On Dilworth k Graphs and their Pairwise Compatibility

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Abstract. The *Dilworth number* of a graph is the size of the largest subset of its nodes in which the close neighborhood of no node contains the neighborhood of another one. In this paper we give a new characterization of Dilworth k graphs, for each value of k, exactly defining their structure. Moreover, we put these graphs in relation with *pairwise compatibility graphs (PCGs)*, i.e. graphs on n nodes that can be generated from an edge-weighted tree T that has n leaves, each representing a different node of the graph; two nodes are adjacent in the graph if and only if the weighted distance in the corresponding T is between two given non-negative real numbers, m and M. When either m or M are not used to eliminate edges from G, the two subclasses *leaf power* and *minimum leaf power graphs* (LPGs and mLPGs, respectively) are defined. Here we prove that graphs that are either LPGs or mLPGs of trees obtained connecting the centers of k stars with a path are Dilworth k graphs. We show that the opposite is true when $k = 1, 2$, but not when $k \geq 3$. Finally, we show that the relations we proved between Dilworth k graphs and chains of k stars hold only for LPGs and mLPGs, but not for PCGs.

Keywords: Graphs with Dilworth number k, leaf power graphs, minimum leaf power graphs, pairwise compatibility graphs.

1 Introduction and preliminary definitions

A graph $G = (V, E)$ is said to be a *difference graph* [15] if there is a positive real number T, the threshold, and for every node v there is a real weight $|a(v)| < T$, such that (v, w) is an edge if and only if $|a(v) - a(w)| \geq T$. Difference graphs are bipartite, so the nodes are partitioned into two stable sets. An example of a difference graph is shown in Figure 1.a.

A graph $G = (V, E)$ is a *threshold graph* [11] if there is a positive real number S, the threshold, and n real weights of the same sign, $|a(v)| < S$, each one associated to a single node v in V, such that (v, w) is an edge if and only if $|a(v) + a(w)| \geq S$. The nodes of a threshold graph can be always partitioned into a clique and a stable set and it is trivial to observe that all the edges connecting the clique and the stable set induce a difference graph. An example of a threshold graph is shown in Figure 1.b.

A graph $G = (V, E)$ is a *threshold signed graph* [2] if there are two positive real numbers S, T, the thresholds, and for every node v there is a real weight $|a(v)| < min(S, T)$

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such that (v, w) is an edge if and only if either $|a(v) + a(w)| \geq S$ or $|a(v) - a(w)| \geq T$. If $S = T$ then the threshold signed graph is simply a threshold graph [20]. A threshold signed graph is constituted by two threshold graphs connected by a set of edges inducing a difference graph. An example of a threshold signed graph is shown in Figure 1.c.

In the following, when we want to highlight function a and thresholds S and T, we will express a threshold and a threshold signed graph as $G = (V, a, S)$ and $G = (V, a, S, T)$, respectively.

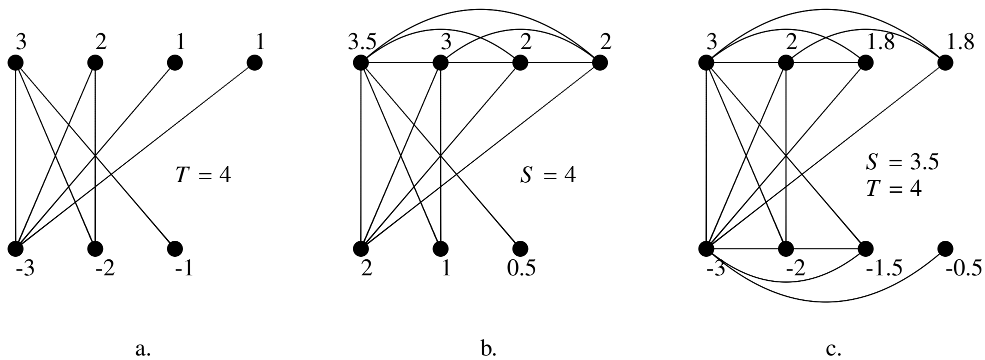

Fig. 1: a. A difference graph; b. A threshold graph; c. A threshold signed graph.

Given a graph $G = (V, E)$, for each node $v \in V$, we call its *open neighborhood* the set $N(v) = \{u \,|\, (u, v) \in E\}$ and its *closed neighborhood* the set $N[v] = N(v) \cup \{v\}$. Two nodes x and y are said to be *comparable* if either $N(y) \subseteq N[x]$ or $N(x) \subseteq N[y]$, otherwise they are said to be *incomparable*. A *chain* is a set of pairwise comparable nodes and the *Dilworth number* of G is the largest number of pairwise incomparable nodes of the graph or, in other words, the minimum size of a partition of its nodes into chains [14].

