The $L(h, k)$-Labelling Problem: An Updated Survey and Annotated Bibliography

Tiziana Calamoneri

Department of Computer Science
"Sapienza" University of Rome - Italy
via Salaria 113, 00198 Roma, Italy. calamo@di.uniroma1.it

November 19, 2014

Abstract

Given any fixed nonnegative integer values h and k, the $L(h, k)$-labelling problem consists in an assignment of nonnegative integers to the nodes of a graph such that adjacent nodes receive values which differ by at least h, and nodes connected by a 2 length path receive values which differ by at least k. The span of an $L(h, k)$-labelling is the difference between the largest and the smallest assigned frequency. The goal of the problem is to find out an $L(h, k)$-labelling with minimum span.

The $L(h, k)$-labelling problem has been intensively studied following many approaches and restricted to many special cases, concerning both the values of h and k and the considered classes of graphs.

This paper reviews the results from previous by published literature, looking at the problem with a graph algorithmic approach. It is an update of a previous survey written by the same author.

Keywords: $L(h, k)$-labelling; frequency assignment; radiocoloring; λ-coloring; distance-2-coloring; D2-vertex coloring

1 Introduction

One of the key topics in graph theory is graph coloring. Fascinating generalizations of the notion of graph coloring are motivated by problems of channel assignment in wireless communications, traffic phasing, fleet maintenance, task assignment, and other applications. (See [1] for a survey.)

While in the classical vertex coloring problem [2] a condition is imposed only on colors of adjacent nodes, many generalizations require colors to respect stronger conditions, e.g. restrictions are imposed on colors both of adjacent nodes and of nodes at distance 2 in the graph.

This paper will focus on a specific graph coloring generalization that arose from a channel assignment problem in radio networks [3]: the $L(h, k)$-*labelling problem*, defined as follows:

Definition 1.1 *Given a graph $G = (V, E)$ and two nonnegative integers h and k, an $L(h, k)$-*labelling *is an assignment of nonnegative integers to the nodes of G such that adjacent nodes* *are labelled using colors at least h apart, and nodes having a common neighbour are labelled using colors at least k apart. The aim of the $L(h,k)$-*labelling problem *is to minimize the* span *$\sigma_{h,k}(G)$, i.e. the difference between the largest and the smallest used colors. The minimum span over all possible labelling functions is denoted by $\lambda_{h,k}(G)$ and is called $\lambda_{h,k}$-*number *of G.*

Observe that this definition imposes a condition on labels of nodes connected by a 2 length path instead of using the concept of *distance* 2, that is very common in the literature. The reason is that this definition works both when $h \geq k$ and when $h < k$. The present formulation allows the nodes of a triangle to be labelled with three colors at least $\max\{h,k\}$ apart from each other, although they are at mutual distance 1; when $h \geq k$ the two definitions coincide.

Furthermore, as the smallest used color is usually 0, an $L(h,k)$-labelling with span $\sigma_{h,k}(G)$ can use $\sigma_{h,k}(G)+1$ different colors; this feature is slightly counter-intuitive, but is kept for historical reasons.

The notion of $L(h,k)$-labelling was introduced by Griggs and Yeh in the special case $h = 2$ and $k = 1$ [4, 5] in connection with the problem of assigning frequencies in a multihop radio network (for a survey on the class of frequency assignment problems, see e.g. [6, 7, 8, 9]), although it has been previously mentioned by Roberts [10] in his summary on T-colorings and investigated in the special case $h = 1$ and $k = 1$ as a combinatorial problem and hence without any reference to channel assignment (see for instance [11]).

After its definition, the $L(h,k)$-labelling problem has been used to model several problems, for certain values of h and k. Some examples are the following: Bertossi and Bonuccelli [12] introduced a kind of integer "control code" assignment in packet radio networks to avoid hidden collisions, equivalent to the $L(0,1)$-labelling problem; channel assignment in optical cluster based networks [13] can be seen either as the $L(0,1)$- or as the $L(1,1)$-labelling problem, depending on the fact that the clusters can contain one ore more nodes; more in general, channel assignment problems, with a channel defined as a frequency, a time slot, a control code, etc., can be modeled by an $L(h,k)$-labelling problem, for convenient values of h and k. Besides the practical aspects, also purely theoretical questions are very interesting. These are only some reasons why there is considerable literature devoted to the study of the $L(h,k)$-labelling problem, following many different approaches, including graph theory and combinatorics [1, 14], simulated annealing [15, 16], genetic algorithms [17, 18], tabu search [19], and neural networks [20, 21]. In all these contexts, the problem has been called with different names; among others, we recall: $L(h,k)$-labelling problem, $L(p,q)$-coloring problem, distance-2-coloring and D2-vertex coloring problem (when $h = k = 1$), radiocoloring problem and λ-coloring problem (when $h = 2$ and $k = 1$).

Many variants of the problem have been introduced in the literature, as well: instead of minimizing the span, seek the $L(h,k)$-labelling that minimizes the *order*, i.e. the number of effectively used colors [3]; given a span σ, decide whether it is possible to $L(h,k)$-label the input graph using all colors between 0 and σ (*no-hole $L(h,k)$-labelling*) [22]; consider the color set as a cyclic interval, i.e. the distance between two labels $i, j \in \{0, 1, \ldots, \sigma\}$ defined as $\min\{|i-j|, \sigma+1-|i-j|\}$ [23]; use a more general model in which the labels and separations are real numbers [24]; generalize the problem to the case when the metric is described by a graph H (*$H(h,k)$-labelling*) [25]; consider the precoloring extension, where some nodes of the graph are given as already (pre)colored, and the question is if this precoloring can be extended to a proper coloring of the entire graph using a given number of colors [26]; consider a one-to-one $L(h,k)$-labelling (*$L'(h,k)$-labelling*) [27]; $L(h,k)$-label a digraph, where the distance from a node x to a node y is the length of a shortest dipath from x to y [28]; study another parameter, called *edge-span*, defined as the minimum, over all feasible labellings, of the $\max\{|f(u)-f(v)| : (u,v) \in E(G)\}$ [29]; impose the labelling to be balanced, i.e. all colors must be used more or less the same number of times (*equitable coloring*) [30].

Some of these generalizations are considered in [31].

The extent of the literature and the huge number of papers concerning the $L(h,k)$-labelling problem have been the main motivation of the surveys [6, 32, 31], each one approaching the problem from a different point of view (operative research, graph algorithms and extremal combinatorial, respectively), but they are all published at least five years ago. Since a substantial progress has been achieved in the last years, the author thinks that an updated survey and annotated bibliography would be useful. The present paper is an update of [32].

In this work, the case $k = 0$, for any fixed h, is not considered as this problem becomes the classical vertex coloring problem. Instead, a particular accent is posed on the special cases $h = 1, 2$ and $k = 1$: the first one is equivalent to the problem of optimally coloring the square of the input graph and the second one has been considered in the seminal works by Roberts, Griggs and Yeh. Both these problems have been intensively studied in the literature.

The decision version of the $L(h,k)$-labelling problem has been proved to be NP-complete, even under restrictive hypotheses. Section 2 lists these results. In Section 3 some general lower and upper bounds on the value of $\lambda_{h,k}$ are summarized.

For some special classes of graphs a labelling can be computed efficiently, while for other classes of graphs only approximate algorithms are known. Both these kinds of results are described in Section 4.

