L(h,1)-Labeling Subclasses of Planar Graphs

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Abstract

$L(h,1)$-*labeling*, $h = 0, 1, 2$, is a class of coloring problems arising from frequency assignment in radio networks, in which adjacent nodes must receive colors that are at least h apart while nodes connected by a two long path must receive different colors. This problem is NP-complete even when limited to planar graphs. Here, we focus on $L(h,1)$-labeling restricted to regular tilings of the plane and to outerplanar graphs. We give a unique parametric algorithm labeling each regular tiling of the plane. For these networks, a channel can be assigned to any node in constant time, provided that relative positions of the node in the network is locally known. Regarding outerplanar graphs with maximum degree Δ, we improve the best known upper bounds from $\Delta + 9$, $\Delta + 5$ and $\Delta + 3$ to $\Delta + 3$, $\Delta + 1$ and Δ colors for the values of h equal to 2, 1 and 0, respectively, for sufficiently large values of Δ. For $h = 0, 1$ this result proves the polinomiality of the problem for outerplanar graphs. Finally, we study the special case $\Delta = 3$, achieving surprising results.

keywords: $L(h,1)$-labeling, radio networks, outerplanar graphs, regular tiling.

1 Introduction

A *radio network* is a network consisting of radio transmitters/receivers distributed over a region. Communication takes place by a node broadcasting a signal over a fixed range (the size of which is proportional to the power of the node's transmitter). Any receiver within the range of the transmitter can get the signal in a single hop; all others receivers will get it in multiple hops. In this context, *radio frequency assignment* is a widely studied research area. The task is to assign radio frequencies to transmitters at different locations without causing interference. This situation can be modeled by a graph, whose nodes are the radio transmitters/receivers, and whose adjacencies indicate possible communications and, hence, interference. Consequently, the problem is closely related to graph coloring, where colors represent possible frequencies.

Among all the problems of radio frequency assignment modeled as coloring of graphs, we are interested in $L(2,1)$-*labeling*, introduced by Griggs and Yeh [12]: 'close' transmitters must receive different frequencies and 'very close' transmitters must receive frequencies that are at least two frequencies apart. In terms of graphs, two transmitters/receivers are 'close' if they are connected by a two long path in the graph and 'very close' if they are adjacent in the graph. The practical reasoning leading to these constraints is that the frequencies of a radio station and its neighbors must be sufficiently different that their signals will not interfere (direct collision); furthermore, a radio station must not receive signals of the same frequency from any of its adjacent neighbors (hidden collision). Here we consider two variants weakening the condition on direct collisions: the $L(1,1)$- and $L(0,1)$-*labeling problems*. In the first problem adjacent stations are required to have different frequencies, while in the second one only hidden collisions are avoided.

Formalizing:

Definition 1.1 *[3] An* $L(h,1)$-labeling *of a graph* $G = (V, E)$, $h = 0, 1, 2$, *is a function* f *from the node set* V *to the set of all nonnegative integers such that*
1. $|f(x) - f(y)| \geq h$ *if* $\{x, y\} \in E$ *and*
2. $|f(x) - f(y)| \geq 1$ *if* $\exists z \in V$ *s.t.* $\{x, z\} \in E$ *and* $\{z, y\} \in E$.

A span *of such a labeling is the difference between the largest and the smallest label. The* $L(h,1)$-number *of* G, *denoted by* $\lambda_{h,1}(G)$, *is the smallest span necessary to* $L(h,1)$*-label* G.

Remark 1 *The number of used colors for a* $L(h,1)$*-labeling of a graph* G *is equal to* $\lambda_{h,1}(G) + 1$, *as the smallest used color is 0.*

For some special classes of graphs – such as paths, cycles, wheels and k-partite graphs – tight bounds for the number of colors necessary for an $L(h,1)$-labeling are known and such a coloring can be computed efficiently [8, 10, 12]. Nevertheless, in general, the decisional version of the problem is NP-complete for all the values of $h = 0, 1, 2$ [1, 12]. Therefore, for many classes of graph – such as chordal graphs [15], interval graphs [8], split graphs [3], unigraphs [6], hypercubes [17], clique graphs of some classes of graphs [7] – approximate bounds have been looked for.

1.1 Our Results

In many real life cases the actual network topologies are planar, because they consist of communication stations located in geographical area with non intersecting communication channel. Since the $L(h,1)$-labeling problem remains NP-complete, even when restricted to planar graphs [1, 11], in this paper we study the $L(h,1)$-labeling problem on two subclasses of planar graphs: the regular tilings of the plane (including cellular networks, that are hexagonal grids, and their interference graphs, that are triangular grids) and the class of outerplanar graphs, interesting because all the nodes lie on the border of the external face.

Concerning regular tilings, we provide a unique parametric algorithm that optimally solves the $L(h,1)$-labeling problem on each of the three regular tilings for all three values of $h = 0, 1, 2$. Furthermore, we give a function of the coordinates of each node in the network conputing the channel in constant time in a distributed fashion.

To the best of our knowledge, in the literature there is a result due to Bertossi, Pinotti and Tan [2], presenting three different algorithms for the three regular tilings of the plane in the special case $h = 2$. Furthermore, a result due to van den Heuvel, Leese and Shepherd [13] – concerning a variation of the problem: the span is cyclic – deals with squared and triangular tilings.

As regards outerplanar graphs, the previous best known results are due to Bodlaender et al. [3] and to Zhou at al. [19]. If $h = 2$, Bodlaender et al. prove that at most $\Delta + 9$ colors are sufficient to $L(2,1)$-label an outerplanar graph with maximum degree Δ and they suspect this bound is not tight. In particular, they leave reducing the additive term 9 as an open problem conjecturing the tightest bound could be $\Delta + 3$. Here this conjecture is proved when the outerplanar graph has maximum degree $\Delta \geq 8$. A linear time algorithm that produces, for outerplanar graphs, an $L(2,1)$-labeling feasible, but not necessarily optimal, is provided. Moreover, for smaller values of Δ, we guarantee that the number of colors used is bounded by 11, which improves anyway the bound $\Delta + 9$. Nevertheless, we conjecture that the bound $\Delta + 3$ holds for any outerplanar graph of degree $\Delta \geq 4$. Indeed, in the special case $\Delta = 3$, we show an outerplanar graph needing $\Delta + 4$ colors and we present an algorithm $L(2,1)$-labeling with at most $\Delta + 6$ colors for any outerplanar graph of degree

3. Understanding whether $\Delta + 6$ is a tight bound or not remains an open problem. Note that our algorithms run in $O(n)$ time improving by a factor Δ the best known results [3].

The paper by Zhou et al. [19] presents a polynomial time algorithm for solving a generalization of the $L(1,1)$-labeling on partial k-trees. With a simple modification, this algorithm can be used to solve also the $L(0,1)$-labeling problem. Hence, this algorithm works on outerplanar graphs in the case $h = 0$ and $h = 1$ since outerplanar graphs are series-parallel graphs, and series-parallel graphs are exactly partial 2-trees [4]. In the present paper, time complexity is reduced from $O(n^3)$ to $O(n)$ for $h = 0, 1$. More precisely, our linear time algorithm guarantees an optimal $L(h,1)$-labeling for outerplanar graphs of degree $\Delta \geq 4$ if $h = 0$ and $\Delta \geq 7$ if $h = 1$ using no more than Δ and $\Delta + 1$ colors, respectively. Consequently, the output is an optimal labeling because Δ and $\Delta + 1$ are proven to be the minimum number of colors necessary to $L(h,1)$-label any graph, $h = 0, 1$, for sufficiently large values of Δ ($\Delta \geq 4$ and $\Delta \geq 7$, respectively).

This paper is organized as follows: the next Section is devoted to the $L(h,1)$-labeling of the regular tilings of the plane. Subsequent sections deal with outerplanar graphs: in Section 3 some preliminary results are stated, then – in Section 4 – an algorithm for $L(h,1)$-labeling outerplanar graphs is provided; finally, the special case $\Delta = 3$ is treated in Section 5. Concluding remarks and some open problems are outlined in Section 6.