Graphs with Dilworth number 1 and 2 have been deeply studied as they correspond to the classes of threshold and of threshold signed graphs, respectively [20]. Graphs with Dilworth number at most 3 have received some attention as subclasses of a class of graphs defined by a special elimination ordering scheme [16] and graphs with Dilworth number 4 have been shown to be a subset of perfectly orderable graphs [21]. Finally, the authors of [1] proved that a large class of vertex subset and vertex partitioning problems can be solved in polynomial time on Dilworth k graphs. To the best of our knowledge, nothing else is known about Dilworth k graphs, when $k \geq 5$, and, unfortunately, it does not seem possible to define Dilworth k graphs by using thresholds and node-weights for $k \geq 3$.

In this paper, we characterize Dilworth k graphs as those graphs whose node set can be partitioned in order to form k threshold graphs and the set of edges between each pair of threshold graphs induces a special difference graph. Although not difficult to obtain, this result is interesting by itself because no other characterization of Dilworth k graphs is known. This is the content of Section 2.

A *star* S_i is the complete bipartite graph $K_{1,i}$: a tree with one internal node c, called *center*, and i leaves. We define a *k-star path* $S_{i_1,\ldots,i_k}$ to be a tree that consists of k stars $S_{i_1}, \ldots, S_{i_k}$ whose centers induce a path (i.e. the centers of stars S_{i_j} and $S_{i_{j+1}}$, for each $j = 1, \ldots, k-1$, are connected by an edge). In other words, $S_{i_1,\ldots,i_k}$ is a caterpillar whose the j-th node on the spine has i_j leaves.

The *merge* of two star-paths $S_{i_1,\ldots,i_r}$ and $S_{j_1,\ldots,j_s}$ is a star-path $S_{k_1,\ldots,k_{r+s}}$ where $i_t = k_t$ for each $t = 1, \ldots, r$ and $j_t = k_{r+t}$ for each $t = 1, \ldots, s$. In other words, $S_{k_1,\ldots,k_{r+s}}$ is obtained by connecting with an edge the center of the r-th star of $S_{i_1,\ldots,i_r}$ with the center of the first star of $S_{j_1,\ldots,j_s}$.

A graph $G = (V, E)$ is a *pairwise compatibility graph (PCG)* [18, 22] if there exists a tree T, a positive edge-weight function w on T and two non-negative real numbers m and M, $m \leq M$, such that V coincides with the set of leaves of T, and the edge (u, v) is in G if and only if $m \leq \text{dist}_{T,w}(u, v) \leq M$, where $\text{dist}_{T,w}(u, v)$ is the sum of the weights of the edges on the unique path from u to v in T. In such a case, we say that G is a PCG of T for m and M; in symbols, $G = PCG(T, w, m, M)$. When the constraints on the distance between the pairs of leaves deal only with M or m, the definition of *leaf power graphs* or *minimum leaf power graphs* arise, respectively. More precisely, a graph $G = (V, E)$ is called a *leaf power graph (LPG)*[3], (respectively *min-leaf power graph (mLPG)*[10]) if there exists a tree T, a positive edge-weight function w on T, and a non-negative real number M (respectively m) such that V coincides with the set of leaves of T, and the edge (u, v) is in G if and only if $\text{dist}_{T,w}(u, v) \leq M$ (respectively $\text{dist}_{T,w}(u, v) \geq m$).

The notions of PCG and LPG have been proposed in relation to sampling problems in phylogenetics but, since then, these classes of graphs have received great interest even from a merely graph-theoretic point of view (e.g. see [5, 6, 10, 12, 17, 19, 22–25]).

It is still unknown an algorithm testing whether a graph is a PCG (or a LPG, or an mLPG) or not. So researchers have concentrated on single classes of graphs, trying to identify their relation with respect to the class of PCGs (or LPGs or mLPGs). Here we try to improve the knowledge on PCGs, LPGs, and mLPGs studying their relations with the class of Dilworth k graphs. It is known that Dilworth 1 graphs are obtained as LPGs of stars [10] and Dilworth 2 graphs are obtained as LPGs of 2-star paths [8]. Here, in Section 3 we prove that a graph that is either LPG or mLPG of a k-star path has Dilworth number at most k. Moreover we show that the opposite is true only when $k = 1, 2$, providing a Dilworth 3 graph that is neither a LPG nor a mLPG.

In Section 4, we highlight that the relations we proved between Dilworth k graphs and k-star paths hold only for LPGs and mLPGs, but not for PCGs, since the PCG of a star can have arbitrarily high Dilworth number.

A last section, listing some open problem concludes the paper.

The reader can refer to [4] for all terminology and definitions not explicitly given in this paper.

2 A Characterization of Dilworth *k* graphs

We start this section deepening the knowledge of the structure of a Dilworth 2 graph. Namely, we prove a new characterization for Dilworth k graphs as a partition into threshold graphs, each pair of which is connected by an edge set that induces a difference graph with a special ordering.

First of all, let us recall some known results from [20] on difference, threshold and threshold signed graphs.

