On the Approximability of the $L(h,k)$-Labelling Problem on Bipartite Graphs ⋆ (Extended Abstract)

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Abstract. Given an undirected graph G, an $L(h,k)$-labelling of G assigns colors to vertices from the integer set $\{0,\ldots,\lambda_{h,k}\}$, such that any two vertices v_i and v_j receive colors $c(v_i)$ and $c(v_j)$ satisfying the following conditions: *i)* if v_i and v_j are adjacent then $|c(v_i)-c(v_j)| \geq h$; *ii)* if v_i and v_j are at distance two then $|c(v_i)-c(v_j)| \geq k$. The aim of the $L(h,k)$-labelling problem is to minimize $\lambda_{h,k}$. In this paper we study the approximability of the $L(h,k)$-labelling problem on bipartite graphs and extend the results to s-partite and general graphs. Indeed, the decision version of this problem is known to be NP-complete in general and, to our knowledge, there are no polynomial solutions, either exact or approximate, for bipartite graphs.

Here, we state some results concerning the approximability of the $L(h,k)$-labelling problem for bipartite graphs, exploiting a novel technique, consisting in computing approximate vertex- and edge-colorings of auxiliary graphs to deduce an $L(h,k)$-labelling for the input bipartite graph. We derive an approximation algorithm with performance ratio bounded by $\frac{4}{3}D^2$, where, D is equal to the minimum even value bounding the minimum of the maximum degrees of the two partitions.

One of the above coloring algorithms is in fact an approximating edge-coloring algorithm for hypergraphs of maximum dimension d, i.e. the maximum edge cardinality, with performance ratio d.

Furthermore, we consider a different approximation technique based on the reduction of the $L(h,k)$-labelling problem to the vertex-coloring of the square of a graph. Using this approach we derive an approximation algorithm with performance ratio bounded by $\min(h,2k)\sqrt{n} + o(k\sqrt{n})$, assuming $h \geq k$. Hence, the first technique is competitive when $D = O(n^{1/4})$.

These algorithms match with a result in [2] stating that $L(1,1)$-labelling n-vertex bipartite graphs is hard to approximate within $n^{1/2-\epsilon}$, for any $\epsilon > 0$, unless NP=ZPP.

We then extend the latter approximation strategy to s-partite graphs, obtaining a $(\min(h,sk)\sqrt{n} + o(sk\sqrt{n}))$-approximation ratio, and to general graphs deriving an $(h\sqrt{n}+o(h\sqrt{n}))$-approximation algorithm, assuming $h \geq k$.

Finally, we prove that the $L(h,k)$-labelling problem is not easier than coloring the square of a graph.

1 Introduction

The *frequency assignment problem (FAP)* in a wireless network is a widely studied problem (see [1, 15] for a survey). A wireless network consists of a set of radio transmitter/receiver stations distributed over a region. Communication takes place by a node broadcasting a signal over a fixed range (whose size is proportional to the power expended by the node's transmitter). Any receiver within the range of the transmitter can receive the signal. In this context, the frequency assignment

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task is to assign radio frequencies to transmitters at different locations without causing interference. This situation can be modelled by a graph G, whose nodes are the radio transmitters/receivers, and the adjacencies indicate possible communications and, hence, interferences. To avoid interference, two adjacent stations must receive far frequencies. Therefore, the problem is closely related to graph coloring, where colors represent possible frequencies.

Many graph coloring models have been proposed to represent the FAP problems depending on the specific features of the problem (i.e. handling the interference, the availability of frequencies etc.) [15]. Among all these models the most widely accepted for the interferences avoidance is the $L(h,k)$-*labelling*, introduced by Griggs and Yeh [12] in the special case $h = 2$ and $k = 1$. The $L(h,k)$-labelling problem is a coloring problem with some constraints araising from practical reasons: a radio station and its neighbors must have far frequencies, at least h apart, so their signals will not interfere (*direct collision*); furthermore, a radio station must have a signal of frequency different at least k from the radio stations adjacent to its neighbors (*hidden collision*). The nature of the environment and the geographical distance are the major factors determining parameters h and k, and usually, $h \geq k$ holds.

It has been proven that the decision version of the $L(h,k)$-labelling problem is NP-complete even for $h \in \{1,2\}$ and $k = 1$ [14, 13, 12]. Therefore, the problem has been widely studied for many specific classes of graphs. For some classes of graphs the problem has been proved to be polynomially solvable and for other classes to be approximable within a constant factor (see [7] for a survey on the $L(h,k)$-labelling, and, for instance, [6, 8, 10, 11, 18] for some specific results).

In this paper we focus on the approximability of the $L(h,k)$-labelling problem on bipartite graphs, for any constant $h \geq 2$ and $k \geq 1$. It is easy to see that when $h = k = 1$, the $L(h,k)$-labelling problem on a graph G is equivalent to vertex-coloring G^2, where G^2 is the square graph of G, and a wide literature is known in this case (e.g., see [3, 14]).