2 Regular Tilings of the Plane

In this section we focus on the problem of $L(h,1)$-labeling any regular tiling, for $h = 0, 1, 2$. More in detail, we show that $\Delta + 2^h - 1$ colors are necessary and sufficient to give an $L(h,1)$-labeling, $0 \leq h \leq 2$, to a regular tiling of degree Δ. Similar results have been independently achieved in [2] for $h = 2$ and in [13] for a variation of the problem restricted to squared and triangular tilings.

Here we provide a unique simpler parametric algorithm that optimally solve the $L(h,1)$-labeling problem on all the three regular tilings for all the three values of $h = 0, 1, 2$. We also show how to convert this algorithm into its constant time distributed version.

The tiling problem consists in covering the plane with copies of the same polygon and it is known that the only regular polygons that can be used in a tiling are hexagons, squares and triangles [14]. Let Δ be the degree of the tiling: for the hexagonal tiling $\Delta = 3$, for the squared tiling $\Delta = 4$ and for the triangular tiling $\Delta = 6$. In these three different cases, it is possible to highlight a common basic element that is a hexagon: in the hexagonal tiling, we take the hexagonal tile (see Fig.1.a). In the squared tiling, the hexagon is generated by two adjacent squares, producing a hexagon with a chord (see Fig. 1.b). Finally in the triangular tiling, the hexagon is generated by a group of 6 triangles, building a wheel (see Fig. 1.c).

Next lemma estimates a lower bound on the number of colors necessary to $L(h,1)$-label regular tilings.

Lemma 2.1 *Any $L(h,1)$-labeling of a degree Δ regular tiling of the plane, $0 \leq h \leq 2$, uses at least $\Delta + 2^h - 1$ colors.*

Proof: All nodes in the tiling have degree 3, 4 or 6, depending from the shape of the tiling (hexagonal, squared or triangular, respectively). Consider any node a of the tiling of degree Δ and let $\alpha_1, \alpha_2, \ldots, \alpha_\Delta$ be the colors assigned by an optimal $L(h,1)$-labeling to the nodes adjacent to a. These colors must be different in view of Property 2. of Def. 1.1, because each pair of these nodes is connected by a two long

Figure 1: Hexagonal basic element H in the three tilings.

path via a. Property 1. suggests how many colors we have to add in order to color a according to the h-value. Let α_0 be the color of a.

If $h = 0$, α_0 may be chosen among the Δ colors $\alpha_1, \alpha_2, \ldots, \alpha_\Delta$.

$h = 1$, implies not to use α_0 for a's adjacent nodes, i.e. to add one new color: in this case globally $\Delta + 1$ colors are required.

Finally, if $h = 2$, we need three colors more in order to guarantee that $|\alpha_0 - \alpha_i| \geq 2$ for any $1 \leq i \leq \Delta$, then $\Delta + 3$ colors are necessary. It is to notice that if α_0 is either the first or the last used color, it seems that $\Delta + 2$ colors are sufficient because one between $\alpha_0 + 1$ and $\alpha_0 - 1$ will be out of the span. Nevertheless, we need $\Delta + 3$ colors, because our reasoning must be done for the general node a and, if a certain node is colored with the first or the last used color, then at least one among its neighbors is not, and this node can be chosen as new a. □

Now we prove that, for each regular tiling, $\Delta + 2^h - 1$ colors are also sufficient.

Lemma 2.2 *There exists an $L(h, 1)$-labeling of the degree Δ regular tiling of the plane, $0 \leq h \leq 2$, using $\Delta + 2^h - 1$ colors.*

Proof: Consider a hexagonal basic element of the tiling H, whose nodes are named according to Fig. 1. Suppose we have provided an $L(h, 1)$-labeling of H with the following properties:

color$(a) + f_1 =$ color(c); color$(f) + f_1 =$ color(d);
color$(b) + f_2 =$ color(f); color$(c) + f_2 =$ color(e);
color$(a) + f_3 =$ color(e); color$(b) + f_3 =$ color(d);
for some $f_1, f_2, f_3 \in Z$ where the sums are computed modulo $\Delta + 2^h - 1$.

The coloring of any hexagonal element H' adjacent to H can be deduced by the coloring of H in the following way: the color of each node in H' is obtained from the color of the corresponding node in H adding one of the functions f_i. The choice of the function depends on the shared edge between H and H'; namely, $(c, d), (f, e)$ and (e, d) imply to sum f_1, f_2 and f_3, respectively, while $(a, f), (b, c)$ and (a, b) imply to sum $-f_1$, $-f_2$ and $-f_3$, respectively.

By iterating this procedure, to all non-labeled hexagonal elements adjacent to some colored hexagonal element, we obtain a feasible $L(h, 1)$-labeling, because of the constraint we imposed to the initial one (see Fig. 2).

Now, we have to provide a coloring for H – among all the feasible ones – satisfying the constraints given at the beginning of the proof. Of course, for each value of h we have a different coloring of H.

Let us consider $h = 2$, first. In this case we have a 'universal' coloring, that is a coloring feasible for all the shapes of tilings: we assign to a, b, c, d, e and f colors $1, 3, 0, 4, 2$ and 5, respectively, and to g – if it exists – color 7, where $f_1 = -1, f_2 = 2$ and $f_3 = 1$.

Figure 2: A feasible $L(2,1)$-labeling of the regular tilings of the plane.

If $h = 1$, we have to distinguish different labelings of H according to the shape of the tiling. In particular, for the hexagonal tiling, the sequence a, b, c, d, e, f can be colored with colors $0, 1, 3, 2, 1, 3$; for the squared tiling the sequence of colors is $0, 1, 4, 2, 1, 3$; finally, in the triangular tiling, H can be colored with the sequence $1, 4, 0, 5, 2, 6$ plus color 3 for node g; in any case it holds $f_1 = -1, f_2 = 2$ and $f_3 = 1$.

Finally, if $h = 0$, a possible coloring for H is $0, 1, 2, 2, 1, 0$, where $f_1 = 2, f_2 = -1$ and $f_3 = 1$, and it is feasible both for the hexagonal and for the squared tiling; for the triangular tiling we can color H with the sequence $0, 1, 4, 3, 2, 5$ plus 0 in the middle, where $f_1 = -2, f_2 = -2$ and $f_3 = 2$. ∎

Theorem 2.3 *For any degree Δ regular tiling of the plane and $h = 0, 1, 2$, $\lambda_{h,1} = \Delta + 2^h - 2$.*

Proof: Lemmas 2.1 and 2.2 prove the assertion. ∎

We remark that our bound on the number of colors has the same elegant appearance as the following general lemma:

Lemma 2.4 *[1, 12, 18] For any graph G, the following lower bound holds:* $\lambda_{h,1} \geq \Delta + h - 1$.

For these networks, a channel can be assigned to any node in constant time, provided that relative positions of the node in the network is locally known. As an example, we will show only the function relative to the hexagonal tiling, in order not to make tedious the reading. Analogous functions can be derived for the other tilings.

Let us consider the hexagonal tiling as in Fig. 3. The general node of coordinates (i, j) must be labeled with color:

0 if either $i = 0 \bmod 3$ and $j = \lfloor i/3 \rfloor + 0 \bmod 4$ or $i = 1 \bmod 3$ and $j = \lfloor i/3 \rfloor + 2 \bmod 4$;

1 if either $i = 0 \bmod 3$ and $j = \lfloor i/3 \rfloor + 1 \bmod 4$ or $i = 2 \bmod 3$ and $j = \lfloor i/3 \rfloor + 0$;

2 if either $i = 1 \bmod 3$ and $j = \lfloor i/3 \rfloor + 3 \bmod 4$ or $i = 2 \bmod 3$ and $j = \lfloor i/3 \rfloor + 1 \bmod 4$;

3 if either $i = 0 \bmod 3$ and $j = \lfloor i/3 \rfloor + 2 \bmod 4$ or $i = 1 \bmod 3$ and $j = \lfloor i/3 \rfloor + 3 + 0 \bmod 4$;

4 if either $i = 0 \bmod 3$ and $j = \lfloor i/3 \rfloor + 3 \bmod 4$ or $i = 2 \bmod 3$ and $j = \lfloor i/3 \rfloor + 2 \bmod 4$;

5 if either $i = 1 \bmod 3$ and $j = \lfloor i/3 \rfloor + 1 \bmod 4$ or $i = 2 \bmod 3$ and $j = \lfloor i/3 \rfloor + 3 \bmod 4$.