Lemma 1 *Let u and w be either any two nodes of a threshold graph or any two nodes in the same set of the partition of a difference graph. It holds that* $deg(u) \leq deg(w)$ *if and only if* $N(u) \subseteq N[w]$.

Consider an edge (u, v) of a threshold signed graph $G = (V, a, S, T)$. It is not difficult to see that only one of the two conditions concerning the thresholds can be satisfied. Indeed, when $a(u)$ and $a(v)$ have the same sign, i.e. $a(u) \cdot a(v) > 0$ it can only be satisfied $|a(u) + a(v)| \geq S$; in this case the edge (u, v) is called *S-edge*. On the contrary, when $a(u)$ and $a(v)$ have different sign, i.e. $a(u) \cdot a(v) < 0$, it can only hold that $|a(u) - a(v)| \geq T$ and the edge is called *T-edge*. In the following, we will consider the two chains,$V_1$and V_2 derived from the partition of the nodes of G into the two sets $V_1 = \{x \in V \text{ s.t. } a(x) < 0\}$ and $V_2 = \{y \in V \text{ s.t. } a(y) > 0\}$.

Next lemma formalizes some concepts already presented in the previous section:

Lemma 2 *Given a Dilworth 2 graph G, its two chains V_1 and V_2 induce each a threshold graph, G_1 and G_2, while the edges connecting nodes in different chains induce a difference graph D. So the graph can be expressed as* $G = (G_1, G_2, D)$.

We have already underlined that the definition of threshold signed graphs and of Dilworth 2 graphs are equivalent. In the following, we will use the terms ”threshold signed” and ”Dilworth 2” depending on what we want to highlight in the graph (either the weight function a and the thresholds, or the structure of two threshold graphs with a difference graph in between).

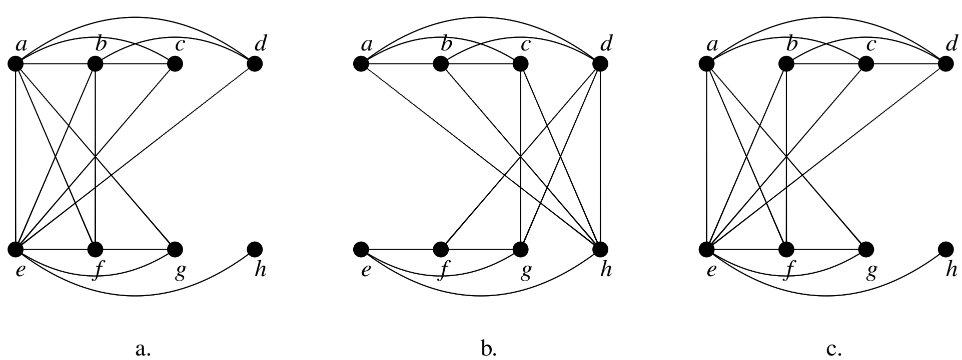

Fig. 2: Three graphs composed by two threshold graphs unified by a difference graph. Only graph in a. is a threshold signed graph.

The opposite of Lemma 2 is not always true, as graphs in Figure 2 show. Indeed, all the three graphs are composed by two threshold graphs unified by a difference graph, but only the graph in Figure 2.a has Dilworth number 2, while graphs in Figure 2.b and 2.c, have higher Dilworth numbers (this can be easily seen by constructing their chains).

This fact suggests us that in a Dilworth 2 graph the difference graph structure must be enriched with a special order guaranteeing that the higher/lower degree nodes in the threshold graphs are also the higher/lower degree nodes in the difference graph.

Given a graph G constituted by two threshold graphs G_1 and G_2 connected by a difference graph D, $G = (G_1, G_2, D)$, let $S_{G_1} = (v^1_1, \ldots, v^1_{|V_1|})$, $S_{G_2} = (v^2_1, \ldots, v^2_{|V_2|})$ and $S_D = (d_1, \ldots, d_n)$ be the ordered sequences of nodes V_1 and V_2 with respect to their degree within G_1, G_2 and D, respectively. So, $\deg|_{G_i}(v^i_1) \geq \ldots \geq \deg|_{G_i}(v^i_{|V_i|})$, $i = 1, 2$, and $\deg|_D(v_1) \geq \ldots \geq \deg|_D(v_n)$. The sequence S_D can be split into two subsequences $S_{D_1} = S_D \cap V_1 = (d^1_1, \ldots, d^1_{|V_1|})$ and $S_{D_2} = S_D \cap V_2 = (d^2_1, \ldots, d^2_{|V_1|})$. As an example, in Figure 2.a the sequences are: $S_{G_1} = abcd$, $S_{G_2} = efgh$, $S_{D_1} = abcd$ and $S_{D_2} = efgh$; in Figure 2.b the sequences are: $S_{G_1} = abcd$, $S_{G_2} = efgh$, $S_{D_1} = dcba$ and $S_{D_2} = hgfe$; finally, in Figure 2.c the sequences are: $S_{G_1} = dcba$, $S_{G_2} = efgh$, $S_{D_1} = abcd$ and $S_{D_2} = efgh$.