Concerning **bipartite graphs**, previous best known results deal only with $h = 2$ and $k = 1$ [5], where the authors prove that the decision version of the $L(2,1)$-labelling problem is NP-complete also for planar bounded degree ($\Delta = 7$) bipartite graphs and they present an infinite class of bipartite graphs requiring at least $\frac{\Delta^2}{4}$ colors, where Δ is the maximum degree. We improve the lower bound by a constant factor, and we present two approximation algorithms whose approximation ratios depend on the degree and on the dimension of the vertex set, respectively. In particular the first algorithm exploits a novel technique, consisting in computing the approximate vertex- and edge-colorings of auxiliary graphs to deduce an $L(h,k)$-labelling for the input bipartite graph. The second approximation technique is based on the reduction of the $L(h,k)$-labelling problem to the vertex-coloring of the square graph. This strategy has the advantage to be extendable to s-partite and general graphs.

The obtained results are listed below:

- We improve the lower bound by a constant factor of $\frac{1}{4}$.
- For n-vertex general bipartite graphs we derive two approximation algorithms with performance ratio:
  - $\frac{4}{3}D^2$, where D is equal to the minimum even value bounding the minimum of the maximum degrees of the two partitions; this result is improved to $\frac{9}{2}$ if one of the two partitions has regular degree 2;
  - $\min(h, 2k)\sqrt{n} + o(k\sqrt{n})$, assuming $h \geq k$.

  Hence, the first technique is competitive when $D = O(n^{1/4})$.

These algorithms match with a result in [2] stating that $L(1,1)$-labelling n-vertex bipartite graphs is hard to approximate within $n^{1/2-\epsilon}$, for any $\epsilon > 0$, unless NP=ZPP.

The technique used to derive the approximation algorithm for general bipartite graphs straightforwardly derives from an edge-coloring approximation algorithm for hypergraphs. More precisely, given an n-vertex hypergraph $\mathcal{H}$ with maximum edge-cardinality d, we describe a d–approximation algorithm for the edge-coloring. To the knowledge of the authors, the best known previous results concern d–uniform hypergraphs, and the approximation ratio is a function of the vertex degree instead of the edge's dimension [4, 16].

Additionally, we extend the last result to s-**partite graphs**, obtaining a $(\min(h, sk)\sqrt{n} + o(sk\sqrt{n}))$-approximation ratio.

Finally, for what concerns **general graphs**:

- We present a $(h\sqrt{n} + o(h\sqrt{n}))$-approximation algorithm, assuming $h \geq k$;
- We prove that the $L(h,k)$-labelling problem is not easier than coloring the square of a graph.

Note that no results are known about the approximability of the $L(h,k)$-labelling problem on general graphs, while for the coloring of square graphs an $O(\sqrt{n})$-approximation algorithm exists [14]. In this context our result strongly relates the approximability of these two problems.

2 Definitions and Preliminary Results

In this section, we recall some basic concepts and known results, and introduce some definitions useful for the rest of the paper.

Definition 1. *An* $L(h,k)$-labelling *of a graph* $G = (V, E)$ *is a function* f *from* V *to the set of all nonnegative integers such that*
1. $|f(x) - f(y)| \geq h$ *if* x *and* y *are at distance* 1 *in* G*;*
2. $|f(x) - f(y)| \geq k$ *if* x *and* y *are at distance* 2 *in* G*;*
for some fixed integer values $h, k \geq 1$*.*

The *span* of an $L(h,k)$-labelling of a graph G is the difference between the maximum and minimum value of f. It is not restrictive to assume that the minimum value is 0, so the span coincides with the maximum value of f.

The $L(h,k)$-*number* of G, denoted by $\lambda^*_{h,k}(G)$ (or simply $\lambda^*(G)$, when the values of h and k are clear from the context), is the minimum span, over all $L(h,k)$-labellings of G. The task of the $L(h,k)$-labelling problem is to determine $\lambda^*(G)$. Although not necessary for our reasonings, we assume $h \geq k$, as suggested by real applications.

Let $G = (V, E)$ be a (multi)graph. In the following we denote by $\Delta(G)$ the maximum degree of G. Consider an optimal vertex-coloring of G and an optimal edge-coloring of G, let $\chi^*(G)$ and $\chi'^*(G)$ be its *chromatic number* and *chromatic index*, respectively, and let $\chi(G)$ and $\chi'(G)$ denote the number of colors used by an approximation algorithm for coloring vertices and edges of graph G, respectively.

Here, we recall some results relating these quantities.