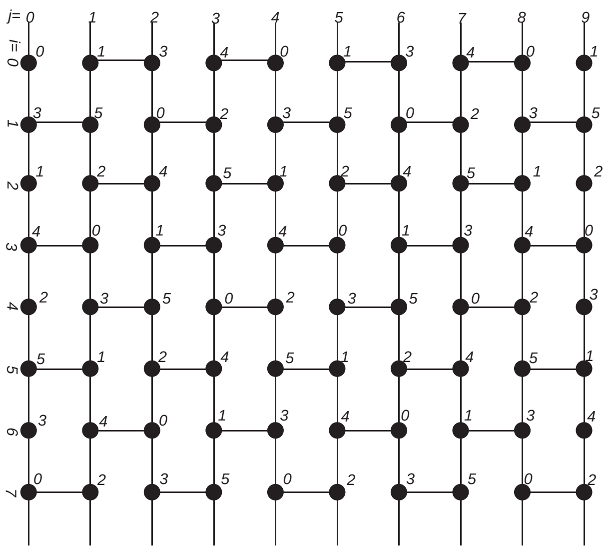

Figure 3: A feasible $L(2, 1)$-labeling of the hexagonal tiling of the plane, where the coordinates of the nodes are highlighted.

3 Preliminary Results

In this section we introduce some notations and two lemmas which the algorithm presented in the next section is based on.

A graph G is called *planar* if it can be represented on a plane by distinct points for nodes and simple curves for edges in such a way that any two such curves do not meet anywhere other than at their endpoints. The representation of G on the plane, according to the mentioned conditions, is called an *embedding.* A graph is *outerplanar* if it can be embedded in the plane so that every node lies on the boundary of the outer face. It follows that, once the first node has been chosen, clockwise order induces a total order on the nodes of the graph.

In the following, we assume that the graphs we handle are loopless, simple and connected.

3.1 Ordered Breadth First Search

Consider an embedding of an outerplanar graph G, choose a node r and induce the total order on the nodes clockwise. Now, compute a Breadth First Search starting from node r in such a way that nodes coming first in the ordering are visited first. In the following we will call *Ordered Breadth First Search (OBFS)* such a computation and *Ordered Breadth First Tree (OBFT)* the (unique) resulting tree (for an example, see Fig. 4.b). The left to right direction on each layer l of the OBFT induces a numbering of the nodes: we will call $v_{l,i}$ a node lying on layer l that occupies the i-th position in the left to right ordering on the layer (see Fig. 4.c).

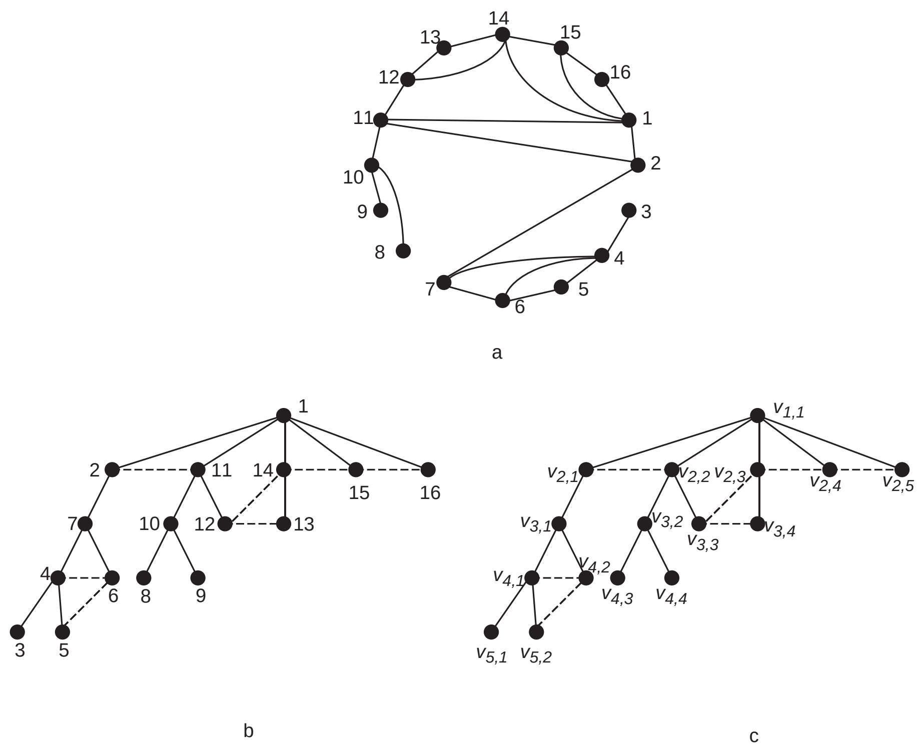

Figure 4: An outerplanar graph and its OBFT.

Before characterizing OBFTs for outerplanar graphs, we have to recall the properties of a general Breadth First Tree.

Fact 3.1 *Let $T = (V, E')$ be a Breadth First Tree for a general graph $G = (V, E)$; for each non tree edge $(v_{l,h}, v_{l',k}), l' \geq l$, it holds:*

- *either* $l' = l$ *or*
- $l' = l - 1$ *and* $r < k$*, where* r *is the index of the father of* $v_{l,h}$ *at layer* $l - 1$*.*

Lemma 3.2 *Every OBFT of an outerplanar graph G has the following properties:*
- if a non-tree edge connects nodes $v_{l,h}$ *and* $v_{l,k}$*,* $h < k$*, then* $k = h+1$ *(e.g. see edges* $(v_{4,1}, v_{4,2})$ *and* $(v_{3,3}, v_{3,4})$ *in Fig. 4.c);*
- if a non-tree edge connects nodes $v_{l,h}$*, child of* $v_{l-1,r}$*, and* $v_{l-1,k}$*, then* $k = r+1$ *and* $v_{l,h}$ *is the rightmost child of* $v_{l-1,r}$ *(e.g. see edges* $(v_{5,2}, v_{4,2})$ *and* $(v_{3,3}, v_{2,3})$ *in Fig. 4.c).*

Proof: We prove the two properties separately, starting from the first one.

Let us suppose, by contradiction, $k > h + 1$.

First, consider the case $v_{l,h}$ and $v_{l,k}$ children of the same node $v_{l-1,r}$; it follows that $v_{l,h+1}$ is child of $v_{l-1,r}$, too. Consider the subgraph induced by $v_{l,h}, v_{l,h+1}$ and $v_{l,k}$, that appear on the outer face of G in this order clockwise for the definition of OBFT. Node $v_{l-1,r}$ can lie either outside or inside this sequence. In the first case a crossing occurs between edges $(v_{l-1,r}, v_{l,h+1})$ and $(v_{l,h}, v_{l,k})$; in the second case $v_{l-1,r}$ cannot lie in the middle of the sequence, otherwise even the root of the tree would lie in the middle of the sequence and the OBFS would visit $v_{l-1,r}$'s children in a different order: in any case, we have a contradiction.