Observe that these orderings are not unique, as nodes with the same degree can be put in any relative order. As a particular case, isolated nodes can be added at the end of a sequence in any order. In the following, two sequences S_{G_i} and $S_D \cap V_i$ are *equal* if $v^i_k = d^i_k$ for all $k = 1, \ldots, |V_i|$.

Lemma 3 *Given a graph G, the following two claims are equivalent:*

a. G has Dilworth number at most 2;
b. The nodes of G can be partitioned into two sets V_1 and V_2, such that V_i, $i = 1, 2$ induces a threshold graph G_i and the edges between V_1 and V_2 induce a difference graph D; furthermore, there exist three ordered sequences S_{G_1}, S_{G_2} and S_D such that $S_{G_i} = S_D \cap V_i$, for $i = 1, 2$.

Proof. First observe that the claim is trivially true for Dilworth 1 graphs because $V_2 = \emptyset$, so in the following we consider the case in which G has Dilworth number exactly 2.

$a. \Rightarrow b.$ G is a Dilworth 2 graph, so Lemma 2 states that the nodes of G can be partitioned into two sets V_1 and V_2, each one inducing a threshold graph, while the edges in between induce a difference graph. So it remains to prove that it is possible to find the three ordered sequences S_{G_i}, $i = 1, 2$ and S_D such that $S_{G_i} = S_{D_i}$, $i = 1, 2$.

By contradiction, assume that it is not possible to find three sequences as required by the theorem. W.l.o.g., let us focus on G_1 and on the two sequences S_{G_1} and $S_{D_1} = S_D \cap V_1$ related to it. Among all such possible sequences, consider those such that $v^1_j = d^1_j$ for every $j < s$, $v^1_s \neq d^1_s$ and s is as large as possible (although, by hypothesis, $s \leq |V_1|$). Let $u = v^1_s$ and $w = d^1_s$.

There must exist an index $r > s$ such that $w = d^1_s = v^1_r$. It follows that $\deg|_{G_1}(u) \geq \deg|_{G_1}(w)$. From the other side, since a Dilworth 2 graph is a threshold signed graph [20], all nodes belonging to the same partition are characterized by having weight values with the same signs and, inside it, two nodes x and y are connected by an edge if $|a(x) + a(y)| \geq S$. From these facts it follows that $a(u) \geq a(w)$.

Analogously, there must exist an index $t > s$ such that $u = v^1_s = d^1_t$. Furthermore, an edge of D connects two nodes x and y if they have weight values with different signs and if $|a(x) - a(y)| \geq T$. From these inequalities, it follows that $a(w) \geq a(u)$.

The two inequalities $a(u) \geq a(w)$ and $a(w) \geq a(u)$ imply $a(u) = a(w)$ and consequently $\deg|_D(u) = \deg|_D(w)$ and hence $\deg|_D(d_s^1) = \ldots = \deg|_D(d_t^1)$. So, in the sequence S_{D_1}, it is possible to swap $d_s^1 = w$ with $d_t^1 = u$ and now $v_s^1 = d_s^1 = u$; but this contradicts that s is as large as possible.

b. ⇒ *a*. Consider any two nodes u and w in the same set V_1. By hypothesis, each of them occupies the same position into the two orderings S_{G_1} and S_{D_1}: let it be $u = v_r^1 = d_r^1$ and $w = v_s^1 = d_s^1$. It is not restrictive to assume $r < s$, that is $\deg|_{G_1}(w) \leq \deg|_{G_1}(u)$ and $\deg|_D(w) \leq \deg|_D(u)$. Since G_1 is a threshold graph, and D is a difference graph, from Lemma 1, it follows that $N|_{G_1}(w) \subseteq N|_{G_1}[u]$ and $N|_D(w) \subseteq N|_D[u]$. In view of the general choice of u and w inside V_1 and observing that $N(w) = N|_{G_1}(w) \cup N|_D(w)$ and $N[u] = N|_{G_1}[u] \cup N|_D[u]$, it follows that V_1 constitutes a unique chain. □

Now we are ready to prove a characterization for Dilworth k graphs.

Theorem 1 *Given a graph $G = (V, E)$, the following two claims are equivalent:*

a. G has Dilworth number at most k;
b. The nodes of G can be partitioned into k sets $V_1, \ldots, V_k$, such that V_i, $i = 1, \ldots, k$ induces a threshold graph and the edges between V_r and V_s, for any $r, s = 1, \ldots, k$, $r < s$, induce a difference graph $D_{r,s}$; furthermore, there exist $\frac{k^2+k}{2}$ ordered sequences S_{G_i}, $i = 1, \ldots, k$ and $S_{D_{r,s}}$, for all $r, s = 1, \ldots k$, $r < s$, such that $S_{G_i} = S_{D_{r,s}} \cap V_i$, for all $r, s = 1, \ldots k$, $r \neq s$ and $i = r, s$.