In the rest of this paper we will consider simple and loopless graphs with n nodes, maximum degree Δ, chromatic number $\chi(G)$, clique number $\omega(G)$ and girth (i.e. the length of the shortest cycle in G) $g(G)$. For all graph theoretic concepts, definitions and graph classes inclusions not given in this review we refer either to [33] or to the related reference.

2 NP-Completeness Results

In this section some general complexity results are listed, divided by different values of h and k. More specific results concerning classes of graphs are given in Section 4.

$L(0,1)$**-labelling.** In [12] the NP-completeness result for the decision version of the $L(0,1)$-labelling problem is derived when the graph is planar by means of a reduction from 3-VERTEX COLORING of straight-line planar graphs.

$L(1,1)$**-labelling.** Also the decision version of the $L(1,1)$-labelling problem, (that is equivalent to the $L(2,1)$-labelling problem where the order must be minimized instead of the span [34]) is proved to be NP-complete with a reduction from 3-SAT [35]. The problem remains NP-complete for unit disk graphs [36], for planar graphs [37] and even for planar graphs of bounded degree [38]. It is also NP-complete to decide whether 4 colors suffice to $L(1,1)$-label a cubic graph. On the contrary, it is polynomial to decide if 3 colors are enough [39].

Studying a completely different problem (Hessian matrices of certain non linear functions), McCormick [35] gives a greedy algorithm that guarantees a $O(\sqrt{n})$-approximation for coloring the square of a graph. The algorithm is based on the greedy technique: consider the nodes in any order, then the color assigned to node v_i is the smallest color that has not been used by any node which is at distance at most 2 from v_i; the performance ratio is obtained by simple considerations on the degree of G and of its square.

Approaching an equivalent scheduling problem, Ramanathan and Lloyd [40] present an approximation algorithm with a performance guarantee of $O(\theta)$, where θ is the thickness of the graph. Intuitively, the thickness of a graph measures "its nearness to planarity". More formally, the *thickness* of a graph $G = (V, E)$ is the minimum number of subsets into which the edge set E must be partitioned so that each subset in the partition forms a planar graph on V.

$L(2,1)$**-labelling.** To decide whether a given graph G admits an $L(2,1)$-labelling of span at most n is NP-complete [4]. This result is obtained by a double reduction: from HAMILTONIAN PATH to the decision problem asking for the existence of an injection $f : V \to [0, n-1]$ such that $|f(x) - f(y)| \geq 2$ whenever $(x, y) \in E$, and from this problem to the decisonal version of the $L(2,1)$-labelling problem. The problem remains NP-complete if we ask whether there exists a labelling of span at most σ, where σ is a fixed constant ≥ 4, while it is polynomial if $\sigma \leq 3$ (this case occurs only when G is a disjoint union of paths of length at most 3). A fortiori, the problem is not fixed parameter tractable [41].

The problems of finding the $\lambda_{2,1}$-number of graphs with diameter 2 [4, 5], planar graphs [42, 43], bipartite, split and chordal graphs [42] are all NP-hard.

Finally, Fiala and Kratochvíl [44] prove that for every integer $p \geq 3$ it is NP-complete to decide whether a p-regular graph admits an $L(2,1)$-labelling of span (at most) $p + 2$.

We conclude this paragraph citing some results where the authors present exact exponential time algorithms for the $L(2,1)$-labelling problem of fixed span σ. In [45], the authors design algorithms that are faster than the naive $O((\sigma + 1)^n)$ algorithm that would try all possible labellings. In the first NP-complete case ($\sigma = 4$), the running time of their algorithm is $O(1.3006^n)$, which beats not only $O(\sigma^n)$ but also $O((\sigma - 1)^n)$. For what concerns the larger values of σ, an exact algorithm for the so called Channel Assignment Problem [46] implies an a $O(4^n)$ algorithm for the $L(2,1)$-labelling problem. This has been improved in [47] to an $O(3.8739^n)$ algorithm. Some modifications of the algorithm and a refinement of the running time analysis has allowed to improve the time complexity to $O(3.2361^n)$ [48]. A lower-bound of $\Omega(3.0731^n)$ on the worst-case running time is also provided. After this result, the base of the exponential running time function seemed hardly decreaseable to a value lower than 3. Nevertheless, in [49] the authors provide a breakthrough in this question by providing an algorithm running in $O(2.6488^n)$ time, based on a reduction of the number of operations performed in the recursive step of the dynamic programming algorithm.

All algorithms mentioned above are based on dynamic programming approach and use exponential memory. The first exact algorithm for the $L(2,1)$-labelling problem with time complexity $O(c^n)$ for some constant c and polynomially bounded space complexity is described in [50] and is based on a divide and conquer approach.

$L(h,k)$**-labelling.** Nobody would expect the $L(h,k)$-labelling problem for $h > k \geq 1$ to be easier than the $L(2,1)$-labelling problem, however, the actual NP-hardness proofs seem tedious and not easily achievable in full generality. In [41] the authors conjecture that for every $h \geq k \geq 1$, there is a σ (depending on h and k) such that deciding whether $\lambda_{h,k}(G) \leq \sigma$ is NP-complete. In support of their conjecture, the authors prove that there is at least one NP-complete fixed parameter instance, namely that it is NP-complete to decide whether

$\lambda_{h,k}(G) \leq h + k\lceil \frac{h}{k} \rceil$ for all fixed $h > k \geq 1$. Under less general conditions they prove that there are infinitely many instances: the problem whether $\lambda_{h,k}(G) \leq h + pk$ is NP-complete for any fixed $p \geq \frac{h}{k}$ and $h > 2k$. It follows that for $k = 1$ (and more generally h divisible by k), there are only finitely many polynomial instances (unless P=NP), namely if $h > 2$ then the decision version of the $L(h, 1)$-labelling problem is NP-complete for every fixed $\sigma \geq 2h$. In this case it is possible a little more: for every $h > 2$, the problem of deciding whether $\lambda_{h,1}(G) \leq \sigma$ is NP-complete if $\sigma \geq h + 5$ while it is polynomially solvable if $\sigma \leq h + 2$.

Open Problem: For $p \geq 5$ this result leaves the cases $\sigma = h + 3$ and $\sigma = h + 4$ as the last open cases for the fixed parameter complexity of the $L(h, 1)$-labelling problem.

More recently, it has been proved [51] that the decisional version of the $L(h, k)$-labelling problem is NP-complete even when restricted to bipartite planar graphs of small maximum degree and for relatively small values of σ.

3 Lower and Upper Bounds

We list here some general bounds on the $\lambda_{h,k}$-number, divided by different values of h and k. Bounds for particular classes of graphs will be given in the corresponding subsections.

$L(0, 1)$**-labelling.** An upper bound on $\lambda_{0,1}(G)$ is $\Delta^2 - \Delta$ for any harph G of maximum degree Δ [52].

