Optimal $L(\delta_1, \delta_2, 1)$-Labeling of Eight-Regular Grids

Tiziana Calamoneri

*Computer Science Department, “Sapienza” University of Rome, Italy

Abstract

Given a graph $G = (V, E)$, an $L(\delta_1, \delta_2, \delta_3)$-labeling is a function $f$ assigning to nodes of $V$ colors from a set $\{0, 1, \ldots, k_f\}$ such that $|f(u) - f(v)| \geq \delta_i$ if $u$ and $v$ are at distance $i$ in $G$. The aim of the $L(\delta_1, \delta_2, \delta_3)$-labeling problem consists of finding a coloring function $f$ such that the value of $k_f$ is minimum. This minimum value is called $\lambda_{\delta_1, \delta_2, \delta_3}(G)$.

In this paper we study this problem on the eight-regular grids for the special values $(\delta_1, \delta_2, \delta_3) = (3, 2, 1)$ and $(\delta_1, \delta_2, \delta_3) = (2, 1, 1)$, providing optimal labelings. Furthermore, exploiting the lower bound technique, we improve the known lower bound on $\lambda_{3,2,1}$ for triangular grids.

Keywords: $L(3, 2, 1)$-labeling, $L(2, 1, 1)$-labeling, eight-regular grids (ERGs), triangular grids, channel assignment problem.

1. Introduction

The enormous growth of wireless networks has made very important the issue of efficiently using the radio spectrum, expensive and scarce resource. The wide class of channel assignment problems (CAPs) consists in assigning channels (frequencies) from the available radio spectrum to the transceivers of the network, so that unconstrained simultaneous transmissions cannot cause interference. The objective is, of course, to minimize the used radio spectrum.

Such kind of problems arose about forty years ago in TV broadcasting, but are still cool due to the wide diffusion of telephone and satellite communication. There are many kinds of CAPs, and all have two things in common: they have to assign to transceivers a set of disjoint channels (frequencies) obtained partitioning the radio spectrum, and they have to respect a set of constraints making possible a communication without interference.

In this paper, we focus on a fixed channel assignment (FCA) strategy, according to which channels are statically assigned to the transceivers for their exclusive and permanent use, and remain stable over time [1]. According to this strategy, if we consider the interference graph of the network, each CAP can be seen as a special node coloring problem. Formally, given a graph $G = (V, E)$ modeling the wireless network, where nodes represent transceivers and edges between two transceivers represent possible interference, a CAP consists of finding an integer $k_f > 0$ so that there is a node coloring function $f : V \to \{0, \ldots, k_f\}$ such that the given separation constraints are satisfied. The objective is to minimize the span of $f$, i.e. the value of $k_f$, while avoiding interference. Since radio signals get attenuated over distance, the same channel can be reused by two transceivers without causing interference between two contemporary transmissions if they are far enough. Furthermore, channels assigned to close transceivers must be separated in value by a gap that is inversely proportional to the distance between them. On the basis of these observations, the CAP called $L(\delta_1, \ldots, \delta_{\sigma-1})$-labeling has

---

©Partially supported by “Sapienza” University of Rome, project
"Graph Labeling for modeling Wireless Networks Problems"

Email address: calamo@di.uniroma1.it (Tiziana Calamoneri)
been introduced and is defined as follows.

Given a graph $G = (V, E)$, an $L(\delta_1, \ldots, \delta_{\sigma-1})$-labeling is a function \( f : V \to \{0, 1, \ldots, k_f\} \) such that \( |f(u) - f(v)| \geq \delta_i \) if $u$ and $v$ are at distance $i$ in the graph, \( i = 1, \ldots, \sigma - 1 \). $\sigma$ is said to be the reuse distance, while $k_f$ is said to be the span of the assignment $f$. The aim of the $L(\delta_1, \ldots, \delta_{\sigma-1})$-labeling problem is to satisfy the distance constraints using the minimum span. This minimum value is denoted by $\lambda_{\delta_1,\ldots,\delta_{\sigma-1}}(G)$.