Proof. a. ⇒ b. If G is a graph with Dilworth number at most k, it is possible to determine its k chains in polynomial time [13]; let $V_1, \ldots, V_k$ be such chains. Consider the graphs induced by these chains. Trivially they have Dilworth number 1 and are threshold graphs. Each graph induced by any pair of two chains V_r and V_s, $r, s = 1, \ldots k$, $r < s$ is a Dilworth 2 graph and, by Lemma 2, the edges connecting nodes of V_r with nodes of V_s induce a difference graph.

In order to prove the claim about the ordered sequences, observe that Lemma 3 can be applied to each pair of chains V_r and V_s. Since r and s vary in all possible ways, we have that $S_{G_i} = S_{D_{r,s}} \cap V_i$, for all $r, s = 1, \ldots k$, $r < s$ and $i = r, s$. Finally, the ordered sequences we considered are one for each threshold graph G_i, $i = 1, \ldots, k$ and one for each difference graph $D_{r,s}$, $r, s = 1, \ldots, k$, $r < s$, that is $k + \frac{k(k-1)}{2} = \frac{k^2+k}{2}$.

b. ⇒ a. Consider the nodes of G as partitioned into k sets, $V_1, \ldots, V_k$. In view of the hypothesis, the graph induced by $V_i \cup V_j$, for each pair $i, j = 1, \ldots, k$, $i < j$ satisfies condition b. of Lemma 3 and hence is a Dilworth 2 graph.

We claim that, even when we put together all the Dilworth 2 graphs into the whole graph G, each set V_i, $i = 1, \ldots k$, forms a unique chain, so deducing that G is a graph with Dilworth number at most k. In order to prove this claim, consider two nodes u and w of the same set V_i, $u = v_x^i$ and $w = v_y^i$. W.l.o.g. let us assume $x < y$. By hypothesis, u precedes w also in all the sequences $S_{D_{i,j}}$, for each $j = 1, \ldots, k$, $i \neq j$. [1] For any other set

[1] For the sake of simplicity, here and in the following we are omitting to differentiate the two cases $i < j$ and $i > j$, leading to the nomenclature $D_{i,j}$ and $D_{j,i}$, respectively. Indeed, we use the condition $i < j$ only to avoid to consider the same set twice, once as $D_{i,j}$ and once as $D_{j,i}$.

V_j, $V_i \cup V_j$ induces a Dilworth 2 graph, so it also holds that $N|_{V_i \cup V_j}(v^i_y) \subseteq N|_{V_i \cup V_j}[v^i_x]$. Since the neighborhood of a node v of set V_i can be expressed as $N(v) = \cup N|_{V_i \cup V_j}(v)$, and the same holds for the closed neighborhood, we consequently have that:

$$N(v^i_y) \bigcup_{j=1,\ldots,k} N|_{V_i \cup V_j}(v^i_y) \subseteq \bigcup_{j=1,\ldots,k} N|_{V_i \cup V_j}[v^i_x] = N[v^i_x]$$

so showing that set V_i constitutes a unique chain of G and proving that G is a graph with Dilworth number at most k. □

Corollary 1 *The class of Dilworth k graphs is self-complementary.*

Proof. When passing from a Dilworth k graph to its complement, the sequences S_{G_i}, $i = 1, \ldots, k$ and $D_{r,s}$, $r, s = 1, \ldots, k$, $r < s$, as characterized in item b. of Theorem 1, reverse their order. The proof of the claim is then easily derived from this property and from the self-complementarity of threshold and difference graphs. □

3 LPGs and mLPGs of *k*-star paths are Dilworth *k* graphs

In this section we highlight the relations between Dilworth k graphs, LPGs and mLPGs of k-star chains.

Theorem 2 *Given a k-star path $S_{i_1,\ldots,i_k}$, $i_1 + \ldots + i_k = n$, an edge-weight function w on $S_{i_1,\ldots,i_k}$ and a value M, the graph $G = LPG(S_{i_1,\ldots,i_k}, w, M)$ is a graph with Dilworth number at most k and the set of leaves of each star induces at most one chain.*

Proof. We will prove the claim by induction on k.

When $k = 1$, the k-star path degenerates in a single star S_n. Consider two leaves of S_n, u and v, such that $w(u) \le w(v)$. For any other leaf x, if $\text{dist}_{S_n,w}(v, x) = (w(v)+w(x)) \le M$ then $\text{dist}_{S_n,w}(u, x) \le M$. It follows that $N(v) \subseteq N[u]$. In general, all nodes belong to the same chain, i.e. G has Dilworth number 1.