$L(1, 1)$**-labelling.** Define $f(\Delta, g)$ as the maximum possible value of $\lambda_{1,1}(G) = \chi(G^2) - 1$ over graphs with maximum degree Δ and girth g. Since the maximum degree of G^2 is at most Δ^2, it follows that $f(\Delta, g) \leq \Delta^2$ for every g. This bound is tight as equality holds for $\Delta = 2$ and $g \leq 5$, as shown by the 5-cycle, for $\Delta = 3$ and $g \leq 5$, as shown by the Petersen graph, and for $\Delta = 7$ and $g \leq 5$, as shown by the Hoffman-Singleton graph. (The *Hoffman-Singleton graph* – see Figure 1 – is the graph on 50 nodes and 175 edges that is the only regular graph of node degree 7, diameter 2, and girth 5; it is the unique $(7, 5)$-cage graph and Moore graph, and contains many copies of the Petersen graph). Moreover, by Brooks theorem (stating that if G is connected then $\chi(G) \leq \Delta(G)$, unless G is complete or G is an odd cycle; cf. e.g. [2]) it follows that the equality can hold only for $g \leq 5$ and only if there exists a Δ-regular graph of diameter 2 on $\Delta^2 + 1$ nodes. If such graph exists then $\Delta \in \{2, 3, 7, 57\}$. It is also possible to see that $f(2, g) = 4$ for all $g \geq 6$. Alon and Mohar [53] prove that $f(\Delta, g)$ is $(1 + o(1))\Delta^2$ if $g = 3, 4, 5$ and is $\Theta(\Delta^2/\log \Delta)$ if $g \geq 7$. In [54] a new approach is followed in order to provide a new upper bound on $\lambda_{1,1}$. Namely, utilizing the probabilistic method, the authors prove that $f(\Delta, g) \leq \left(1 - \frac{2}{3e^6}\right)\Delta^2$ if the graph is regular of girth $g \geq 7$ and Δ sufficiently large.

$L(2, 1)$**-labelling.** As in the case of $\lambda_{1,1}$, even for the bounds on $\lambda_{2,1}$, Δ is the most common used parameter.

The obvious lower bound for $\lambda_{2,1}(G)$ is $\Delta + 1$, achieved for the star $K_{1,\Delta}$, but Griggs and Yeh [4] describe a graph requiring span $\Delta^2 - \Delta$. This graph is the *incidence graph of a projective plane* $\pi(n)$ of order n, i.e. the bipartite graph $G = (U \cup V, E)$ such that:
i. $|U| = |V| = n^2 + n + 1$,
ii. each $u \in U$ corresponds to a point p_u in $\pi(n)$ and each $v \in V$ corresponds to a line l_v in $\pi(n)$, and
iii. $E = \{(u, v) : u \in U, v \in V$ such that $p_u \in l_v$ in $\pi(n)\}$.
By definition of $\pi(n)$, G is $(n + 1)$-regular.

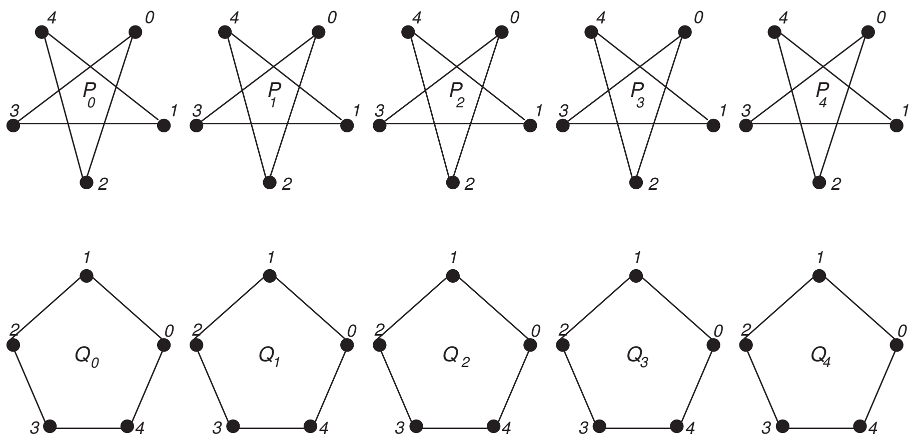

Figure 1: The Hoffman-Singleton graph, constructed from the 10 5-cycles illustrated, with node i of P_j joined to node $i + jk \pmod 5$ of Q_k.

By a simple greedy algorithm, that labels each node with the smallest color that does not induce conflicts, the same authors prove that $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta$ and improve this upper bound to $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta - 3$ when G is 3-connected and $\lambda_{2,1}(G) \leq \Delta^2$ when G is of diameter 2. They conjecture $\lambda_{2,1}(G) \leq \Delta^2$ for any graph G. This conjecture has been motivation of some research since. In fact, we can claim that this is the most famous open problem in this area for more than fifteen years. In his invited talk, during the conference CIAC 2010, Reed has claimed that the Griggs and Yeh's conjecture has raised an interest that can be compared with the one given to the Four Colors' conjecture.

Observe that the upper bound set by the conjecture would be tight: there are graphs with degree Δ, diameter 2 and $\Delta^2 + 1$ nodes, namely the 5-cycle, the Petersen graph and the Hoffman-Singleton graph, so the span of every $L(2, 1)$-labelling is at least Δ^2. Nevertheless, notice that the conjecture is not true for $\Delta = 1$. For example, $\Delta(K_2) = 1$ but $\lambda_{2,1}(K_2) = 2$.

Open Problem: The Moore graphs (i.e. graphs having the minimum number of nodes possible for a regular graph with given diameter and maximum degree) are at the moment the only graphs known to require span Δ^2, and it is an open problem to understand if there are infinitely many graphs G satisfying $\lambda_{2,1}(G) > \Delta^2 - o(\Delta)$.

Using constructive labelling schemes, Jonas [55] improves the upper bound by showing that $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta - 4$ if $\Delta \geq 2$ and, successively, Chang and Kuo [27] further decrease the bound to $\Delta^2 + \Delta$. The algorithm by Chang and Kuo is funded on the concept of *2-stable set* of a graph G, that is a subset S of $V(G)$ such that every two distinct nodes in S are of distance greater than 2. At each step i of the algorithm, a subset of nodes S_i is built, and all nodes of S_i are labelled with i. S_i is a maximal 2-stable set of the set of unlabelled nodes at distance ≥ 2 from any node in S_{i-1}.

Then, it has been proven the analogue of Brook's theorem for some channel assignment problems, deriving as corollary of a more general result that $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 1$ for any graph G [56], and successively Gonçalves [57] proved $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 2$ using the algorithm of Chang and Kuo. In 2008, Havet, Reed and Sereni [58] have finally proven that the Griggs and Yeh's conjecture is true, for sufficiently large values of Δ (about $\Delta \geq 10^{69}$). This is the first paper addressing the solution of the conjecture for general graphs, although it does not completely close it. The same authors prove in [59] that for every graph G of maximum degree Δ, $\lambda_{2,1}(G) \leq \Delta^2 + c$, for some constant c. Of course, the bound of $\Delta^2 + c$ is better than $\Delta^2 + \Delta - 1$ as Δ increases enough, but c is unfortunately a rather huge number. The interest of this result lies in having exactly determined the growth of the function to be added to Δ^2. Now, it is a very interesting issue to find a tight value for c (hopefully as close to 0 as possible).

Open Problem: The Griggs and Yeh's conjecture is still unproved for $\Delta < 10^{69}$ and hence it is not considered closed yet.

For what concerns the relationship between $\lambda_{2,1}$ and the graph parameters, it is not difficult to see that $\lambda_{2,1} \geq 2\omega(G) - 2$.

A relationship between $\lambda_{2,1}$ and the chromatic number of G is stated in [4]: $\lambda_{2,1}(G) \leq n + \chi(G) - 2$ and for complete k-partite graphs the equality holds, i.e. $\lambda_{2,1}(G) = n + k - 2$.