Theorem 1. *[19, 17] The chromatic index of any graph* G *of maximum degree* $\Delta(G)$ *satisfies* $\Delta(G) \leq \chi'^*(G) \leq \Delta(G) + 1$*. If* G *is a multigraph then* $\chi'^*(G) \leq \frac{3}{2}\Delta(G)$*.*

Since the proof is constructive, we have the following result:

Corollary 1. *There is an algorithm for coloring the edges of any graph (multigraph, respectively)* G *with maximum degree* $\Delta(G)$*, that guarantees a performance ratio of* $1 + \frac{1}{\Delta(G)}$ *(*$\frac{3}{2}$*, respectively).*

Theorem 2. *[9] There is a simple greedy algorithm for coloring the vertices of any (multi)graph* G *with maximum degree* $\Delta(G)$*, with at most* $\Delta(G) + 1$ *colors (i.e.* $\chi(G) \leq \Delta(G) + 1$*).*

Corollary 2. *For any (multi)graph* G*,* $\chi(G) \leq \chi'(G) + 1$*.*

Let $B = (U \cup V, E)$ be a bipartite graph. The sets U and V are defined as *upper* and *lower set*, respectively, in view of the usual graphical representation of bipartite graphs, although – of course – they can be freely interchanged. Let Δ_U and Δ_V be the maximum degrees of the vertices in the upper and lower set, respectively, and let $\delta(x)$ denote the degree of a vertex x. We introduce the following three structures.

The first one is a multigraph associated with a bipartite graph with one partition containing only vertices of degree exactly two.

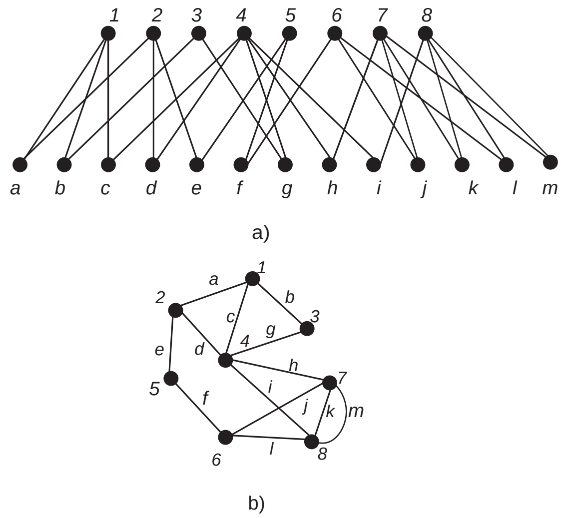

Fig. 1. a) A bipartite graph B; (b) Its incident graph $I(B)$.

Definition 2. *Let $B = (U \cup V, E)$ be a bipartite graph with one partition containing only vertices of degree exactly two (w.l.o.g. let it be V). The* incidence graph *$I(B) = (U, E')$ is a multigraph defined as follows:*

i. *The vertex set corresponds to the upper set U of B;*
ii. *The edge set E' corresponds to the lower set V of B. For every vertex $e \in V$, such that $(u, e) \in E \wedge (v, e) \in E$, there exists an edge $(u, v) \in E'$.*

The incidence graph of the bipartite graph in Fig. 1 (a) is shown in Fig. 1 (b).

It is straightforward to see that all vertices in $I(B)$ have the same degree as they have in the upper set of B. Observe that any (multi)graph is the incidence graph of a bipartite graph with all vertices in the lower set of degree 2.

The second structure is a generalization of the incidence graph, extended to even-degree bipartite graphs. Assume that each vertex x of the lower set V in the bipartite graph $B = (U \cup V, E)$ has even degree $\delta(x)$, and that an ordering of the edges incident at each vertex is given. Let $\langle e_1^x, e_2^x, \ldots, e_{\delta(x)}^x \rangle$ denote the ordered sequence of edges incident at $x \in V$.

Definition 3. *Given a bipartite graph $B = (U \cup V, E)$ as above, the* extended incidence graph *$Ext(B) = (U, E'')$ of B is a (multi)graph defined as follows:*

– *the vertex set corresponds to the upper set U of B;*
– *for every vertex $x \in V$ and each couple $e_{2j+1}^x = \{x, u\} \in E$ and $e_{2j+2}^x = \{x, v\} \in E$, where $j \in \{0, \ldots, \delta(x)/2 - 1\}$, there exists an edge $(u, v) \in E''$.*

Fig. 2 (b) shows the extended incidence graph of the graph in Fig. 2 (a).

Each vertex of $Ext(B)$ maintains the same degree it has in B, therefore $\Delta(Ext(B)) = \Delta(B)$. Furthermore, observe that a vertex $x \in V$ generates in $Ext(B)$ a set of $\delta(x)/2$ edges, denoted *Set(x)*. In Fig. 2 (b), $Set(a) = \{a', a''\}$, $Set(b) = \{b'\}$, $Set(c) = \{c', c''\}$, $Set(d) = \{d', d''\}$, $Set(e) = \{e'\}$, and, $Set(f) = \{f', f''\}$.