By inductive hypothesis, assume now that $LPG(S_{i_1,\ldots,i_{k-1}}, w, M)$ is a graph with Dilworth number at most $k-1$, where the set of leaves of each star induces at most one chain, and consider the k-star path $S_{i_1,\ldots,i_k}$ obtained by merging a star S_{i_k} and the previous $(k-1)$-star path. We have to prove that the addition of S_{i_k} and of the edge (c_{k-1}, c_k) does not modify in any way the existing relations between any pair of nodes in $S_{i_1,\ldots,i_{k-1}}$ and add at most one new chain.

In other words, for each u and v in $S_{i_1,\ldots,i_k}$, considering w.l.o.g. $w(u) \le w(v)$, we have to prove the following assertions:

- if u and v are both in S_{i_k} then $N(v) \subseteq N[u]$;
- if u and v are both in $S_{i_1,\ldots,i_{k-1}}$ and $N|_{V_1 \cup \ldots V_{k-1}}(v) \subseteq N|_{V_1 \cup \ldots V_{k-1}}[u]$ then $N|_{V_1 \cup \ldots V_k}(v) \subseteq N|_{V_1 \cup \ldots V_k}[u]$;
- if u in $S_{i_1,\ldots,i_{k-1}}$ and v in S_{i_k} (or, vice-versa, v in $S_{i_1,\ldots,i_{k-1}}$ and u in S_{i_k}) then either $N(v) \subseteq N[u]$ or the neighborhoods of u and v are incomparable.

Let u and v be two leaves in S_{i_k}. For any other leaf x in S_{i_k}, if $\text{dist}_{S_{i_k},w}(v, x) = (w(v) + w(x)) \le M$ then $\text{dist}_{S_{i_k},w}(u, x) \le M$ and $N(v) \subseteq N[u]$. In general, all nodes in S_{i_k} belong to the same chain. For any other node $x \in V_{\bar{\imath}}$ (where $\bar{\imath} \neq i_k$), if $x \in N(v)$ then $\text{dist}_{S_{i_1,\ldots,i_k},w}(x, v) = w(x) + w(\{c_{\bar{\imath}}, c_{\bar{\imath}+1}\}) + \ldots + w(\{c_{k-1}, c_k\}) + w(v) \le M$ and so $\text{dist}_{S_{i_1,\ldots,i_k},w}(x, u) \le M$, that is $x \in N[u]$, so confirming that u and v belong to the same chain and that at most one new chain is introduced by the nodes in S_{i_k}.

Let u and v be two leaves in $S_{i_1,\ldots,i_{k-1}}$. If all nodes in S_{i_k} are not adjacent to any node in $S_{i_1,\ldots,i_{k-1}}$, then trivially the second assertion is true. Let v belong to $S_{\bar{\imath}}$, $1 \le \bar{\imath} \le k-1$. Now, let us consider $x \in V_k$, s.t. $x \in N(v)$. This means that $\text{dist}_{S_{i_1,\ldots,i_k},w}(x, v) = w(v) + w(\{c_{\bar{\imath}}, c_{\bar{\imath}+1}\}) + \ldots + w(\{c_{k-1}, c_k\}) + w(x) \le M$, and hence it holds that $\text{dist}_{S_{i_1,\ldots,i_k},w}(x, u) \le M$.

The third assertion is trivially true. We want only to point out that the incomparability implies that the Dilworth number of $LPG(S_{i_1,\ldots,i_k}, w, M)$ is strictly greater than the Dilworth number of $LPG(S_{i_1,\ldots,i_{k-1}}, w, M)$. □

The following theorem can be proved with considerations that are very similar to those exploited in the previous proof. Nevertheless, we prefer to present a different approach.

Theorem 3 *Given a k-star path $S_{i_1,\ldots,i_k}$, $i_1 + \ldots + i_k = n$, an edge-weight function w on $S_{i_1,\ldots,i_k}$ and a value m, the graph $G = mLPG(S_{i_1,\ldots,i_k}, w, m)$ is a graph with Dilworth number at most k.*

Proof. W.l.o.g., let w assume integer values (if not, it is known [7] that it is possible to find a new edge-weight function w' and a new value m' such that $G = mLPG(S_{i_1,\ldots,i_k}, w', m')$ and w' has integer values). Consider the graph $\bar{G} = LPG(S_{i_1,\ldots,i_k}, w, m-1)$. It is easy to see that $\bar{G}$ is in fact the complement of G.

From Theorem 2, $\bar{G}$ is a graph with Dilworth number at most k because it is a LPG of a k-star path. In view of Corollary 1, the class of Dilworth k graphs is self-complementary, and so G is a graph with Dilworth number at most k. □

The opposite of the previous theorems holds when $k = 1, 2$, in view of the following constructions:

- Let G be a threshold graph on n nodes and let B_i, $i = 1, \ldots, r$ be the set of its nodes having degree i. $G = (V, a, S)$ is a leaf power graph of a star S_n, the weight of each edge $\{c, v_i\}$, $w(\{c, v_i\})$, is equal to j if $v_i \in B_j$, $1 \le j \le r$ and M coincides with $r + 1$, i.e. $G = LPG(S_n, w, M)$. G is even a minimum leaf power graph of the same star S_n, the wight of each edge $\{c, v_i\}$, $w'(\{c, v_i\})$ is equal to $r + 1 - j$ if $v_i \in B_j$, $1 \le j \le r$ and $m = r + 1$, i.e. $G = mLPG(S_n, w', m)$. [10].
- A threshold signed graph $G = (V, a, S, T)$, $V = V_1 \cup V_2$, is a leaf power graph of a 2-star path $S_{|V_1|,|V_2|}$ whose nodes are $c, v_{i_1}, \ldots, v_{i_{|V_1|}}$ and $d, u_{i_1}, \ldots, u_{i_{|V_2|}}$. The weight of each edge $\{c, v_i\}$, $w(\{c, v_i\})$, is equal to $-a(v_i)$ and the weight of each edge $\{d, u_i\}$, $w(\{d, u_i\})$, is equal to $a(u_i)$. Finally, the weight of the edge $\{c, d\}$ is equal to $S - T$ and M coincides with S, i.e. $G = LPG(S_{|V_1|,|V_2|}, w, S)$. It is even possible to determine an edge-weight function w' and a value m such that $G = mLPG(S_{|V_1|,|V_2|}, w', m)$ [8].

Hence we have:

Corollary 2 *Given an n node graph G and a k-star path, with $k = 1, 2$, it holds:*

- *G is a Dilworth 1 graph if and only if $G = LPG(S_n, w, M)$ ($G = mLPG(S_n, w', m)$) for some edge-weight function w (w′) on S_n and some value of M (m).*
- *G is a Dilworth 2 graph if and only if $G = LPG(S_{i_1,i_2}, w, M)$ ($G = mLPG(S_{i_1,i_2}, w', m)$) for some edge-weight function w (w′) on S_{i_1,i_2} and some value of M (m).*

In the following we will show that the opposite of Theorem 2 is not true for $k \geq 3$, because there exists a Dilworth 3 graph that is not LPG. Before proving the existence of such graph, let us recall two lemmas from [12].

Lemma 4 *Let G, be the cycle of four nodes a,b,c,d. $G = PCG(T, w, m, M)$ for some tree T, edge-weight w and non-negative real numbers m and M. Then $dist_{T,w}(a, c)$ and $dist_{T,w}(b, d)$ cannot be both greater than M.*

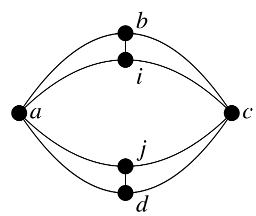

Fig. 3: The graph H.

Lemma 5 *Let H be the graph depicted in Figure 3, where the nodes are a,b,c,d,i, j. $H = PCG(T, w, m, M)$ for some tree T, edge-weight w and non-negative real numbers m and M. Then at least one of $dist_{T,w}(a, c)$, $dist_{T,w}(b, d)$, $dist_{T,w}(i, d)$, $dist_{T,w}(j, b)$, $dist_{T,w}(i, j)$ must be greater than M.*

Lemma 6 *Graph H has Dilworth number 3 and is neither a LPG nor a mLPG.*

Proof. It is easy to see that H is a Dilworth 3 graph, indeed its three chains are $\{a, c\}$, $\{b, i\}$ and $\{d, j\}$ and it is not possible to merge them into only two chains.

Assume now, by contradiction that H is either a LPG or a mLPG, that is either m or M are set in order not to exclude any edge. Lemma 4 implies that at least one of the non-existing chords $\{a, c\}$ and $\{b, d\}$ must necessarily be excluded from H by using m; from the other side, Lemma 5 ensures that at least one of the non-edges of H must be excluded by using M. It follows that both m and M are necessary and hence H is neither a LPG nor a mLPG. □

4 PCGs of a star can have arbitrarily large Dilworth number

In the previous section we have proved that LPGs and mLPGs of k-star paths are Dilworth k graphs. We wonder whether PCGs of k-star paths are Dilworth k graphs, too. The answer is negative, indeed even the PCG of a single star can have arbitrarily high Dilworth number, as proved in the following theorem.

Theorem 4 *There exists an edge-weight function w of an n leaf star, and two non-negative numbers m and M such that the n node graph $G = PCG(S_n, w, m, M)$ has Dilworth number at least $n/3$.*

Proof. Consider an n leaf star where, for the sake of simplicity, n is a multiple of 3, and partition its leaves into three equally sized sets: $K = \{k_1, \ldots, k_{n/3}\}$, $S_1 = \{s_1, \ldots, s_{n/3}\}$ and $S_2 = \{t_1, \ldots, t_{n/3}\}$. Define the edge-weight function w as follows: $w(\{c, s_i\}) = i$, $w(\{c, k_i\}) = n/3 + i$, and $w(\{c, t_i\}) = 2n/3 + i$, for each $i = 1, \ldots n/3$. Let $m = 2n/3 + 1$ and $M = 4n/3 + 1$.