Now, let $v_{l,h}$ and $v_{l,k}$ be children of two different nodes, $v_{l-1,r}$ and $v_{l-1,s}$, respectively, $r < s$. It is not restrictive to suppose $v_{l,h+1}$ child of $v_{l-1,r}$. Indeed, $v_{l,h+1}$ child of $v_{l-1,s}$ leads to analogous reasonings, and $v_{l,h+1}$ child of another node $v_{l-1,t}$ moves the role of $v_{l-1,r}$ to the first common ancestor of $v_{l-1,r}$ and of $v_{l-1,t}$. Again, $v_{l,h}, v_{l,h+1}$ and $v_{l,k}$ must be in this order clockwise on the external face of G and $v_{l-1,r}$ lies in the middle of the sequence, otherwise edges $(v_{l-1,r}, v_{l,h+1})$ and $(v_{l,h}, v_{l,k})$ would cross. Suppose first $v_{l-1,r}$ is between $v_{l,h+1}$ and $v_{l,k}$: $v_{l-1,s}$ can be positioned either before or behind $v_{l,k}$. If $v_{l-1,s}$ is behind $v_{l,k}$ then we have an absurd because it is not possible to position on the outer face of G any common ancestor of $v_{l-1,r}$ and $v_{l-1,s}$ without introducing crossings; hence, let $v_{l-1,s}$ be before $v_{l,k}$. Also this case is not possible since any common ancestor w of $v_{l-1,r}$ and $v_{l-1,s}$ (included the root) cannot lie both outside and inside the sequence. Indeed, w outside the sequence would generate crossings in G or would contradict the assumption $r < s$. Indeed, in order to avoid crossings in G, w must be between $v_{l-1,r}$ and $v_{l-1,s}$. This fact and the definition of OBFT imply that $s < r$.

Suppose now $v_{l-1,r}$ between $v_{l,h}$ and $v_{l,h+1}$. Then, the only possible position for $v_{l-1,s}$ is between $v_{l,h+1}$ and $v_{l,k}$; hence, any common ancestor of $v_{l-1,r}$ and $v_{l-1,s}$ (included the root of the OBFT) must lie in the same interval, i.e. again $r < s$, a contradiction.

Now, we prove the second property, and show separately the two conditions. First we show that it must be $k = r + 1$, and then that $v_{l,h}$ must be the rightmost child.

Suppose $k > r + 1$, then we have the ordered sequence $v_{l-1,r}, v_{l-1,r+1}, v_{l-1,k}$ on the outer face of G. As in the previous case, it is not restrictive to assume that these three nodes are children of the same father $v_{l-2,p}$. Consequently, $v_{l-2,p}$ lies outside the sequence $v_{l-1,r}, v_{l-1,r+1}, v_{l-1,k}$. It is easy to see that, anywhere $v_{l,h}$ is positioned, it is impossible to insert both edge $(v_{l-1,r}, v_{l,h})$ and edge $(v_{l,h}, v_{l-1,k})$ without introducing any crossing. An absurd arises from considering $k > r + 1$.

Now, it remains to prove that $v_{l,h}$ is the rightmost child of $v_{l-1,r}$. If we suppose the existence of a right sibling $v_{l,h+1}$ of $v_{l,h}$, the sequence $v_{l,h}, v_{l,h+1}$ would be ordered clockwise on the outer face of G. Node $v_{l-1,r}$ can lie either outside or inside the sequence. If it is outside, we have the ordered sequence $v_{l-1,r}, v_{l,h}, v_{l,h+1}$. If $v_{l-1,r+1}$ lies outside the sequence, edge $(v_{l,h}, v_{l-1,r+1})$ introduces a crossing. Also $v_{l-1,r+1}$ in the middle of the sequence leads to a contradiction, since every common ancestor of $v_{l-1,r}$ and $v_{l-1,r+1}$ must lie between $v_{l-1,r+1}$ and $v_{l,h+1}$, and therefore $v_{l,h+1}$ would come before $v_{l,h}$ in the ordering. It remain to consider the case in which $v_{l-1,r}$ lies between $v_{l,h}$ and $v_{l,h+1}$. Now, $v_{l-1,r+1}$ must lie outside the sequence, and any common ancestor of $v_{l-1,r}$ and $v_{l-1,r+1}$ must do the same. A contradiction holds because this configuration leads $v_{l-1,r+1}$ to come before $v_{l-1,r}$ in the ordering. $\square$

3.2 Graph W_Δ

Our $L(h,1)$-labeling algorithm of outerplanar graphs is based on the coloring of simple substructures, that are studied in the following.

Let $W_\Delta(V,E)$ be the outerplanar graph defined as follows:
$V = \{v_0, v_1, \ldots, v_\Delta\}; |V| = \Delta + 1$ and $E = \{(v_0, v_i), 1 \le i \le \Delta\} \cup \{(v_i, v_{i+1}), 1 \le i \le \Delta - 1\}$ (see Figs. 5 and 6).

Lemma 3.3 *For any W_Δ and $h = 0, 1$, $\lambda_{h,1}(W_\Delta) = \Delta + h - 1$.*

Proof: Let us prove first that $\Delta + h$ colors are necessary to $L(h,1)$-label $W_\Delta, h = 0, 1$. Let $\alpha_0, \alpha_1, \ldots, \alpha_\Delta$ be the colors assigned to v_0, v_1, …, v_Δ, respectively. Observe that:
- $\alpha_i \ne \alpha_j$ for each $1 \le i, j \le \Delta$ for Property 2. of Def. 1.1, as each pair of nodes v_i and v_j are connected by a two long path via v_0 (Δ colors);
- $|\alpha_0 - \alpha_i| \ge h$ for each $1 \le i \le \Delta$ for Property 1. of Def. 1.1, as v_0 is adjacent to each v_i (at least h more colors). Therefore, at least $\Delta + h$ colors are necessary. This number of colors is also sufficient, indeed possible $L(h,1)$-labelings of W_Δ are the following:
- if $h = 1$, let $\alpha_0, \alpha_1, \ldots, \alpha_\Delta$ be different colors from $0, 1, \ldots, \Delta$ in any order;
- if $h = 0$, let $\alpha_1, \alpha_2, \ldots, \alpha_\Delta$ be different colors from $0, 1, \ldots, \Delta - 1$ in any order; let α_0 be any already used color. As an example see Fig. 5. $\square$

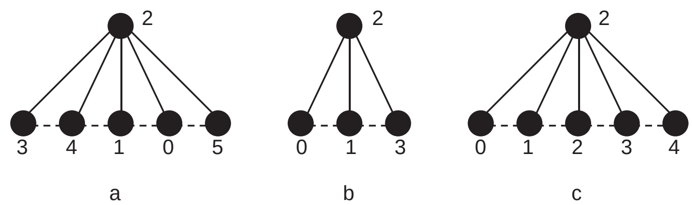

Figure 5: Two possible $L(1,1)$-labelings of W_Δ (a. and b.) and an $L(0,1)$-labeling of W_Δ (c.).

Lemma 3.4 *For any $W_\Delta, \Delta \ge 4$, $\lambda_{2,1}(W_\Delta) = \Delta + 1$.*

Proof: The proof of the necessity is exactly the same as that one of the previous Lemma, substituting 2 to h.

The sufficiency is shown providing some possible $L(2,1)$-labelings. When $\Delta = 4$, an optimal $L(2,1)$-labeling is shown in Fig. 6.a. For $\Delta \ge 5$, a possible labeling rule is the following:
- set $\alpha_0 = 0$; this choice inhibits colors 0 and 1 to nodes $v_1, \ldots, v_\Delta$;
- label the sequence $v_1, v_2, \ldots, v_\Delta$ with $\Delta+1, \Delta-1, \Delta-3, \ldots, 3, \Delta, \Delta-2, \Delta-4, \ldots, 2$ if Δ is even and $\Delta+1, \Delta-1, \Delta-3, \ldots, 2, \Delta, \Delta-2, \Delta-4, \ldots, 3$ if Δ is odd (see Fig. 6.b).