Note that, the extended incident graph represents a hypergraph where each $v \in V$ is a hyperedge.

The third structure is associated to a general bipartite graph and represents nodes in the upper set and the relations they have through nodes in the lower set.

Definition 4. *The* node-graph *$N(B) = (U, E''')$ of a bipartite graph $B = (U \cup V, E)$ is a simple graph defined as follows:*

– *the vertex set corresponds to the upper set U of B;*

– *an edge* $(u, v) \in E'''$ *if and only if there exists a vertex* x *in the lower set of* B *such that* $(u, x) \in E$ *and* $(x, v) \in E$.

In Fig. 2 (c) the node-graph of the graph in Fig. 2 (a) is depicted. The maximum degree $\Delta(N(B))$ of $N(B)$ is bounded by $min\{\Delta_U \cdot (\Delta_V - 1), |U|\}$.

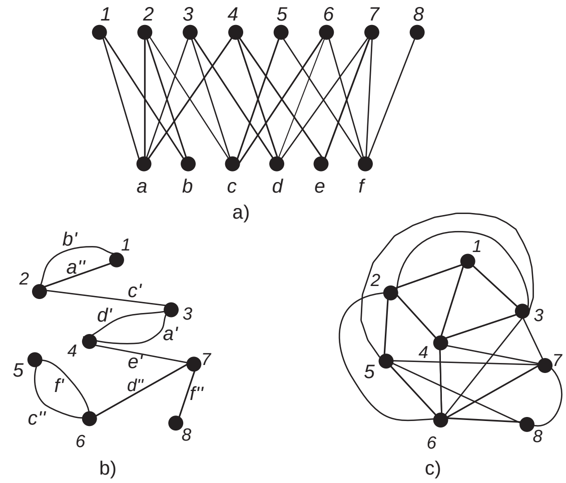

Fig. 2. (a) A bipartite graph B; (b) The extended incidence graph $Ext(B)$; (c) The node graph $N(B)$;

3 Bipartite Graphs: Lower Bound

In [5], Bodlaender et al. proved that for every $\Delta \geq 2$, there is a bipartite graph with maximum degree Δ such that $\lambda^*_{0,1} \geq \frac{\Delta^2}{4}$. Since $\lambda^*_{0,1} \leq \lambda^*_{h,1}$, for any $h > 0$, the value $\frac{\Delta^2}{4}$ represents a lower bound for $\lambda^*_{h,k}$ of bipartite graphs. We improve this bound to $\Delta^2 - \Delta + 1$.

Definition 5. *Let* $\mathcal{B}$ *be the class of bipartite graphs* $B = (U \cup V, E)$ *satisfying the following conditions:*

a. *Both the upper set* U *and the lower set* V *have* $q^2 + q + 1$ *vertices, for any integer* $q \geq 0$;
b. *Any vertex* $u \in U \cup V$ *has exactly degree* $\Delta = q + 1$;
c. *Given* $u, v \in U$ *(or,* $u, v \in V$*), then* $|Adj(v) \cap Adj(u)| = 1$.

We show that the cardinality of $\mathcal{B}$ is infinite. Let us consider a projective plane $\mathcal{P}$ of order q, for some prime power q. Let U represent points and V lines of $\mathcal{P}$. For $u \in U$ and $v \in V$, $(u, v) \in E$ if and only if point u belongs to line v. It is straightforward to verify that the bipartite graph so defined satisfies the constraints in Definition 5 and, since all vertices in U (or in V) are at distance 2, we have $\lambda^*_{0,1}(B) \geq |U| = q^2 + q + 1 = \Delta^2 - \Delta + 1$.

Note that the class $\mathcal{B}$ has been introduced in [12] without deriving the lower bound. It is easy to see that this lower bound applies to s-partite graphs, for any s.

4 First approximation algorithm for bipartite graphs

The first algorithm we propose reduces the $L(h, k)$-labelling problem to both the edge- and the vertex-coloring of the (extended) incidence and the node-graphs associated with a given bipartite graph.

To make the exposition easier, the presentation proceeds in three steps. First we analyze bipartite graphs with vertices in the lower partition of regular degree 2. Then, we extend the technique to bipartite graphs with even-degree lower set. Finally, we generalize the algorithm to any bipartite graph.

4.1 Lower Set of Regular Degree 2

Let $B = (U \cup V, E)$ be a bipartite graph with all vertices in the lower set having regular degree 2. Let $I(B) = (U, E')$ be the corresponding incidence graph.

Let c_1 and c_1' be two functions for vertex- and edge-coloring $I(B)$, requiring $\chi_1(I(B))$ and $\chi_1'(I(B))$ colors, respectively.