The decisional version of the $L(\delta_1, \ldots, \delta_{\sigma-1})$-labeling problem is NP-complete for general graphs even for small values of $\sigma$ and of $\delta_i, i = 1, 2, \ldots, \sigma - 1$ [11, 12]. This motivates seeking optimal solutions on particular classes of graphs, for special values of $\sigma$ and of $\delta_i, i = 1, 2, \ldots, \sigma - 1$.

Among the possible values of these parameters, an extensive study has been done in the case in which the reuse distance $\sigma$ is 3 (for a comprehensive survey, see [4]). However, in many networks such a small reuse distance is not feasible and $\sigma$ is expected to be larger [14]. So, the study of the case $\sigma = 4$ becomes necessary. In view of the extent of the results when $\delta_1, \delta_2$ and $\delta_3$ vary (for each choice of values to the triple $(\delta_1, \delta_2, \delta_3)$ there is a different optimal coloring assignment), the literature concentrates on special values of these parameters: $\delta_1 = 2, \delta_2 = \delta_3 = 1$ and $\delta_1 = \sigma - i, i = 1, 2, 3$. For surveys on these cases, see [3, 5, 6].

Cellular networks often cover service areas by nearly congruent polygonal cells, with each transceiver at the center of the cell it covers. Varying the mutual position and the power of the transceiver, different regular grids can model the network.

In the following table we recall all the known results for what concerns the hexagonal grid $G_3$ (each transceiver interferes with three neighbors, also called honeycomb grid), the squared grid $G_4$ (each transceiver interferes with four neighbors), the triangular grid $G_6$ (each transceiver interferes with six neighbors, also called cellular grid) and the eight-regular grid (ERG) $G_8$ (each transceiver interferes with eight neighbors).

<table>
<thead>
<tr>
<th>Grid type</th>
<th>$\lambda_{2,1,1}$</th>
<th>$\lambda_{3,2,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>triangular grid $G_6$</td>
<td>11 [2]</td>
<td>$16 \leq \lambda_{3,2,1} \leq 19$ [9]</td>
</tr>
<tr>
<td>eight-regular grid $G_8$</td>
<td>18 $\leq \lambda_{3,2,1} \leq 30$ [8]</td>
<td></td>
</tr>
</tbody>
</table>

Both problems are closed on $G_3$ and $G_4$. On the contrary, lower and upper bound on $\lambda_{3,2,1}$ do not coincide in the case of eight-regular grids (ERGs) and triangular grids while - to the best of our knowledge - the $L(2,1,1)$-labeling problem on ERGs has never been studied.

In this paper, we study the $L(3,2,1)$- and $L(2,1,1)$-labeling problem on ERGs, and we completely solve it, by providing coinciding lower and upper bounds on $\lambda_{3,2,1}(G_8)$ and on $\lambda_{2,1,1}(G_8)$. In particular, we show that $\lambda_{3,2,1}(G_8) = 23$ and $\lambda_{2,1,1}(G_8) = 15$.

Furthermore, by applying the lower bound technique to the triangular grid, we make closer upper and lower bounds on $\lambda_{3,2,1}(G_6)$, showing that $17 \leq \lambda_{3,2,1}(G_6) \leq 19$.

We discuss some considerations that lead us to conjecture that the exact value coincides with the upper bound, i.e. that $\lambda_{3,2,1}(G_6) = 19$.

## 2. $L(3,2,1)$- and $L(2,1,1)$-Labelings of ERGs

We consider the ERG topology $G_8$, that can be obtained by a squared grid by adding two edges to connect the pairs of diagonal nodes of every square (see Fig. 1.a). As we are not interested in what happens on the boundary of the grid, we assume that it is infinite.

In this section we prove the main result of the paper, that is $\lambda_{3,2,1}(G_8) = 23$, by showing that 23 is both a lower and an upper bound for $\lambda_{3,2,1}(G_8)$. We remind that the best previously known result is $18 \leq \lambda_{3,2,1}(G_8) \leq 30$ [8] and the authors conjecture that the value 30 is optimal;
here we improve both the upper and the lower bound, and disprove the conjecture.

Subsequently, we exploit the same techniques to prove that $\lambda_{2,1,1}(G_8) = 15$.