Finally, observe that if $\Delta \le 3$, $\Delta + 2$ colors are not enough to color the graph because there is no way to satisfy the condition $|\alpha_i - \alpha_{i+1}| \ge 2, i = 1, 2$ (see Fig. 6.c). $\square$

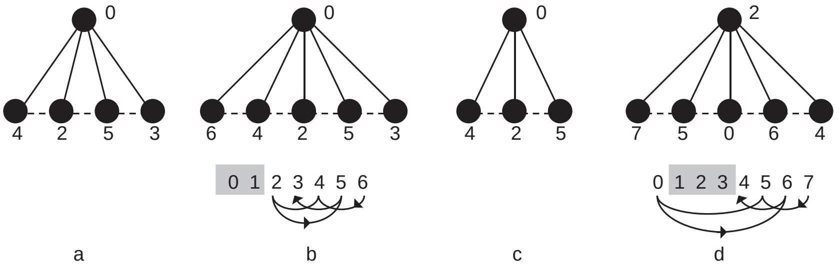

Figure 6: W_Δ with some possible $L(2,1)$-labelings.

It is to remark that Lemma 3.4 works only assigning either color 0 or color $\Delta+1$ to v_0. Indeed, if v_0 were colored with any color in the range $[1, \Delta]$ one more color would be necessary.

Corollary 3.5 *For any $W_\Delta, \Delta \geq 4$, if v_0, the node of degree Δ, has an assigned color α_0 different from 0 and from $\Delta+1$, $\lambda_{2,1}(W_\Delta) = \Delta+2$.*

Proof: Labels $\alpha_0 - 1, \alpha_0$ and α_0+1 are inhibited to nodes $\alpha_i, i = 1, \ldots \Delta$. Then the necessity follows.

For the sufficiency, we label $v_1, \ldots v_\Delta$ with Δ colors different from α_0, $\alpha_0 - 1$ and $\alpha_0 + 1$ and we assign to two consecutive nodes, among $v_1, \ldots v_\Delta$, colors at distance at least two. A possible way to label is depicted in Fig. 6.d. $\square$

Now, we want to highlight a general $L(2,1)$-labeling scheme for $W_\Delta, \Delta \geq 4$, summarizing the results in the last two claims.

Given a $W_\Delta, \Delta \geq 4$, whose v_0 has already been colored, and $\Delta + k$ consecutive available colors, $k \geq 0$, if k colors are forbidden for $v_1, \ldots, v_\Delta$, let us call $c_1, \ldots, c_\Delta$ the Δ non forbidden colors in increasing order. Then, assign to $v_1, \ldots, v_\Delta$ colors:
$c_\Delta, c_{\Delta-2}, \ldots, c_2, c_{\Delta-1}, c_{\Delta-3}, \ldots, c_1$ if Δ is even and
$c_\Delta, c_{\Delta-2}, \ldots, c_1, c_{\Delta-1}, c_{\Delta-3}, \ldots, c_2$ if Δ is odd.

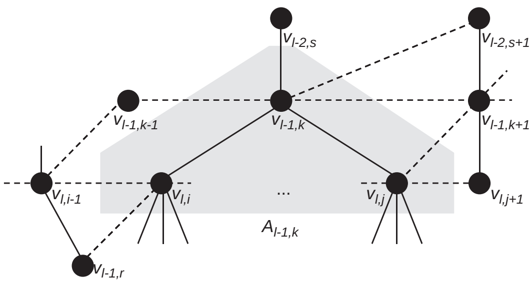

Figure 7: Subgraph $A_{l-1,k}$.

Before concluding this section, let us consider a special subgraph of W_Δ, that will be essential for the algorithm described in the next section.

Let G be an outerplanar graph of maximum degree Δ and T be one of its OBFTs. Call $A_{l-1,k}$ the subgraph of G induced by a general node $v_{l-1,k}$ and by the group of all its children in T, $v_{l,i}, \ldots, v_{l,j}$ (see Fig. 7).

For each $A_{l-1,k}$, we want to highlight the fan of edges outgoing from it:

a. one tree edge connecting $v_{l-1,k}$ with its father;

b. at most three non tree edges connecting $v_{l-1,k}$ with two nodes at the same layer $l-1$ and with one node at layer $l-2$;

c. at most two non tree edges connecting the leftmost sibling $v_{l,i}$ with $v_{l,i-1}$ and with the rightmost child of $v_{l,i-1}$, $v_{l+1,r}$;

d. at most one non tree edge connecting the rightmost sibling $v_{l,j}$ with $v_{l-1,k+1}$;

e. at most one non tree edge connecting the rightmost sibling $v_{l,j}$ with $v_{l,j+1}$;

f. all the tree edges from $v_{l,i}, \ldots, v_{l,j}$ to their children.

For the sake of completeness, we have listed all the edges incident to $A_{l-1,k}$; nevertheless, the edges of kind e and f will not be used by our algorithm.

Remark 2 *$A_{l-1,k}$ is a subgraph of W_Δ, therefore for it Cor. 3.5 holds* a fortiori.

4 An Algorithm for Outerplanar Graphs

In this section, we present a linear time algorithm for $L(h,1)$-labeling an outerplanar graph. For $h=2$, we prove that $\Delta+3$ colors are always sufficient if $\Delta \geq 8$. In this way we prove the conjecture left open in [3]. Although, for small degree, we leave the conjecture open, our algorithm improves anyway the previously known results, guaranteeing 11 colors instead of $\Delta+9$, (≥ 13).

For $h=0,1$, we prove that the problem is polynomial and we provide an optimal coloring for sufficiently large values of Δ ($\Delta \geq 4$ and $\Delta \geq 7$, respectively).

Using the results of the previous section, we can describe the following algorithm, finding an $L(2,1)$-labeling of an outerplanar graph:

Algorithm `Label Outerplanar Graphs`
Input: An outerplanar graph G of maximum degree Δ;
Output: An $L(2,1)$-labeling of G;

1. Consider a maximum degree node v and run an OBFS starting from v.
2. Label v with color 0.
3. Label layer 1 according to Lemma 3.4.
4. Repeat for each layer $l \geq 2$, from left to right, from top to down:
   - Ordinately consider $v_{l-1,k}$ and subgraph $A_{l-1,k}$: $v_{l-1,k}$ has already been colored, while $v_{l,i}, \ldots, v_{l,j}$ must still be labeled.
   - Label nodes $v_{l,i}, \ldots, v_{l,j}$ according to Cor. 3.5 (cf. Remark 2), eliminating from the feasible colors all colors forbidden by edges of kind a, b, c and d;

Theorem 4.1 *Given an outerplanar graph G of degree Δ, algorithm* `Label Outerplanar Graphs` *correctly computes an $L(2,1)$-labeling; if $\Delta \geq 8$ the algorithm uses at most $\Delta+3$ colors, otherwise it uses at most 11 colors.*

Proof: The correctness of the algorithm directly descends from Lemma 3.4 and from Corollary 3.5.

For what concerns the number of used colors, we will prove by induction on the number of colored nodes that $\Delta+3$ colors are always enough if $\Delta \geq 8$ and, to do that, we will distinguish how each kind of edges acts on the number of necessary colors. We consider only edges from a to d, since the edges of kind e and f do not connect any node of $A_{l-1,k}$ to colored nodes and thus can be left out.

Basis: Our algorithm uses $\Delta + 2$ colors to correctly color layers 1 and 2 of the tree in view of Lemma 3.4.

Induction: Assume we have already colored with $\Delta + 3$ colors all nodes in layers $1, \ldots, l-1$, and all nodes in layer $l, l \geq 3$, from left to node $v_{l,i-1}$, while $v_{l,i}$ has to be colored. We will prove that we do not need to add further colors.

It is not restrictive to assume that $v_{l-1,k}$ has degree Δ in G; indeed, if its degree is lower, then less colors are enough. First observe that, as stated in Cor. 3.5, at most three colors must be eliminated from the $\Delta + 3$ available colors, to label $v_{l,i}, \ldots, v_{l,j}$.

Edge of kind a forbids one color more, since the father of $v_{l-1,k}$ is connected by a two long path to each $v_{l,t}, i \leq t \leq j$. So we have at least $(\Delta + 3) - 3 - 1 = \Delta - 1$ available colors.

Now, let us consider the x non tree edges of kind b, $0 \leq x \leq 3$. The $\Delta - x - 1$ siblings $v_{l,i}, \ldots, v_{l,j}$ must avoid the color of the x nodes adjacent to their father, hence have $(\Delta - 1) - x$ available colors, that are enough in view of the reasonings after Cor. 3.5 if $\Delta - 1 - x \geq 4$, i.e. $\Delta \geq 8$.