These two colorings can be exploited to deduce an $L(h,k)$-labelling of B in the following way:

- label each vertex v in the upper set of B with $k(c_1(v) - 1)$;
- label each vertex e in the lower set B with $(\chi_1(I(B)) - 1)k + h + k(c_1'(e) - 1)$,

where the term -1 is due to the fact that the smallest value for c_1 and c_1' is 1, while for the $L(h,k)$-labelling it is 0. The term $(\chi_1(I(B)) - 1)k$ represents the largest color used in the upper set, and term h is the separation value.

It is easy to prove that the labelling produced is indeed an $L(h,k)$-labelling. The span of this labelling is $k(\chi_1(I(B)) + \chi_1'(I(B))) + h - 2k$.

Hence, from Corollary 2, we have:

$$\lambda(B) \leq 2k\chi_1'(I(B)) + h - k. \quad (1)$$

Consider now an optimal $L(h,k)$-labelling of B and call $f(v)$ the color assigned to vertex v in B. We can use this labelling to achieve a feasible edge-coloring for $I(B)$: consider each vertex e in the lower set of B and its corresponding edge e of $I(B)$; label e in $I(B)$ with $\lfloor f(e)/k \rfloor + 1$.

This labelling is a feasible edge-coloring of $I(B)$ since, for any pair of adjacent edges e and e' in $I(B)$, they are at distance two in V (via the vertex in the upper set corresponding to the common end-point) and, hence, they have labels at least k apart, so $\lfloor f(e)/k \rfloor + 1$ and $\lfloor f(e')/k \rfloor + 1$ are different. Therefore, for the number $\chi_2'(I(B))$ of used colors it holds:

$$\chi'^* \leq \chi_2'(I(B)) \leq \frac{\lambda^*(B)}{k} + 1 \quad (2)$$

Furthermore, if we consider any non trivial bipartite graph B, i.e. a connected graph with at least three vertices, with maximum degree $\Delta(B)$ then $\lambda^*(B) \geq h + (\Delta(B) - 1)k$. Since, in this case, $\Delta(B) \geq 2$, we have $\lambda^*(I(B)) \geq h + k$. On the other hand, since $k \leq h$, then $k \leq \lambda^*(I(B))/2$.

From Equation 1, Corollary 1 and Equation 2, and from the above observations, we get:

$$\lambda(B) \leq 3\lambda^*(B) + 2k + h \leq \frac{9}{2}\lambda^*(B).$$

The previous reasonings lead us to the following theorem:

Theorem 3. *The $L(h,k)$-labelling problem on bipartite graphs with a partition of regular degree 2 is $\frac{9}{2}$–approximable.*

4.2 Lower Set of Even Degree

In this subsection we extend the results of the previous section to bipartite graphs having vertices in the lower set of even degree.

Let $B = (U \cup V, E)$ be a bipartite graph, with vertices in the lower set of even degree. Let Δ_U and Δ_V be the maximum degree of the upper and lower set, respectively. Consider the associated node graph $N(B)$ and the extended incidence graph $Ext(B)$.

Observe that there exists a vertex-coloring function of $N(B)$ using at most $\chi(N(B)) \leq \Delta(N(B)) + 1 \leq \Delta_U \cdot (\Delta_V - 1) + 1$ colors.

Consider now the trivial greedy algorithm to color edges of $Ext(B)$ that sequentially considers each *Set(v)* and assigns to all its edges the same smallest feasible color.

Lemma 1. *The greedy algorithm for edge-coloring $Ext(B)$ uses at most $\chi'(Ext(B)) \leq \Delta_V \cdot \Delta_U - \Delta_V + 1$ colors.*

Proof. Let us consider *Set(x)* for some $x \in V$. *Set(x)* has at most $\Delta_V/2$ vertex disjoint edges, each one of them incident at most $2(\Delta_U - 1)$ further edges; hence, to color *Set(x)* at most $2 \cdot \Delta_V/2 \cdot (\Delta_U - 1)$ colors must be avoided. The proof follows.

Corollary 3. *$Ext(B)$ can be edge-colored so that all edges in $Set(x)$, for any x, receive the same color, with a guaranteed performance ratio of Δ_V.*

Proof. The claim follows from Lemma 1 and from the obvious inequality
$\chi'^*(Ext(B)) \geq \Delta(Ext(B)) = \Delta_U$.

An immediate consequence of the above result is shown in the following theorem.