In [60] the authors investigate the relationship between $\lambda_{2,1}(G)$ and another graph invariant, i.e. the path covering number c of the complement graph G^C (the *path covering number* of a graph is the smallest number of node-disjoint paths needed to cover the nodes of the graph) proving that $\lambda_{2,1}(G) = n + c(G^C) - 2$ if and only if $c(G^C) \geq 1$.

It is quite simple to see that $\lambda_{2,1}$ and $\lambda_{1,1}$ are related: the number of colors necessary for an $L(2,1)$-labelling of a graph G is at least $\lambda_{1,1} + 1$, and conversely from an optimal $L(1,1)$-labelling of G we can easily obtain an $L(2,1)$-labelling of G with colors between 0 and $2\lambda_{1,1} - 1$. Hence, an algorithm solving the $L(1,1)$-labelling problem for a class of graphs also provides a 2-approximation for the $L(2,1)$-labelling problem.

In [61], Balakrishnan and Deo give upper and lower bounds on the sum and product of the $\lambda_{2,1}$-number of an n node graph and that of its complement:

$$2\sqrt{n} - 2 \leq \lambda_{2,1}(G) + \lambda_{2,1}(G^C) \leq 3n - 3$$

$$0 \leq \lambda_{2,1}(G) \cdot \lambda_{2,1}(G^C) \leq \left(\frac{3n-3}{2}\right)^2$$

These bounds are similar to the well consolidated bounds given by Nordhaus and Gaddum [62] on the chromatic number of a graph and that of its complement.

***L*(*h*, *k*)-labelling.** Passing to the general $L(h,k)$-labelling problem, for any positive integers $h \geq k$, $\lambda_{h,k} \geq h + (\Delta + 1)k$ [63], as a generalization of the known results for $h = 2$, it is easy to state that $\lambda_{h,1}(G) \leq \Delta^2 + (h-1)\Delta$ for any graph of maximum degree Δ. This bound has been improved to $\lambda_{h,1}(G) \leq \Delta^2 + (h-1)\Delta - 2$ when $\Delta \geq 3$ [57]. Furthermore, $\lim_{h\to\infty} \frac{\lambda_{h+1,1}(G)}{\lambda_{h,1}(G)} = 1$ [64]. Havet, Reed and Sereni [58] generalize their proof of the Griggs and Yeh's conjecture for sufficiently large values of Δ to any h and k, proving that $\lambda_{h,1}(G) \leq \Delta^2$ for any $\Delta \geq 10^{69}$ and $\lambda_{h,1} \leq \Delta^2 + c(h)$ for every integer Δ and for an opportune constant $c(h)$, depending on the parameter h.

It is easy to see that $\lambda_{h,k}(G) \geq h + (\Delta - 1)k$ for $h \geq k$. Moreover, if $h > k$ and the equality holds in the previous formula and $h > k$, then for any $L(h,k)$-labelling of G, each node of degree Δ must be labeled 0 (or $h + (\Delta - 1)k$) and its neighbors must be labeled $h + ik$ (or ik) for $i = 0, 1, \ldots \Delta - 1$.

The structures of graphs with $\Delta \geq 1$ and $\lambda_{h,k}(G) = h + (\Delta - 1)k$ are studied in [63] and they are called $\lambda_{h,k}$*-minimal graphs*.

A basic result, implicitly taken into account in any work on the $L(h,k)$-labelling, states that for all G, there exists an optimal $L(h,k)$-labelling of G such that each label is of the form $\alpha h + \beta k$, α and β being non negative integers. Hence, in particular, $\lambda_{h,k}(G) = \alpha h + \beta k$ for some non negative integers α and β [65]. It is also worthy to notice that, for any positive integer c, $c \cdot \lambda_{h,k}(G) = \lambda_{ch,ck}(G)$ and that if $h' \geq h$ and $k' \geq k$, then for any graph G, $\lambda_{h',k'}(G) \geq \lambda_{h,k}(G)$ [65]. Finally, let G have maximum degree Δ. Suppose there is a node with Δ neighbors, each of which has degree Δ. Then $\lambda_{h,k}(G) \geq h + (2\Delta - 2)k$ if $h \geq \Delta k$ and $\lambda_{h,k}(G) \geq 2h + (\Delta - 2)k$ if $h \leq \Delta k$. Of course, these lower bounds fit particularly well for regular graphs.

In [66], by stating a strong relationship between the $L(h,k)$-labelling problem and the problem of coloring the square of a graph, it is exploited the algorithm by McCormick for approximating the $L(1,1)$-labelling problem [35] to provide a $(h\sqrt{n} + o(h\sqrt{n}))$-approximation algorithm for the $L(h,k)$-labelling problem. This result has been improved by Halldórson [67] proving that the performance ratio of the First Fit algorithm (consisting in processing the nodes in an arbitrary order, so that each node is assigned the smallest color compatible with its neighborhood) is at most $O(\min(\Delta, \frac{h}{k} + \sqrt{n}))$. This is tight within a constant factor, for all values of the parameters. The author shows that this is close to the best possible, as it is NP-hard to approximate the $L(h,k)$-labelling problem within a factor of $n^{1/2-\epsilon}$ for any $\epsilon > 0$ and h in the range $[n^{1/2-\epsilon}, n]$. On the positive side, it is never harder to approximate than the ordinary vertex coloring problem, hence an upper bound of $O(n(\log\log n)^2/\log^3 n)$ holds [68].

4 Known Results on Graph Classes

In view of the hardness results described in Section 2 and of the gap between upper and lower bounds on the $\lambda_{h,k}$-number listed in Section 3, further bounds, exact results and approximation algorithms have been found by restricting the classes of graphs under consideration. This is the topic of the present section.

The $L(h,k)$-labelling problem has been intensively studied on various graph classes in its general version (any h and k) but above all in some of its specializations (e.g. $h = 2, 1$ and $k = 1$); some of these classes have been considered because they well model real networks, others for their theoretical interest.

In the following we analyze a number of graph classes, describe the known results concerning each of them, and propose some interesting problems still open. This section is organized listing first the graph classes for which exact results are known, and then the graph classes for which only approximate bounds and labeling algorithms have been found. In view of this organization, not all correlated classes are treated in consecutive subsections.

4.1 Paths, Cycles, Cliques and Wheels

Let P_n, C_n and K_n be a *path*, a *cycle* and a *clique*, respectively, of n nodes. The *wheel* W_n is obtained by C_n by adding a new node adjacent to all nodes in C_n.

Paths (i.e. buses), cycles (i.e. rings), cliques (i.e. completely connected networks) and wheels are the simplest and most common networks one can consider; the decision version of the $L(h,k)$-labelling problem is polynomially solvable on each of them.

$L(0,1)$**-labelling.** Optimal $L(0,1)$-labellings are known for paths, needing 2 colors (i.e.

$\lambda_{0,1}(P_n) = 1$) [69], and for cycles, having $\lambda_{0,1}(C_n)$ equal to 1 if n is multiple of 4, and 2 otherwise [12]. In Figure 2 these labellings are shown.

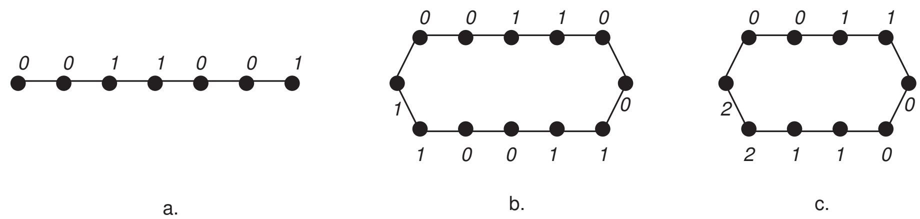

Figure 2: a. $L(0,1)$-labelling of a path; b. $L(0,1)$-labelling of a cycle whose number of nodes is multiple of 4; c. $L(0,1)$-labelling of a cycle whose number of nodes is not multiple of 4.