We fix an arbitrary point to be the origin $O$ of a coordinate system, whose axes are parallel to the edges of the underlying square grid, so that the position of each node is defined by a couple of integer coordinates (see Fig. 1.a).

![Figure 1: a. An ERG $G_8$ where the coordinates of some nodes are highlighted; b. the subgraph $G_{4\times4}$ of $G_8$.](image)

The result $\lambda_{3,2,1}(G_8) = 23$ comes out from the following two theorems.

**Theorem 1.** $\lambda_{3,2,1}(G_8) \geq 23$.

**Proof.** Consider an optimal $L(3,2,1)$-labeling $f$ of $G_8$ with span $\lambda_{3,2,1}(G_8) = \lambda$ and consider one of its $(4 \times 4)$-subgrids. Then, the colors 0 and $\lambda$ are not used in at least two of the four disjoint $(2 \times 2)$-subgrids of the selected $(4 \times 4)$-subgrid. Name the nodes of this $(2 \times 2)$-subgrid with $x_1$, $x_2$, $x_3$ and $x_4$ and consider the $(4 \times 4)$-subgrid having them as internal nodes. Call this subgrid $G_{4\times4}$ and refer to Figure 1.b. All the nodes in $G_{4\times4}$ are at mutual distance at most three. It follows that $f$ must use at least 16 distinct colors to label $G_{4\times4}$.

Nodes $x_1, \ldots, x_4$ are all at mutual distance at most two from any other node in $G_{4\times4}$. This means that if $x_i$ is colored by color $c_i$, $i = 1, \ldots, 4$, then neither color $c_i - 1$ nor color $c_i + 1$ can be used for any other node in $G_{4\times4}$, and these colors are thrown away, i.e. they cannot be used for any node in $G_{4\times4}$ (although they could be used to color nodes that are outside this subgraph). Furthermore, nodes $x_1, \ldots, x_4$ are all adjacent, so colors $c_1, \ldots, c_4$ must be at mutual distance at least three, and it cannot happen that for example $c_i + 1$ coincides with $c_j - 1$ for some $i, j \in \{1, 2, 3, 4\}$.

Since all $c_i$’s, $i = 1, \ldots, 4$, are different from 0 and $\lambda$, exactly 8 colors are thrown away and so at least $16 + 8 = 24$ colors, from 0 to 23, are used by $f$ for $G_{4\times4}$. This concludes the proof.

**Theorem 2.** $\lambda_{3,2,1}(G_8) \leq 23$.

**Proof.** We provide a feasible labeling using at most 24 colors, from 0 to 23. Let $f$ be defined as follows.

$$f(x,y) = \begin{cases} 
4x \mod 24 & \text{if } y \equiv 0 \mod 4 \\
(4x + 15) \mod 24 & \text{if } y \equiv 1 \mod 4 \\
(4x + 2) \mod 24 & \text{if } y \equiv 2 \mod 4 \\
(4x + 13) \mod 24 & \text{if } y \equiv 3 \mod 4 
\end{cases}$$

Observe that this function is well defined because to each node in $G_8$ corresponds one color; furthermore, function $f$ defines a pattern of $6 \times 4$ colors that is replicated both horizontally and vertically, as highlighted in Figure 2.a.

Consider any two nodes of $G_8$ $a = (x_a, y_a)$ and $b = (x_b, y_b)$; their distance is $\max(|x_a - x_b|, |y_a - y_b|)$. It is not difficult, though boring, to exhaust all cases of the possible mutual positions of $a$ and $b$ and of possible remainders of the division between $y_a$ and 4, so formally proving that the colors assigned above always produce a feasible $L(3,2,1)$-labeling. For the sake of brevity, we omit this case by case proof, even because it is easy to verify on Figure 2.a that function $f$ provides a valid $L(3,2,1)$-labeling.

Now, we focus our attention on the $L(2,1,1)$-labeling of ERGs. The following two theorems prove that $\lambda_{2,1,1}(G_8) = 15$.

**Theorem 3.** $\lambda_{2,1,1}(G_8) \geq 15$. 
Figure 2: a. An optimal $L(3,2,1)$-labeling of $G_8$; b. an optimal $L(2,1,1)$-labeling of $G_8$.