Theorem 4. *Given an n–vertex hypergraph $\mathcal{H}$ of dimension d, then there exists an approximation algorithm coloring edges of $\mathcal{H}$ with guaranteed approximation ratio of d.*

We are now ready to derive a feasible $L(h,k)$-labelling of B as follows. Let c_1 be a vertex-coloring function for $N(B)$ and let c'_1 be an edge-coloring function for $Ext(B)$, with the property that edges of *Set(x)* have all the same color, for each x. Let $\chi_1(N(B))$ and $\chi'_1(Ext(B))$ denote the number of colors required.
Proceed as follows:
– label each vertex v in the upper set of B with $k(c_1(v) - 1)$;
– label each vertex x in the lower set of B with $(\chi_1(N(B)) - 1)k + h + k(c'_1(Set(x)) - 1)$.
By the definitions of $N(B)$, $Ext(B)$, and the specific edge-coloring function c'_1, the labelling obtained is feasible and its span is $\lambda(B) = k\chi_1(N(B)) + k\chi'_1(Ext(B)) + h - 2k$.

Reminding that $\chi_1(N(B)) \leq \Delta(N(B)) + 1 \leq \Delta_U(\Delta_V - 1) + 1$ and that $\chi'_1(Ext(B)) \geq \Delta_U$ we have:

$$\lambda(B) \leq k\chi'_1(Ext(B))\Delta_V + h - k. \tag{3}$$

Similarly to Subsection 4.1, we assume to have an optimal $L(h,k)$-labelling for B with span $\lambda^*(B)$ and deduce a feasible edge-coloring for $Ext(B)$ with the property that all edges in *Set(x)* have the same color, for any x. Let $\chi'_2(Ext(B))$ be the number of used colors. It follows that:

$$\chi'^*(Ext(B)) \leq \chi'_2(Ext(B)) \leq \frac{\lambda^*(B)}{k} + 1. \tag{4}$$

Considering that, for the class of graphs under consideration $\Delta \geq 3$ (if $\Delta = 2$ we have already given a result), we have $\lambda^*(B) \geq h + 2k$. Additionally, since $h \geq k$, then $k \leq \frac{\lambda^*(B)}{3}$. From Equation 3, Corollary 3, Equation 4, and the above observations, we have:

$$\begin{aligned}\lambda(B) &\leq k\left(\frac{\lambda^*(B)}{k} + 1\right)\Delta_V^2 + h - k \leq \\ &\leq \Delta_V^2\lambda^*(B) + (\Delta_V^2 - 3)\frac{\lambda^*(B)}{3} + \lambda^*(B) = \\ &= \frac{4}{3}\Delta_V^2\lambda^*(B).\end{aligned} \tag{5}$$

From the above discussion we have the following theorem:

Theorem 5. *The $L(h,k)$-labelling problem on bipartite graphs with all vertices in the lower set of even degree is $\frac{4}{3}\Delta_V^2$–approximable, where Δ_V is the maximum degree of the lower set.*

4.3 General Bipartite Graphs

In this subsection we further extend the previous results obtaining an approximation algorithm guaranteeing a performance ratio of $\frac{4}{3}D^2$ for each bipartite graph where D is the smallest even value bounding the minimum of the maximum degrees of the two partition.

Let $B = (U \cup V, E)$ be a bipartite graph. W.l.o.g. let $\Delta_V = \min \Delta_U, \Delta_V$, hence D is either Δ_V or $\Delta_V + 1$. Consider all vertices in the lower set with odd degree, and for each such vertex x, add a dummy vertex v_x in the upper set and a dummy edge (x, v_x) in E. In this way a new bipartite graph $B' = (U' \cup V, E')$ is generated and its lower set has maximum degree D and all vertices in V have even degree. Hence, we can consider graphs $N(B)$ and $Ext(B')$ and apply Theorem 5. Indeed, it is easy to see that a feasible $L(h,k)$-labelling for B' is a $L(h,k)$-labelling for B, also. Obviously, the viceversa could not be true. Hence, we have the following theorem:

Theorem 6. *The $L(h,k)$-labelling problem on bipartite graphs is $\frac{4}{3}D^2$–approximable, where D is the smallest even value bounding the minimum of the maximum degrees of the two partitions.*

5 Second Approximation Algorithm

In this section we propose another approximation algorithm for $\lambda^*_{h,k}(B)$ on a general bipartite graph B with a ratio $\min(h, 2k)\sqrt{n} + o(k\sqrt{n})$, not depending on the degree of the graph.

In [14] the author proves the existence of an approximation algorithm for vertex-coloring the square G^2 of any n vertex graph G with performance ratio $\sqrt{n-1}+1$.

Consider any bipartite graph $B = (U \cup V, E)$. We remind that any $L(1,1)$-labelling of B is a vertex-coloring of B^2, and it partitions the vertex set in classes such that vertices in different classes have different colors. Furthermore, if a class contains nodes of both U and V, we can split it into two classes so that each of them contains only elements in the upper (lower) set. Let $L_{1,1}(U)$ and $L_{1,1}(V)$ be the number of classes covering U and V, respectively. It holds $L_{1,1}(U)+L_{1,1}(V) \leq 2(\lambda_{1,1}(B)+1)$, where the term $+1$ derives from the fact that the smallest color of an $L(1,1)$-labelling is 0. Additionally, observe that $\lambda^*_{h,k}(B) \geq \lambda^*_{1,1}(B)$. We are now ready to describe the algorithm.