It is easy to check that $\lambda_{0,1}(K_n) = \lambda_{1,1}(K_n) = n-1$ and that $\lambda_{0,1}(W_n) = \lambda_{1,1}(W_n) = n$ according to our definition of $L(h,k)$-labelling; on the contrary, $\lambda_{0,1}(W_n) = \lfloor \frac{(n-1)}{2} \rfloor$ according to the definition based on the concept of distance.

Polynomial results concerning the $L(0,1)$-labelling are found for a class of cycle-related graphs such as *cacti* [70]. (A *cactus* is a connected finite graph in which every edge is contained in at most one cycle.)

$L(1,1)$**-labelling.** $\lambda_{1,1}(P_2) = 1$ and $\lambda_{1,1}(P_n) = 2$ for each $n \geq 3$; $\lambda_{1,1}(C_n)$ is 2 if n is a multiple of 3 and it is 3 otherwise [71] (see Figure 3).

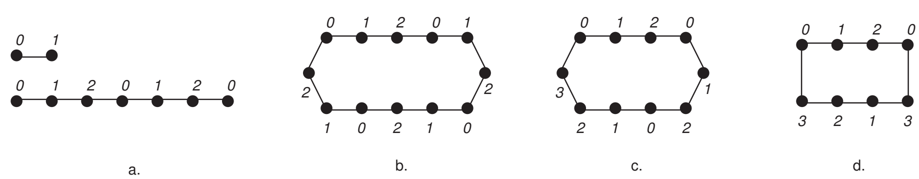

Figure 3: a. $L(1,1)$-labelling of a path; b. $L(1,1)$-labelling of a cycle whose number of nodes is multiple of 3; c. and d. $L(1,1)$-labellings of cycles whose numbers of nodes are not multiple of 3.

$L(2,1)$**-labelling.** It is simple to prove that $\lambda_{2,1}(P_1) = 0$, $\lambda_{2,1}(P_2) = 2$, $\lambda_{2,1}(P_3) = \lambda_{2,1}(P_4) = 3$, and $\lambda_{2,1}(P_n) = 4$ for $n \geq 5$, that $\lambda_{2,1}(K_n) = 2(n-1)$, that $\lambda_{2,1}(W_3) = \lambda_{2,1}(W_4) = 6$ and $\lambda_{2,1}(W_n) = n+1$ for each $n \geq 5$. Finally, $\lambda_{2,1}(C_n) = 4$ for each $n \geq 3$ [4, 5].

As an example, we recall here how to label a cycle. If $n \leq 4$ the result is trivial to verify. Thus, suppose that $n \geq 5$, and C_n must contain a P_5 as a subgraph, hence $\lambda_{2,1}(C_n) \geq \lambda_{2,1}(P_5) = 4$. Now, let us show an $L(2,1)$-labelling l of C_n with span 4. Let $v_0, \ldots, v_{n-1}$ be nodes of C_n such that v_i is adjacent to v_{i+1}, $0 \leq i \leq n-2$ and v_0 is adjacent to v_{n-1}. Then, consider the following labelling:

if $n \equiv 0 \bmod 3$ then

$$l(v_i) = \begin{cases} 0 & \text{if } i \equiv 0 \bmod 3 \\ 2 & \text{if } i \equiv 1 \bmod 3 \\ 4 & \text{if } i \equiv 2 \bmod 3 \end{cases}$$

if $n \equiv 1 \bmod 3$ then redefine l at $v_{n-4}, \ldots, v_{n-1}$ as

$$l(v_i) = \begin{cases} 0 & \text{if } i \equiv n-4 \\ 3 & \text{if } i \equiv n-3 \\ 1 & \text{if } i \equiv n-2 \\ 4 & \text{if } i \equiv n-1 \end{cases}$$

if $n \equiv 2 \bmod 3$ then redefine l at v_{n-2} and at v_{n-1} as

$$l(v_i) = \begin{cases} 1 & \text{if } i = n-2 \\ 3 & \text{if } i = n-1 \end{cases}$$

A *unicycle* – respectively, *bicycle* – is a connected graph having only one – respectively, two – cycles. Polynomial results concerning $L(2,1)$-labelling are found for classes of *cacti*, *unicycles* and *bicycles* in [55].

$L(h,k)$**-labelling.** Georges and Mauro [65] evaluate the span of cycles and paths for any h and k (with $h \geq k$) showing that

$$\lambda_{h,k}(P_n) = \begin{cases} 0 & \text{if } n = 1 \\ h & \text{if } n = 2 \\ h+k & \text{if } n = 3 \text{ or } 4 \\ h+2k & \text{if } n \geq 5 \text{ and } h \geq 2k \\ 2h & \text{if } n \geq 5 \text{ and } h \leq 2k \end{cases}$$

and

$$\lambda_{h,k}(C_n) = \begin{cases} 2h & \text{if } n \text{ odd, } n \geq 3 \text{ and} \\ & h \geq 2k, \text{ or} \\ & \text{if } n \equiv 0 \bmod 3 \text{ and} \\ & h \leq 2k \\ h+2k & \text{if } n \equiv 0 \bmod 4 \text{ and} \\ & h \geq 2k, \text{ or} \\ & \text{if } n \not\equiv 0 \bmod 3,\ n \neq 5 \\ & \text{and } h \leq 2k \\ 2h & \text{if } n \equiv 2 \bmod 4 \text{ and} \\ & h \leq 3k \\ h+3k & \text{if } n \equiv 2 \bmod 4 \text{ and} \\ & h \geq 3k \\ 2h & \text{if } n \geq 5 \text{ and } h \leq 2k \\ 4k & \text{if } n = 5 \end{cases}$$

It is straightforward to see that an optimal $L(h,k)$-labelling of an n node clique requires span $(n-1)h$, for each $h \geq k$ and that $\lambda_{h,k}(W_n) = n + h - 1$ for sufficiently large values of n and $h \geq k$.

Griggs and Jin [72] extended the results on paths, cycles and wheels to values of h and k such that $h \leq k$; as an example, we list these results in the case of paths:

$$\lambda_{h,k}(P_n) = \begin{cases} 0 & \text{if } n = 1 \\ h & \text{if } n = 2 \\ k & \text{if } n = 3 \text{ and } 0 \le h/k \le 1/2 \\ 2h & \text{if } n = 3 \text{ and } 1/2 \le h/k \le 1 \\ h+k & \text{if } n = 4,5,6 \text{ or if } n \ge 7 \text{ and } 0 \le h/k \le 1/2 \\ 3h & \text{if } n \ge 7 \text{ and } 1/2 \le h/k \le 2/3 \\ 2k & \text{if } n \ge 7 \text{ and } 2/3 \le h/k \le 1. \end{cases}$$

Finally, we point out that the $\lambda_{h,1}$-number of cacti is investigated in [73] while the $\lambda_{h,k}$-number of r-paths (i.e. of graphs on nodes $\ldots, v_{-3}, v_{-2}, v_{-1}, v_0, v_1, v_2, v_3, \ldots$ such that v_s is adjacent to v_t iff $|s-t| \le r-1$, $r \ge 2$) is investigated in [74] and in [75] (where an error of [64] is fixed).