Let us suppose that there exists an edge of kind c and, in particular $v_{l,i}$ is directly connected to $v_{l,i-1}$. Then, $v_{l,i+1}$ must be colored differently from $v_{l,i-1}$, while $v_{l,i}$ needs a label distant at least two from $v_{l,i-1}$ and different from the colors of $v_{l,i-1}$'s adjacent nodes at layers l and $l-1$ (at most three). So, $v_{l,i}$ can choose its color among $(\Delta - 1 - x) - 3 - 3$, $v_{l,i+1}$ among $(\Delta - 1 - x) - 1$ and all other nodes are colored according to Cor. 3.5. Observe that from these considerations it follows $\Delta \geq 11$. If, on the contrary, $v_{l,i}$ is connected to $v_{l,i-1}$ by means of a two long path, it must avoid the color of $v_{l,i-1}$. It is to notice that if $v_{l,i}$ is connected to $v_{l,i-1}$ both by an edge and a two long path, then this second connection does not add any constraints.

Finally, let us assume there exists the edge of kind d. Then $v_{l,j-1}$ needs to be colored differently from $v_{l-1,k+1}$, while, $v_{l,j}$ must avoid the three colors including the color of $v_{l-1,k+1}$ and all the colors of $v_{l-1,k+1}$'s adjacent nodes at layers $l-1$ and $l-2$ (at most three). Hence, $(\Delta - 1 - x) - 1$ colors are feasible to color $v_{l,j-1}$ and only $(\Delta - 1 - x) - 3 - 3$ to color $v_{l,j}$. Again, we can apply Cor. 3.5 and label all the siblings, if $\Delta \geq 11$.

The limitation $\Delta \geq 11$ can be improved to $\Delta \geq 8$ by observing that, in the previous reasonings, nodes $v_{l-2,s+1}$, $v_{l-1,k-1}$ and $v_{l-1,k+1}$ are considered twice.

If $3 \leq \Delta \leq 7$, all the previous reasonings continue to hold if we substitute to $\Delta + 3$ value 8+3=11 for the colors anyway available. $\square$

Better results can be derived if the input graph is triangulated:

Theorem 4.2 *Given an outerplanar triangulated graph G of degree Δ, algorithm* `Label Outerplanar Graphs` *correctly computes an $L(2,1)$-labeling; if $\Delta \geq 8$ the algorithm uses at most $\Delta + 2$ colors, otherwise it uses 10 colors.*

Proof: It is possible to repeat the proof of the previous theorem, observing that $v_{l-1,k}$ and $v_{l-1,k+1}$ must be connected, since G is triangulated. It follows that $v_{l-1,k}$ cannot have more than $\Delta - 2$ children and therefore $\Delta + 2$ colors are enough. $\square$

Theorem 4.3 *Given an outerplanar graph G, Algorithm* `Label Outerplanar Graphs` *runs in $O(n)$ time, independently from its degree Δ.*

Proof: The algorithm runs in $O(n)$ steps, each one labeling a group of siblings, children of the general $v_{l-1,k}$. Each step takes time proportional both to the number of the neighbors of $A_{l-1,k}$ and to the time necessary to guarantee the $L(2,1)$-constraints.

The number of the neighbors is constant and reaches at most 12: 1 (in view of edge a) + 3 (in view of edges b) + 4 (in view of edges c, considering also at most three neighbors of $v_{l,i-1}$) + 4 (in view of edge d, considering also at most three neighbors of $v_{l-1,k}$).

It remains to show how a shrewd implementation allows one to check the $L(2,1)$-constraints in $O(1)$ time. Let the set of available colors be maintained in an ordered linked list, whose records are also pointed by an array. For each already colored neighbor, its color is eliminated from the list in constant time by means of the array; once the colors of all the neighbors have been checked, the first available color is pointed by the head of the list. $\square$

An obvious lower bound for any degree Δ graph is $\lambda_{2,1} \geq \Delta + 1$. Then, for outerplanar triangulated graphs, $\Delta \geq 8$, our algorithm provides an optimal $L(2,1)$-labeling, while for outerplanar graphs, $\Delta \geq 8$, it generates a labeling at most one color far from optimum. Our conjecture is that Algorithm `Label Outerplanar Graphs` gives an optimal solution also in this latter case.

By slightly modifying Algorithm `Label Outerplanar Graphs`, Theorem 4.1, Theorem 4.3 and their proofs, we obtain the following results:

Theorem 4.4 *Given an outerplanar graph G of degree Δ, there exists an algorithm that correctly computes an $L(h,1)$-labeling in polynomial time, $h = 0, 1$. If $\Delta \geq 4$ and $h = 0$ the algorithm uses at most Δ colors, otherwise it uses 4 colors. If $\Delta \geq 7$ and $h = 1$ the algorithm uses at most $\Delta + 1$ colors, otherwise it uses $\Delta + 3$ colors.*

The previous theorem improves the known bounds of $\Delta + 3$ and $\Delta + 5$ colors to $L(0,1)$- and $L(1,1)$-label outerplanar graphs, respectively. Furthermore, observe that at least Δ and $\Delta + 1$ colors are necessary to $L(0,1)$- and $L(1,1)$-label any degree Δ graph; hence, this theorem states the optimality of the solution found by the algorithm, when $\Delta \geq 4$ and $h = 0$ or $\Delta \geq 7$ and $h = 1$.

It has been proven that the $L(h,1)$-labeling is NP-complete both in general [1] and when restricted to planar graphs [3, 11]. In this paper we have proven that:

Corollary 4.5 *The $L(h,1)$-labeling problem on a degree Δ outerplanar graph G is in P, if:*

- $h = 0$ *and* $\Delta \geq 4$,
- $h = 1$ *and* $\Delta \geq 7$, *and*
- $h = 2$ *and* $\Delta \geq 8$ *and G is triangulated.*

5 The special case $\Delta = 3$

In the previous subsection we proved that, for rather big values of Δ, $\Delta+3$ colors are enough to $L(2,1)$-label a degree Δ outerplanar graph. From the other side, if $\Delta = 2$, it is known that any $L(2,1)$-labeling needs $\Delta + 3$ colors [12]. For $4 \leq \Delta \leq 7$ we are not able to prove the same bound, nevertheless we conjecture that it holds. Different considerations must be done in the special case $\Delta = 3$: indeed, it is possible to prove in an exhaustive way that the graph in Fig. 8, requires $\Delta + 4$ colors. This implies that the inequality $\lambda_{2,1} \leq \Delta + 2$ is false, for $\Delta = 3$. More formally:

Figure 8: A cubic outerplanar graph for which $\Delta + 3$ colors are not enough.

Theorem 5.1 *For the class of outerplanar graphs with* $\Delta = 3$*,* $\Delta + 3$ *colors are not always sufficient.*

In the following we call *cubic outerplanar graphs* the graphs belonging to the class of outerplanar graphs with maximum degree $\Delta = 3$. Their behavior is completely different from the other outerplanar graphs even if they are very simple in structure and must have at least two nodes of degree lower than three.

In the following we propose an algorithm running on cubic outerplanar graphs that improves the results of Thm. 4.1 when $\Delta = 3$: $L(2,1)$-labeling is guaranteed with at most 9 (instead of 11) colors, and at most 8 colors if the graph is triangle-free.

It remains an open problem to understand if this result is tight or not.

Before detailing the algorithm we need to do a pre-computation on the considered graph. Let $G = (V, E)$ be a cubic outerplanar graph, embedded with all its nodes on a circle and all its edges inside the circle (see Fig. 9.a). Since we deal with the embedding of G, it makes sense to speak about its faces. There exists at least an internal face, say $\tilde{f}$, otherwise G is a tree and it is known that it can be optimally $L(2,1)$-labeled with at most $\Delta + 3$ colors [12].