Run algorithm described in [14] on B to obtain an $L(1,1)$-labelling such that:

$$\frac{\lambda_{1,1}(B)}{\lambda^*_{1,1}(B)} \leq \sqrt{n-1}+1.$$

If $2k \leq h$, consider the classes induced by colors and split them into classes separating vertices of U and of V.

Number all classes in U, starting from 0, and all classes in V, starting again from 0. Then, for each vertex $v \in U$ belonging to class numbered $f(v)$, label v with $kf(v)$. Finally, for each vertex $v \in V$ belonging to class numbered $f(v)$, label v with $k(L_{1,1}(U)-1)+h+kf(v)$. The computed labelling is a feasible $L(h,k)$-labelling of B and has span $\lambda_{h,k}(B) \leq k(L_{1,1}(U)-1)+k(L_{1,1}(V)-1)+h \leq 2k\lambda_{1,1}(B)+h$. The performance ratio of the previous algorithm is:

$$\frac{\lambda_{h,k}(B)}{\lambda^*_{h,k}(B)} \leq \frac{2k\lambda_{1,1}(B)+h}{\lambda^*_{h,k}(B)} \leq$$

$$\leq \frac{2k\lambda_{1,1}(B)}{\lambda^*_{1,1}(B)} + \frac{h}{\lambda^*_{h,k}(B)} \leq 2k(\sqrt{n-1}+1)+1$$

as $\lambda^*_{h,k}(B)$ is at least h for each non trivial graph with at least two vertices.

If $2k > h$, instead of labelling nodes as above described, we proceed as follows: Consider the classes induced by colors and label each node in the class colored $f(v)$ with $hf(v)$. The produced labelling is feasible and its span is $\lambda_{h,k}(B) \leq h\lambda_{1,1}(B)$. Hence, the performance ratio is $h\sqrt{n-1}+h$.

The above discussion leads to the following theorem:

Theorem 7. *Given a n-vertex bipartite graph B, there exists a polynomial time approximation algorithm for computing an $L(h,k)$-labelling of B with $min(h, 2k)\sqrt{n} + o(k\sqrt{n})$ guaranteed performance ratio.*

Observe that the $\frac{4}{3}D^2$-approximation algorithm is better than this one when $D = O(n^{1/4})$.

Furthermore, it is easy to generalize the above strategy to s-partite graph. In this case we have the following theorem:

Theorem 8. *Given an n-vertex s-partite graph G, there exists a polynomial time approximation algorithm for computing an $L(h,k)$-labelling of G with $min(h, sk)\sqrt{n} + o(sk\sqrt{n})$ guaranteed performance ratio.*

Proof. The proof easily derives by generalizing Theorem 7, and it is omitted in this extended abstract for the sake of brevity.

6 General Graphs

In this section, we show a result stating the strong tie between the $L(h,k)$-labelling problem and the problem of coloring the vertices of the square of a graph with the minimum number of colors.

Theorem 9. *Let be given any value $\alpha > 1$ and a graph G. If there exists an algorithm finding an approximate vertex-coloring of G^2 with approximation ratio $\frac{\chi(G^2)}{\chi^*(G^2)} \leq \alpha$, then there exists an algorithm finding an approximate $L(h,k)$-labelling of G with approximation ratio $\frac{\lambda_{h,k}(G)}{\lambda^*_{h,k}(G)} \leq h\alpha$.*

Conversely, let be given any value $\beta > 1$ and a graph G. If there exists an algorithm finding an approximate $L(h,k)$-labelling with approximation ratio $\frac{\lambda_{h,k}(G)}{\lambda^*_{h,k}(G)} \leq \beta$, then there exists an algorithm finding an approximate vertex-coloring of G^2 with approximation ratio $\frac{\chi(G^2)}{\chi^*(G^2)} \leq h\beta$.

Proof. Suppose there exists an algorithm coloring vertices of G^2 with performance ratio $\frac{\chi(G^2)}{\chi^*(G^2)} \leq \alpha$, for any graph $G = (V,E)$. Let $f(v)$ be the color assigned to vertex v. Then a feasible $L(h,k)$-labelling for G is obtained by assigning label $h(f(v)-1)$ to v, assuming $h \geq k$. It is easy to see that such a labelling is a feasible $L(h,k)$-labelling and its performance ratio is:

$$\frac{\lambda_{h,k}(G)}{\lambda^*_{h,k}(G)} \leq \frac{h(\lambda_{1,1}(G))}{\lambda^*_{1,1}(G)} \leq h\alpha.$$

Conversely, suppose there exists an approximation algorithm for $L(h,k)$-labelling with performance ratio $\frac{\lambda_{h,k}(G)}{\lambda^*_{h,k}(G)} \leq \beta$ for any graph G.