4.2 Regular Grids

Let G_Δ, $\Delta = 3, 4, 6, 8$, denote the hexagonal, squared, triangular and octagonal grid, respectively. Portions of these grids are shown in Figure 4.

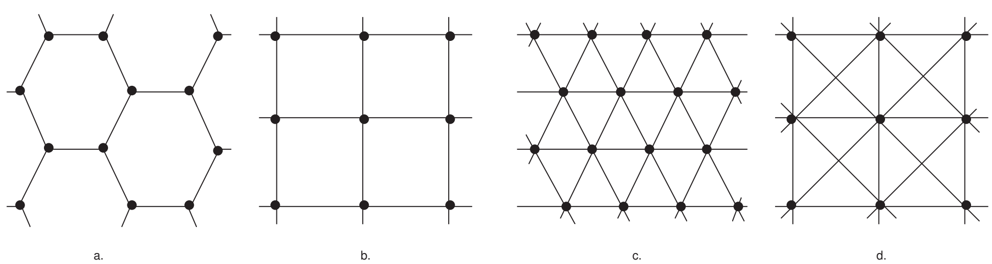

Figure 4: A portion of: a. hexagonal grid G_3; b. squared grid G_4; c. triangular grid G_6; d. octagonal grid G_8.

The hexagonal grid is a natural model for cellular networks, and its interference graphs is the triangular grid, also called *cellular graph*, according to the notation introduced in [76].

The $L(h,k)$-labelling problem has been extensively studied on regular grids, and shown to be polynomially solvable. More detailed results are given in the following. Note that some grids are equivalent to some special products of paths, so other related results can be found in Subsection 4.3.

$L(0,1)$-, $L(1,1)$- and $L(2,1)$-labelling. An optimal $L(0,1)$-labelling for squared grids and an optimal $L(1,1)$-labelling for hexagonal, squared and triangular grids are given in [69] and in [71], respectively.

The $\lambda_{2,1}$-number of regular grids has been proved to be $\lambda_{2,1}(G_\Delta) = \Delta + 2$ by means of optimal labelling algorithms [77] . All of these algoithms are based on the replication of a labelling pattern, depending on Δ. An example is given in Figure 5.

A natural generalization of squared grids is obtained by adding wrap-around edges on each row and column: these graphs are known as *tori*. In spite of the similarity between squared grids and tori, the presence of wrap-around edges prevents the labelling of the squared grid from being extended to tori unless both the number of rows and the number of columns are multiples of 5. If this is not the case, an $L(1,1)$-labelling exists using at most 8 colors, which

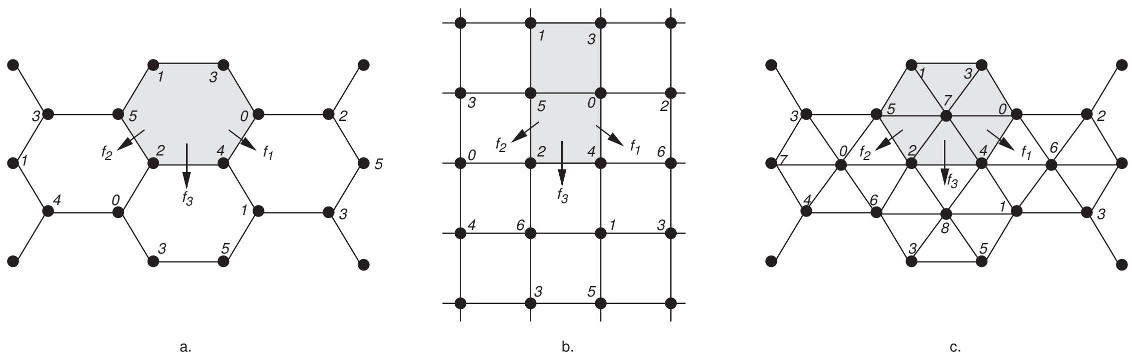

Figure 5: An example of $L(2,1)$-labelling of regular grids by means of labelling patterns; function f_1 subtracts 1 to each label, f_2 sums 2 and f_3 sums 1; all the operations must be considered mod $(\lambda_{2,1}(G_\Delta) + 1)$.

is nearly-optimal since 6 is a lower bound [78]. Exact results concerning the $\lambda_{2,1}$-number of tori are reported in Subsection 4.3.

Open Problem: To find the optimal $L(1,1)$-labelling of tori is an interesting issue.

$L(h,k)$**-labelling.** In 1995 Georges and Mauro [65] give some results concerning the $L(h,k)$-labelling of squared grids as a special result of their investigation on the $\lambda_{h,k}$-number of product of paths. Only in 2006 the problem has been systematically handled and the union of the results presented in [79] and in [80] provides the exact value of function $\lambda_{h,k}(G_\Delta)$, where $\Delta = 3, 4, 6, 8$ for almost all values of h and k.

The exact results are obtained by means of two series of proofs: lower bounds proofs, based on exhaustive considerations, deducing that $\lambda_{h,k}(G_\Delta)$ cannot be less than certain values, and upper bounds proofs, based on labelling schemes. Of course, the results obtained for any h and k include as special case the previous ones for $h = 1, 2$ and $k = 1$.

All the aforementioned results lead to assign a color to any node in constant time in a distributed fashion, provided that the relative positions of the nodes in the grid are locally known.

Later, Griggs and Jin [81] close all gaps for the squared and hexagonal grids, and all gaps except when $k/2 \le h \le 4k/5$ for the triangular grids; they do not handle the octagonal grid. This grid has been investigated in [82], where some previous results have been improved. A summary of all the known results is plotted in Figure 6.

Open Problem: It remains to compute the exact value of $\lambda_{h,k}(G_\Delta)$ in those intervals where lower and upper bounds do not coincide when $\Delta = 6$ and $\Delta = 8$.

Open Problem: Almost all the proofs for the lower bounds are based on exhaustive reasonings, and so are very long and difficult to follow. Furthermore, the range of h/k is divided into several intervals, and a different proof is given for each Δ and for each interval. It would be interesting to design a new proof technique in order to simplify all these proofs and to propose a unifying approach useful to reduce the number of the proofs and to increase their elegance.

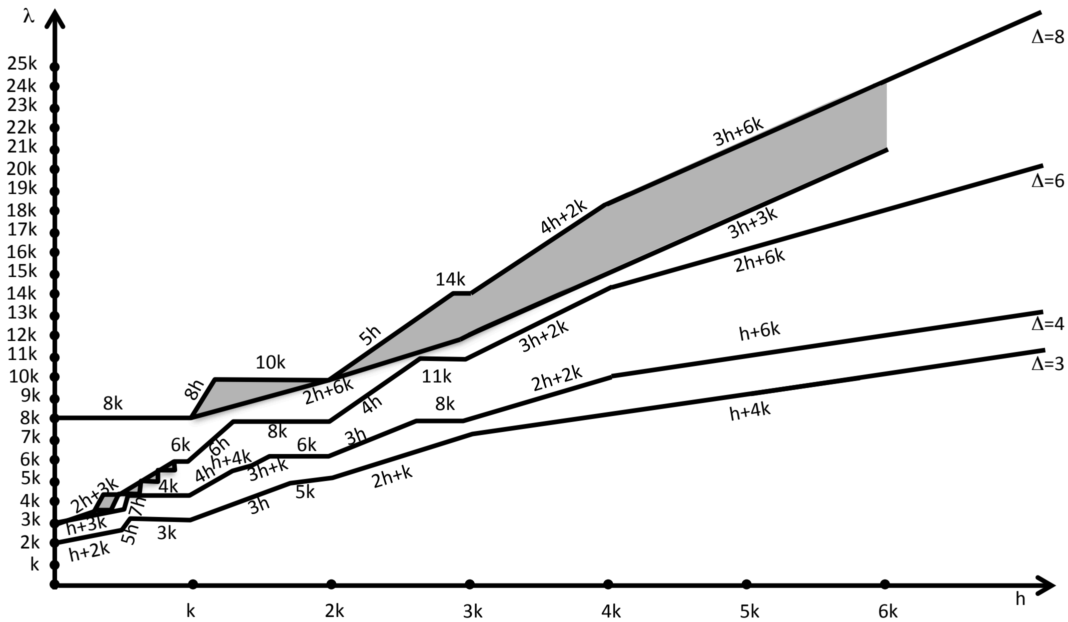

Figure 6: State of the art concerning the $L(h, k)$-labelling of regular grids. Grey areas represent the gaps to be still closed.