In agreement with [9], where PCGs of stars are studied, K induces a clique while S_1 and S_2 induce a stable set; these sets are pairwise connected by difference graphs.

In order to prove our claim, we focus on the two difference graphs between K and S_1 and between K and S_2. For any two nodes in K, k_i and k_j, with $i < j$, it holds:

$$N|_{K\cup S_1}(k_i) \subseteq N|_{K\cup S_1}[k_j]$$
$$N|_{K\cup S_2}(k_j) \subseteq N|_{K\cup S_2}[k_i].$$

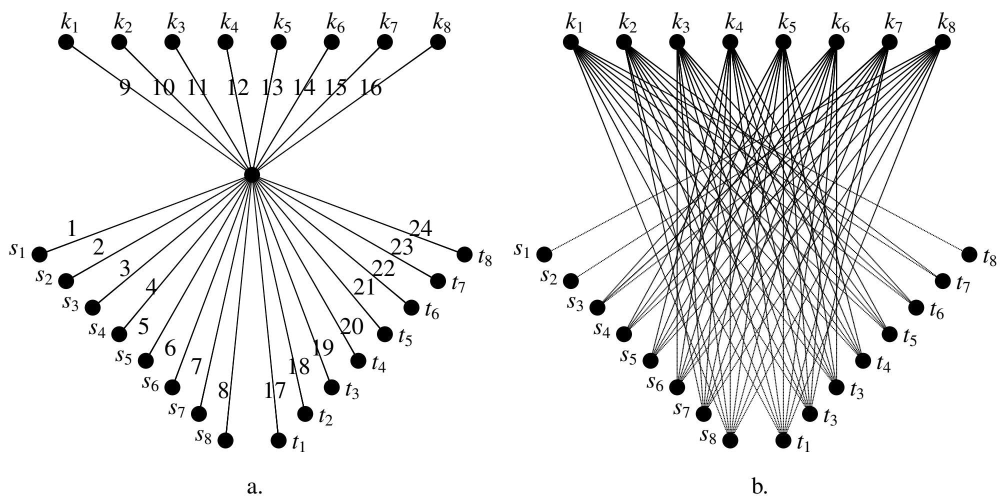

Fig. 4: a. An edge-weighted star S_{24} where w is defined according to the proof of Theorem 4; b. The resulting PCG when $m = 17$ and $M = 33$, having Dilworth number at least 8. For the sake of clarity, the edges between nodes in K and between S_1 and S_2 have been omitted.

It is easy to see that both the set inclusions are in fact strict, i.e there exist two nodes $s_p \in S_1$ and $t_q \in S_2$ such that $s_p \in N|_{K\cup S_1}(k_j) \setminus N|_{K\cup S_1}(k_i)$ while $t_q \in N|_{K\cup S_2}(k_i) \setminus N|_{K\cup S_2}(k_j)$. For example, choose $p = n/3 - i$ and $q = n - j + 2$. It holds that $N(k_i) = N|_{K\cup S_1}(k_i) \cup N|_{K\cup S_2}(k_i)$ and $N(k_j) = N|_{K\cup S_1}(k_j) \cup N|_{K\cup S_2}(k_j)$, hence $s_p \in N(k_j)$ but $s_p \notin N(k_i)$, while $t_q \in N(k_i)$ but $t_q \notin N(k_j)$; it follows that k_i and k_j do not belong to the same chain.

In general, each node of K is in a different chain, so that the Dilworth number of G is at least $|K| = n/3$.

In Figure 4.a a graph with $n = 24$ nodes and Dilworth number at least $n/3 = 8$ is depicted. Figure 4.b represents the star that witnesses that G is PCG. □

5 Open Problems

From the results of this paper, many questions naturally arise. We list here some of them, sure that this is not an exhaustive list.

In this paper, we preliminary provide a characterization of Dilworth k graphs, a result that is interesting by itself because no other characterization is known in the literature for this class of graphs. It deserves to be investigated whether this characterization is useful to provide a fast recognition algorithm for Dilworth k graphs.

Then, we exploit this characterization to prove that graphs that are either LPGs or mLPGs of k-star paths are Dilworth k graphs, but the opposite is not true, unless $k = 1$ or $k = 2$, indeed we provide a simple counterexample (a graph with Dilworth number 3 that is neither a LPG nor a mLPG). It is natural to wonder if graphs having Dilworth number equal to 3 are Pairwise Compatibility Graphs ore not. In fact, it would be of interest to understand which is the smallest Dilworth number of a graph that is not PCG. It is neither clear whether there exists a Dilworth 3 graph that is a LPG but not a LPG of a 3-star path.

Finally, we show that the relation we highlighted between Dilworth k graphs and k-star paths hold only for LPGs and mLPGS, but not for PCGs; indeed the PCG of a star can have arbitrarily high Dilworth number. We wonder if the Dilworth number of an arbitrary LPG is unbounded or not.

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