Since $h \geq k \geq 1$, a $L(h,k)$-labelling is always a feasible $L(1,1)$-labelling for G. On the other hand, using a similar reasoning as in Section 3, $\lambda^*_{1,1}(G) \geq \lambda^*_{h,k}/h$. It follows that:

$$\frac{\chi(G^2)}{\chi^*(G^2)} \leq \frac{h\lambda_{h,k}(G)}{\lambda^*_{h,k}(G)} \leq h\beta$$

Corollary 4. *The problem of vertex-coloring the square of a graph with the minimum number of colors is in APX if and only if the problem of $L(h,k)$-labelling a graph is in APX, for each constant value h and $k \leq h$.*

Note that for a not constant h, the $L(h,k)$-labelling problem is not easier than the vertex-coloring of the square of a graph.

From Theorem 9 and considering the approximation algorithm described in [14], we can state the following theorem:

Theorem 10. *Given a n-vertex graph G, there exists a polynomial time approximation algorithm for computing an $L(h,k)$-labelling of G with $h(\sqrt{n-1}+1)$ guaranteed performance ratio.*

References

1. K. I. Aardal, S. P. M. van Hoesel, A. M. C. A. Koster, C. Mannino, and A. Sassano. Models and solutions techniques for frequency assignmet problems. Technical report, Konrad-Zuse-Zentrum *für* Informationtechnik Berlin, December 2001.
2. G. Agnarsson, R. Greenlaw, and M.M. Halldorson. On powers of chordal graphs and their colorings. *Congr. Numerant.*, 144:41–65, 2000.
3. G. Agnarsson and M. M. Halldórsson. Coloring powers of planar graphs. In *Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 654–662, N.Y., January 9–11 2000. ACM Press.
4. N. Alon and J. H. Kim. On the degree, size, and chromatic index of a uniform hypergraph. *J. Combin. Theory Ser. A*, 77:165–170, 1997.
5. H. L. Bodlaender, T. Kloks, R. B. Tan, and J. van Leeuwen. λ-coloring of graphs. *Proc. of STACS 2000. Lecture Notes in Computer Science*, 1770:395–406, 2000.

6. T. Calamoneri. Exact solution of a class of frequency assignment problems in cellular networks (extended abstract). *Proc. of the ICTCS 2003. Lecture Notes in Computer Science*, 2841:163–173, 2003.
7. T. Calamoneri. The $l(h,k)$-labelling problem: A survey. Technical Report 04/2004, Dept. of Computer Science, University of Rome *La Sapienza*, 2004.
8. T. Calamoneri, A. Pelc, and R. Petreschi. Labelling trees with a condition at distance two. In *Proc. of R.C. Bose Centenary Symp. on Discr. Math. and Applications*, 2002. Accepted to Discrete Mathematics.
9. R. Diestel. *Graph theory*. Springer-Verlag, New York, 1997. Translated from the 1996 German original.
10. J. P. Georges and D. W. Mauro. Some results on λ_k^j-numbers of the products of complete graphs. *Congr. Numer.*, 140:141–160, 1999.
11. J. P. Georges, D. W. Mauro, and M. I. Stein. Labeling products of complete graphs with a condition at distance two. *SIAM Journal on Discrete Mathematics*, 14(1):28–35, February 2001.
12. J. R. Griggs and R. K. Yeh. Labelling graphs with a condition at distance 2. *SIAM Journal on Discrete Mathematics*, 5(4):586–595, November 1992.
13. Y. L. Lin and S. S. Skiena. Algorithms for square roots of graphs. *SIAM Journal on Discrete Mathematics*, 8(1):99–118, February 1995.
14. S. T. McCormick. Optimal approximation of sparse Hessians and its equivalence to a graph coloring problem. *Mathematical Programming*, 26(2):153–171, 1983.
15. R. A. Murphey, P. M. Pardalos, and M. G. C. Resende. Frequency assignment problems. In D.-Z. Du and P. M. Pardalos, editors, *Handbook of combinatorial optimization*, volume Supplement A. Kluwer Academic Publishers, 1999.
16. N. Pippinger and J. Spencer. Asymptotic behaviour of the chromatic index for hypergraphs. *J. Combin. Theory Ser. A*, 51:24–42, 1989.
17. C. E. Shannon. A theorem on coloring the lines of a network. *J. Math. Phys*, 28:148–151, 1949.
18. J. van den Heuvel, R. A. Leese, and M. A. Shepherd. Graph labeling and radio channel assignment. *Journal of Graph Theory*, 29, 1998.
19. V. G. Vizing. On an estimate of the chromatic class of a p-graph (in russian). *Diskret. Analiz*, 3:23–30, 1964.