**Proof.** Observe that the nodes of the subgraph $G_{4\times 4}$ of $G_8$, depicted in Figure 1 are at mutual distance at most three, so they must receive all different colors by any $L(2,1,1)$-labeling. Hence, any such labeling uses at least 16 colors, from 0 to 15, i.e. $\lambda_{2,1,1}(G_8) \geq 15$.  

**Theorem 4.** $\lambda_{2,1,1}(G_8) \leq 15$.

**Proof.** The proof of this statement provides a labeling of $G_8$ through a pattern of $4 \times 4$ colors that is replicated both horizontally and vertically, as depicted in Figure 2.b and formalized as follows:

$$f(x,y) = \begin{cases} 
2x \mod 8 & \text{if } y \equiv 0 \mod 4 \\
2x \mod 8 + 9 & \text{if } y \equiv 1 \mod 4 \\
2x \mod 8 + 1 & \text{if } y \equiv 2 \mod 4 \\
2x \mod 8 + 8 & \text{if } y \equiv 3 \mod 4 
\end{cases}$$

It is not difficult to exhaustively prove that this labeling is feasible.

**3. $L(3,2,1)$-Labeling of Triangular Grids**

We consider the infinite triangular grid topology $G_6$ (shown in Fig. 3.a).

It is known that $16 \leq \lambda_{3,2,1}(G_6) \leq 19$ [7]. In this section we improve the lower bound from 16 to 17. Furthermore, due to some considerations, we conjecture that the exact value is 19.

**Theorem 5.** $\lambda_{3,2,1}(G_6) \geq 17$.

**Proof.** We exploit the same technique we used to prove Theorem 1 so we omit here some details of the same discussions that are obvious after reading that proof. Let $f$ be an optimal $L(3,2,1)$-labeling of $G_6$ with span $\lambda$ and consider one of the infinite number of subgraphs of $G_6$ shaped as in Figure 3.b. The 12 nodes of this subgraph are all at mutual distance at most three and so they require at least 12 distinct colors. If colors 0 and $\lambda$ are among these colors, it is anyway possible to find inside this subgraph a triangle whose nodes are labeled neither with 0 nor with $\lambda$. Call $x_1$, $x_2$, $x_3$ the nodes of this triangle.
Let $G_{3\times3}$ be the subgraph of $G_6$ shaped as in Figure 3.b and having the three nodes $x_1$, $x_2$ and $x_3$ as internal nodes. $x_1$, $x_2$ and $x_3$ are at distance at most two from any other nodes of $G_{3\times3}$, and so we have to throw away the 6 colors adjacent to the colors assigned to these three nodes. It follows that $G_{3\times3}$ needs $12+6=18$ colors, from 0 to 17.

Observe that if we try to exploit the same technique we used to prove Theorem 2 to obtain an improved upper bound, we got an $L(3,2,1)$-labeling needing a number of colors that is multiple of 4. In particular, it produces a feasible labeling using 20 colors, meaning that $\lambda(G_6) \leq 19$, that is also the value provided in [7].

Alternatively, we could think of applying a technique similar to the one in [2] exploited for the $L(2,1,1)$-labeling of $G_6$ that consists of covering $G_6$ with an infinite number of disjoint tiles having the shape of $G_{3\times3}$ and equally labeled. Even this method does not produce any improved result, since if we assign to any node $x$ of a $G_{3\times3}$ a color $c$, colors $c-1$ and $c+1$ cannot be used for any other node of $G_{3\times3}$ as they are at distance at most two either from $x$ or from a translational copy of $x$ in another close tile.

From the other side, the authors of [13] characterize all the optimal channel assignment schemes for triangular grids showing that they must be based on a replicated pattern. For these reasons, we state the following conjecture:

**Conjecture 6.** $\lambda_{3,2,1}(G_6) = 19$. 

Hence, we think that it should be possible to improve the lower bound, and we leave this issue as open problem.

**Acknowledgments**

The author would like to thank the anonymous reviewers for their careful reading of this paper and for their constructive suggestions, contributing to improve the clarity of the exposition.

**References**