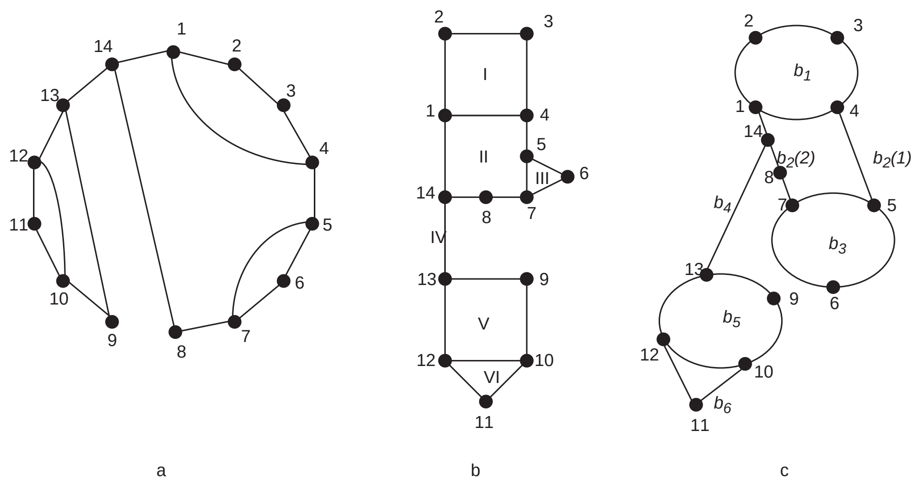

Figure 9: A cubic outerplanar graph represented with its nodes in circle, and highlighting its blocks.

We want to visit all the faces of G (not included the external one) starting from $\tilde{f}$ and to induce an order on them. Let G' be the portion of G already visited at the current step. At the beginning G' is empty. At the first step we visit $\tilde{f}$ and put it in G', i.e. we put in G' all its nodes and edges. If there exists a not yet visited face

f having at least an edge in common with G', then we visit f and we add it to G' (see face II in Fig. 9.b). If no face has such a property, then we look for a face f having only one node in common with G' (see face V in Fig. 9.b); if neither such a face exists, we visit an edge incident to some node of G' and add it to G' (see edge IV in Fig. 9.b).

This step is iterated until all the graph has been visited, i.e. $G' = G$. At the end of the procedure we have partitioned the graph into cycles and paths (*blocks* in general), sorted according to the visiting order (see Fig. 9.b). Observe that two cycles in the sequence can share an edge (and consequently two nodes) but not a unique node, as $\Delta = 3$. We want to transform this representation in order to guarantee that all the cycles are edge-disjoint. Face $\tilde{f}$ is a cycle. Let us call $R_1 = \{\tilde{f}\}$ the current transformed structure. For each successive block b_i in the order, if it has no edges in common with R_{i-1}, then $R_i = R_{i-1} \cup \{b_i\}$ (see Fig. 9.c, block b_5). Let b_i share an edge e with R_{i-1}. Then b_i is a cycle; let us distinguish two cases according to that e belongs to a cycle of R_{i-1} or not. If e is in a cycle of R_{i-1}, let us call p_i the path resulting by removing e from b_i; $R_i = R_{i-1} \cup \{p_i\}$ (see Fig. 9.c block b_6). If e is in a path $p_j, j < i$, of R_{i-1}, break p_j into two subpaths $p_j(1)$ and $p_j(2)$ eliminating e from p_j; $R_i = R_{i-1} - \{p_j\} \cup \{p_j(1)\} \cup \{p_j(2)\} \cup \{b_i\}$ (see Fig. 9.c, subpaths $b_2(1)$ and $b_2(2)$ and block b_3). In this way, G is represented by B blocks, that are either disjoint cycles or paths; on this representation, the previous order holds, even if paths belonging to the same face take consecutive, though arbitrary, numbers (see Fig. 9.c).

The following two properties hold:

Property 1: In R_i, each block b_i is connected to R_{i-1} by means of at most two nodes.

Property 2: Let x and y be two nodes belonging to the same block b_i, and let b_i be a cycle; if a path external to b_i connects x and y, then x and y are adjacent in the cycle b_i.

The first property descends from the definition of the order of the blocks; the second one holds since G is outerplanar and in view of the construction of this representation. The previous properties lie on the basis of the correctness of algorithm `Label Cubic Outerplanar Graphs`.

Before detailing the algorithm that $L(2,1)$-labels G, we need some preliminaries lemmas:

Lemma 5.2 *Given a simple path $P(n)$ with n nodes, if one of the endpoints of $P(n)$ is already colored with a color in $0 \ldots 4$, then 5 colors are enough to $L(2,1)$-label $P(n)$.*

Proof: Let $P(n)$ and $C(n)$ be constituted by nodes $v_0, v_1, \ldots, v_{n-1}$, and let v_0 be colored with color c. An easy modification of the $L(2,1)$-labeling of any simple path [12, 18] leads to the following rule: $color(v_i) = (c + 2i) \bmod 5$. □

Lemma 5.3 *Given a simple cycle $C(m)$ with m nodes, if either one or two of its adjacent nodes are already colored with color in the range $0 \ldots 5$, then 6 colors are enough to $L(2,1)$-label it, if m is at least 4; 7 colors are enough if $m = 3$.*

Proof: The statement can be proved in a exhaustive way, using the $L(2,1)$-labeling of simple cycles [12, 18]. Observe that 7 colors are necessary only in the case $m = 3$ and when the two already colored nodes have labels 1 and 4, respectively. □

The algorithm we now propose $L(2,1)$-labels any cubic outerplanar graph G considering cycles and edges in their order, and labeling their nodes according to the ordering of the blocks.

Algorithm `Label Cubic Outerplanar Graphs`
Input: An outerplanar graph G of maximum degree 3;
Output: An $L(2,1)$-labeling of G;

1. Divide G into B ordered blocks and individuate the edge disjoint blocks to be considered as cycles;
2. b_1 is a cycle: label it with the first 5 colors;
3. FOR $i = 2$ TO B DO
   For each colored node connected by a two long path to any node not yet colored in b_i, eliminate its color from the sequence of available colors;
   case 1 (b_i cycle)
     IF b_i is a cycle and either one or two of its adjacent nodes are already colored
       THEN Label b_i according to Lemma 5.3;
   case 2 (b_i path, one endpoint colored)
     IF b_i is a path and one of its endpoints is already colored
       THEN Label b_i according to Lemma 5.2;
   case 3 (b_i path, two endpoints colored)
     IF b_i is a path and both its endpoints are already colored
       THEN Consider b_i as a cycle and label it according to case 1;

Theorem 5.4 *Given a cubic outerplanar graph G, algorithm* `Label Cubic Outerplanar Graphs` *correctly computes an $L(2,1)$-labeling in polynomial time using at most $\Delta+6$ colors; if G is triangle free, then the number of necessary colors is $\Delta+5$.*

Proof: It is easy to see that algorithm `Label Cubic Outerplanar Graphs` runs in polynomial time.

The correctness of the algorithm descends from Properties 1 and 2; infact, Property 1 guarantees that in b_i no more than two nodes are already colored at steps $1, 2, \ldots, i-1$, while Property 2 ensures that, in b_i, the two already colored nodes are either endpoints (if b_i is a path) or adjacent nodes (if b_i is a cycle). Hence, algorithm `Label Cubic Outerplanar Graphs` covers all the possible cases.

