'| + \alpha + \beta}{|V_K| + |V_C| + |S'| + \beta} \leq 1 + \frac{|V_K| + \alpha}{|V_K| + |V_C| + |S'| + \beta} < \frac{3}{2}.
  \]

Observe that when the unigraph is constituted only by its crown our algorithm provides the optimum labeling, according to Theorems of Subsection 5.1. Furthermore, it is not difficult to see that, if the input unigraph is either a threshold or a matrogenic graph then our algorithm behaves exactly in the same way as the known algorithms specifically designed for these classes of graphs and hence, in the case of threshold graphs, it provides the optimum labeling.

7 Concluding remarks and open problems

In this paper we have answered the open problem left in [6] to present an $L(2,1)$-labeling for unigraphs. In Theorem 6.2 we prove that its approximation ratio is $3/2$, nevertheless a large number of examples show that our algorithm discards very few colors, thus achieving a number of used colors which is very close to the optimal value, so we suspect that its performance is even better.

We would like to conclude this paper with two considerations concerning the number of colors used by our algorithm in comparison to the minimum value $\lambda$.

First, observe that the number of colors used by any optimal $L(2,1)$-labeling must respect the following facts:

1. the nodes in $V_K$ must be labeled with colors at a mutual distance of at most two;
2. the nodes in $V_C$ must all be different from each other and at distance of at least two from the colors used for $V_K$ (except for the special case when the unigraph coincides with its crown);
3. the nodes in $V_S \cap P_i$, $i > 1$ must all use different colors.

The foregoing facts imply that our algorithm may use a larger number of colors than is strictly necessary in the worst case; this number may be equal to the number of discarded colors.

Second, it is important to note that the use of optimal labelings for all pieces allows us to get a minimal (not a minimum) labeling of the whole unigraph so we cannot guarantee it is optimal. The reason for this is that a different arrangement of colors of the nodes in the clique may lead to less new colors being used in the second REPEAT cycle. From this consideration we conjecture that the $L(2,1)$-labeling problem may be NP-hard for unigraphs, and we leave the proof of this as an open problem.
References