A generalization of the squared grid is the d-dimensional grid. A motivation for studying higher dimensional grids is that when the networks of several service providers overlap geographically, they must use different channels for their clients. The overall network can then be modelled in a suitably higher dimension. Optimal $L(2,1)$-labellings for d-dimensional square grids, for each $d \geq 1$ are presented in [83]. These results are extended to any h, k for each $d \geq 1$ in [84]. The authors give lower and upper bounds on $\lambda_{h,k}$ for d-dimensional grids, and show that in some cases these bounds coincide. In particular, in the case $k = 1$, the results are optimal.

Open Problem: It is still an open question to find optimal, or nearly-optimal, labellings for higher dimensional triangular grids, even for special values of h and k. Nevertheless, it is in the opinion of the author that such a result would be only technical, without any particular practical relevance.

4.3 Products of Graphs

The *cartesian product* (or simply *product*) $G \square H$ and the *direct product* $G \times H$ of graphs G and H are defined as follows: $V(G \square H) = V(G \times H)$ is equal to the cartesian product of $V(G)$ and $V(H)$; $E(G \square H) = \{((x_1, x_2), (y_1, y_2)) : (x_1, y_1) \in E(G)$ and $x_2 = y_2$, or $(x_2, y_2) \in E(H)$ and $x_1 = y_1\}$; $E(G \times H) = \{((x_1, x_2), (y_1, y_2)) : (x_1, y_1) \in E(G)$ and $(x_2, y_2) \in E(H)\}$. $G \square H$ and $G \times H$ are mutually nonisomorphic with the sole exception of when G and H are odd cycles of the same size. The *strong product* $G \otimes H$ of G and H has the same node set as the other two products and the edge set is the union of $E(G \square H)$ and $E(G \times H)$. In Figure 7 the product of P_3 and P_4 is depicted, according to each one of the three just defined products.

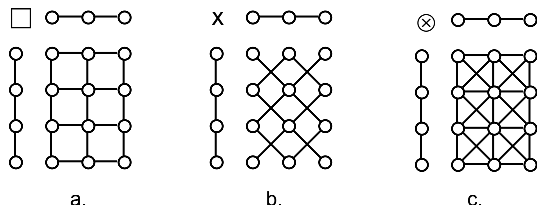

Figure 7: Product of P_3 and P_4, where the product is: a. the cartesian product; b. the direct product and c. the strong product.

Product graphs have been considered in the attempt of gaining global information from the factors. Many interesting wireless networks have simple factors, such as paths and cycles.

Observe that any d-dimensional grid is the cartesian product of d paths, any d-dimensional torus is the cartesian product of d cycles and any octagonal grid is the strong product of two paths; so there is some intersection between the results summarized in this section and in the previous one. Nevertheless, they have been described separately in order to highlight the independence of the approaches and of the methods for achieving the results. The same reasoning holds for the cartesian product of complete graphs, i.e. the Hamming graph and for the cartesian product of n K_2 graphs, i.e. the n-dimensional hypercube: we will detail the results on these graphs in Subsection 4.8.

$L(2,1)$**-labelling.** Exact values for the $\lambda_{2,1}$-numbers for the cartesian product of two paths for all values of m and n are given in [85]. Namely, the authors prove that $\lambda_{2,1}(P_m \square P_n)$ is equal to 5 if $n = 2$ and $m \geq 4$ and it is equal to 6 if $n, m \geq 4$ or if $n \geq 3$ and $m \geq 5$.

In the same paper the $L(2,1)$-labelling of the cartesian product of several paths $P = \prod_{i=1}^{n} P_{p_i}$ is also considered. For certain values of p_i exact values of $\lambda_{2,1}(P)$ are obtained. From this result, they derive an upper bound for the span of the hypercube Q_n.

In [86] the $\lambda_{2,1}$-numbers for the product of paths and cycles are studied: bounds for $\lambda_{2,1}(C_m \square P_n)$ and $\lambda_{2,1}(C_m \square C_n)$ are given, and they are exact results for some special values of m and n. The authors of [87] and [88] independently achieve the same issue of completing the previous results, and determine $\lambda_{2,1}(C_m \square P_n)$ for all values of m and n: $\lambda_{2,1}(C_m \square P_n)$ is either 5 or 6 or 7, according to the values of m and n.

Kuo and Yan [88] determine $\lambda_{2,1}(C_m \square C_n)$ with $m = 3$ or m multiple of 4 or 5. Finally, in [89] the previous partial results on $\lambda_{2,1}(C_m \square C_n)$ are completed; for all values of m and n, $\lambda_{2,1}(C_m \square C_n)$ is either 6 or 7 or 8, according to the values of m and n.

Exact results for the $L(2,1)$-labelling of the product of complete graphs $K_m \square K_n$ and of $K_p \square \ldots \square K_p$ repeated q times, when p is prime are given in [90].

Concerning the direct and strong products, Jha obtains the $\lambda_{2,1}$-number of some infinite families of products of several cycles [91, 92]. Exact values for $C_3 \otimes C_n$, $C_4 \otimes C_n$ and improved bounds for $C_n \otimes C_m$ are presented in [93, 94].

In [95], the $\lambda_{2,1}$-number is computed for $C_m \times C_n$ for some special values of m and n.

Papers [96] and [97] handle the $L(2,1)$-labelling of graphs that are the direct/strong product and the cartesian product of general non trivial graphs. The authors prove that for all the three classes of graphs the conjecture by Griggs and Yeh is true, except in the special case in which one of the two factor has degree 1 (end hence it is $K_2 = P_2$). In [98] the previous upper bounds for direct and strong product of graphs are improved. The main tool for this purpose is a more refined analysis of neighborhoods in product graphs. In [99], the authors correct a mistake in a proof of [97], and study another product similar to the cartesian product called *composition*. Finally, the authors of [100] derive alternative upper bounds on $\lambda_{2,1}$ of graphs $G \star K_2$ (where $\star$ is one product among direct, Cartesian, strong) improving in most cases the previously known values.

$L(h,k)$**-labelling.** In [101] exact values for the $\lambda_{h,1}$-number of $C_{c_1} \times \ldots \times C_{c_n}$ and of $C_{c_1} \square \ldots \square C_{c_n}$ are provided, if there are certain conditions on h and on the length of the cycles $c_1, \ldots, c_n$.