It remains to prove that $\Delta+6$ colors are always enough to $L(2,1)$-label G ($\Delta+5$ if G is triangle free). Let us assume to have $\Delta+6 = 9$ colors. Only 5 of them are necessary to color b_1. Consider now the general block b_i. If case 1 holds and only one node of B_i is colored, then b_i is hanged to R_{i-1} by means of a path through a node v (see Fig. 10.a). Let x be the adjacent node of v not belonging to b_i; of course, x is already colored, and it is 2 far from the other adjacent nodes of v in b_i. Therefore, the algorithm eliminates its color; among the 8 remaining colors, in view of Lemma 5.3, 6 colors are enough to $L(2,1)$-label b_i. If b_i has two adjacent nodes colored, the color of at most two nodes must be eliminated (see Fig. 10.b); if b_i is a triangle, all the 7 remaining colors could be necessary, if b_i is a larger cycle, 6 colors are always enough (cf. Lemma 5.3). Analogous considerations hold in case 3, i.e. if b_i is a path connected to R_{i-1} by means of its two endpoints (see Fig. 10.c). Finally, if b_i is a path and for it case 2 holds, then at most two colors must be eliminated and, among the remaining ones, no more than 5 are used to color b_i. In any case, no more than 9 colors are used and, if G is triangle free, 8 colors are enough. □

Observe that it is possible to slightly modify algorithm `Label Cubic Outerplanar Graphs` so that it outputs an $L(h,1)$-labeling, $h = 0, 1$, as we did for algorithm `Label`

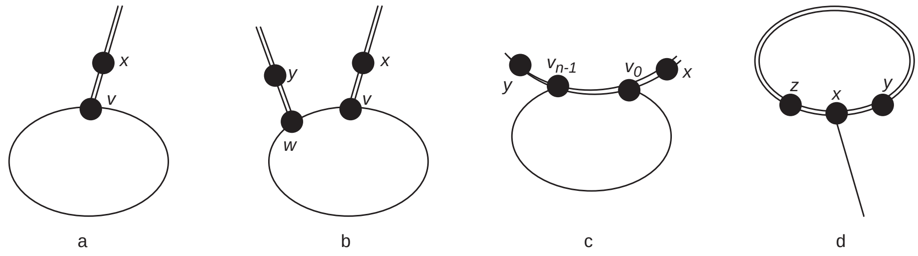

Figure 10: All the possibilities of coloring a piece of a degree 3 outerplanar graph. Double lines represents the part of the graph already colored, simple lines the part to be colored.

`Outerplanar Graphs`. Nevertheless, in this case, the bounds on the number of colors are exactly the same as those stated in Thm. 4.4, i.e. $4+h$ colors. Graph in Fig. 8 requires exactly $4+h$ colors to be $L(h,1)$-labeled, $h=0,1$.

6 Conclusions and Open Problems

In this paper we study the $L(h,1)$-labeling problem of two subclasses of planar graphs: the regular tilings of the plane and the outerplanar graphs.

For regular tiling, we present a simple unique parametric algorithm to solve the $L(h,1)$-problem for each $h=0,1,2$ and for each shape of the tiling (triangular, squared and hexagonal). Furthermore, we give a function of the coordinates of each node in the network conputing the channel to be assigned in constant time in a distributed fashion.

Concerning outerplanar graphs, we give an algorithm that $L(2,1)$-labels a degree $\Delta \geq 8$ outerplanar graph using no more than $\Delta+3$ colors, and at most 11 colors otherwise.

Although for $4 \leq \Delta \leq 7$ our bound on the number of colors is rather high, nevertheless we conjecture that the bound $\Delta+3$ holds for any outerplanar graph of degree $\Delta \geq 4$. On the contrary, in the special case $\Delta = 3$, in Fig. 8 we show a cubic outerplanar graph needing $\Delta+4$ colors. Furthermore, we provide an algorithm $L(2,1)$-labeling a cubic outerplanar graph using no more than $9 = \Delta+6$ colors. It remains an open problem to understand whether $\Delta+6$ is a tight bound or not.

Both in the case $h=0$ and $h=1$ algorithm `Label Outerplanar Graphs` guarantees an optimal labeling for $\Delta \geq 4$ and $\Delta \geq 7$ respectively, proving also the polinomiality of the problem, when restricted to outerplanar graphs. It remains an open problem to understand if this algorithm can be distributed.

Fig. 11 summarizes all the known results and open problems related to the number of colors. Starting from these results, the experimental result by Bruce and Hoffmann [5] has improved the lower bounds for small values of Δ ($3 \leq \Delta \leq 6$). Observe that the boundary line between values 7 and 8 for Δ is recurrent in the literature when planarity is involved. We cite as examples:

- it is NP-complete to decide if the inequality $\lambda_{2,1} \leq 8$ holds for planar graphs with $\Delta < 8$ [3];

- it is stated the conjecture that the chromatic number $\chi(G^2) \leq \Delta+5$ if $4 \leq \Delta \leq 7$ and $\chi(G^2) \leq \lfloor 3/2\Delta \rfloor + 1$ otherwise [16];

- the chromatic index of a degree Δ planar graph is Δ if $\Delta \geq 8$ and becomes $\Delta+1$ if $\Delta \leq 7$ [9].

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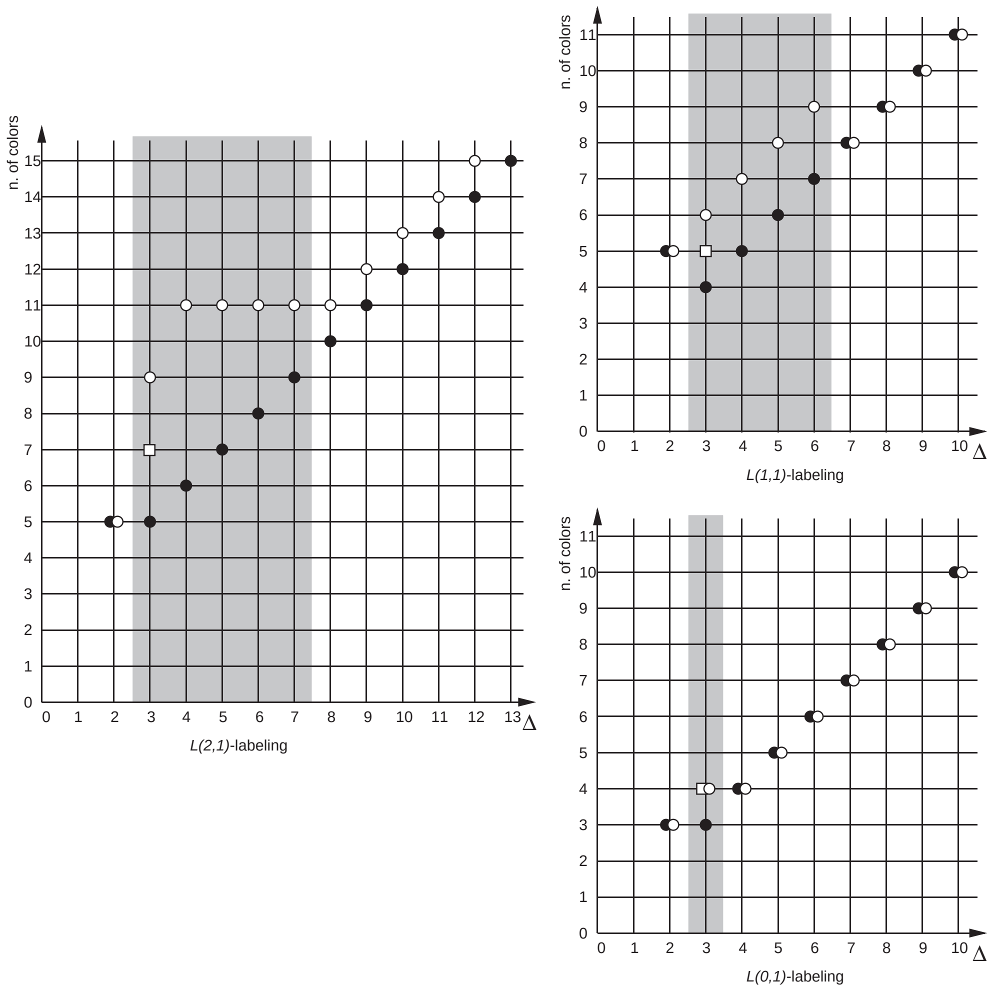

Figure 11: Summary of the known results and of the open problems: maximum degree Δ of an outerplanar graph versus the number of used colors. $\bullet$ indicates the lower bound on the number of used colors, $\circ$ indicates the value that nowadays is guaranteed, $\square$ indicates the existence of at least a graph requiring its y coordinate number of colors.