In [102, 90] the $L(h,k)$-labelling problem of products of complete graphs is considered ($h \geq k$) and the following exact results are given, where $2 \leq n < m$:
$\lambda_{h,k}(K_n \square K_m) = (m-1)h + (n-1)k$ if $\frac{h}{k} > n$;
$\lambda_{h,k}(K_n \square K_m) = (mn-1)k$ if $\frac{h}{k} \leq n$;
$\lambda_{h,k}(K_n \square K_n) = (n-1)h + (2n-1)k$ if $\frac{h}{k} > n-1$;
$\lambda_{h,k}(K_n \square K_n) = (n^2-1)h$ if $\frac{h}{k} \leq n-1$.

In [103] the $\lambda_{h,k}$-number of the cartesian product $\prod_{i=1}^{n} K_{t_i}$ is exactly determined for $n \geq 3$ and relatively prime $t_1, \ldots, t_n$, where $2 \leq t_1 < t_2 < \ldots < t_n$.

In [104] all previous results are extended more generally, indeed the authors consider graph $K_{n_1} \square K_{n_2} \square \ldots \square K_{n_q}$ for any value of $n_1, n_2, \ldots, n_q$, and compute exact values of its $\lambda_{h,k}$-number, for all values of h and k such that $\frac{h}{k} \leq n-q+1$ and $\frac{h}{k} \geq qn-2q+2$, where $2 \leq q \leq p$, being p the minimum prime factor of n.

We underline that many technical papers have appeared, concerning the product of graphs. For example, in [105] the $L(h,1)$-labelling of the Cartesian product of a cycle and a path is handled, while the authors of [106] determine the $\lambda_{h,k}$-number of graphs that are the direct product of complete graphs, with certain conditions on h and k.

Open Problem: It remains an open problem to complete the previous results, but also this result would be especially technical.

Moreover, the $L(2,1)$-labelling of special kinds of graph product has been investigated. Among them, we remind:

- the *amalgamation* of graphs (let $G_1, \ldots, G_p$ be $p \geq 2$ graphs each containing a fixed indued subgraph isomorphic to a graph G_0; the amalgamation of $G_1, \ldots, G_p$ along G_0 is the simple graph obtained by identifying $G_1, \ldots, G_p$ along G_0 at the nodes in the fixed subgraphs isomorphic to G_0 in each $G_1, \ldots, G_p$, respectively), studied in [108, 109, 110];

- the *modular product* of two graphs (the modular product of two graphs G and H is the graph with node set $V(G) \times V(H)$, in which a node (v,w) is adjacent to a node (v',w') iff either (i) $v = v'$ and w is adjacent to w', or (ii) $w = w'$ and v is adjacent to v', or (iii) v is adjacent to v' and w is adjacent to w', or (iv) v is not adjacent to v' and w is not adjacent to w'), studied in [111];

- the *skew product* of two graphs (the skew product of G and H is the graph with node set $V(G) \times V(H)$, in which the node (x_1,x_2) is adjacent to the node (y_1,y_2) iff either $x_1 = y_1$ and $(x_2,y_2) \in E(H)$ or $(x_1,y_1) \in E(G)$ and $(x_2,y_2) \in E(H)$), studied in [112, 113, 114].

We conclude this subsection by dealing with the r-th power of a graph G, written G^r, defined as a graph on the same node set as G, such that two nodes are adjacent if and only if their distance in G is at most r. We have already spoken about the correspondence of the vertex coloring of G^2 and the $L(1,1)$-labelling of G. Here we remind that two papers deal with the $L(h,1)$-labelling of powers of paths: in [64] $\lambda_{h,1}(P_n^r)$ is determined, while in [107] it is proven that some of the previous results are incorrect and new bounds are presented. These bounds are function of h, r and n.

4.4 Trees

Let T be any tree with maximum degree Δ.

$L(0,1)$- and $L(1,1)$-labelling. Bertossi e Bonuccelli [12] investigate the $L(0,1)$-labelling problem on complete binary trees, proving that 3 colors suffice. An optimum labelling can be found as follows. Assign first labels 0, 1 and 2, respectively, to the root, its left child and its right child. Then, consider the nodes by increasing levels: if a node has been assigned label c, then assign the remaining two colors to its grandchildren, but giving different to brother grandchildren. The above procedure can be generalized to find an optimum $L(1,1)$-labelling for complete $(\Delta-1)$-ary trees, requiring span Δ. It is straightforward to see that when $\Delta = 3$ and $\Delta = 2$ this result gives the $\lambda_{0,1}$-number for complete binary trees and paths, respectively.

It is shown in [65] that for any T, $\lambda_{1,1}(T)$ is equal to Δ.

$L(2,1)$-labelling. Given any tree T, Griggs and Yeh [4] show that $\lambda_{2,1}(T)$ is either $\Delta+1$ or $\Delta+2$, and conjecture that recognising the two classes is NP-hard. Chang and Kuo [27] disprove this conjecture by providing a polynomial time algorithm based on dynamic programming. The algorithm consists in calculating a certain function s for all nodes of the tree. It starts from the leaves and works toward the root. For any node v, whose children are $v_1, v_2, \ldots, v_k$, the algorithm uses $s(v_1), \ldots, s(v_k)$ to calculate $s(v)$, and to do that, it needs to construct a bipartite graph and to find a maximum matching. The algorithm runs in $O(\Delta^{4.5}n)$ time, where Δ is the maximum degree of tree T and n is the number of nodes, hence the time complexity is $O(n^{5.5})$ in the worst case. In this time complexity, its $\Delta^{2.5}$ factor comes from the complexity of solving the bipartite matching problem, and its $\Delta^2 n$ factor from the number of iterations for solving bipartite matchings.

The Chang and Kuo's algorithm can be also used to optimally solve the problem for a slightly wider class of graphs, i.e. p-almost trees, for fixed values of p [25]. (A *p-almost tree* is a connected graph with $n+p-1$ edges.) More precisely, the authors prove that $\lambda_{2,1}(G) \leq \sigma$ can be tested in $O(\sigma^{2p+9/2}n)$ time, for each p-almost tree G and each given σ. In [115] an $O(\min\{n^{1.75}, \Delta^{1.5}n\})$ time algorithm has been proposed. It is based on the similar dynamic programming framework to Chang and Kuo's algorithm, but achieves its efficiency by reducing heavy computation of bipartite matching and by using amortized analysis.

Obviously, Chang and Kuo's algorithm runs in linear time if $\Delta = O(1)$. It has also been proven [116] that the $L(2,1)$-labelling problem on trees can be also solved in linear time if $\Delta = \Omega(\sqrt{n})$. It follows that the worst running time of the algorithm in [115] is $O(n^{1.75})$. Hasunama, Ishii, Ono and Uno have presented a linear time algorithm for $L(2,1)$-labelling of trees [115], which finally settles the problem of improving the complexity to linear time and closes the question.

A tree T is of *type 1* if $\lambda_{2,1}(T) = \Delta+1$ and of *type 2* if $\lambda_{2,1}(T) = \Delta+2$. It seems that characterizing all type 1 (2) trees is very difficult. In [117] a sufficient condition for a tree T

to be of type 1 is given. Namely, the author proves that if a tree T contains no two nodes of maximum degree at distance either 1, 2 or 4, then $\lambda_{2,1}(T) = \Delta + 1$.

$L(h,k)$**-labelling.** Chang and Kuo's algorithm can be generalized to trees and to p-almost trees to polynomially determine the exact $\lambda_{h,1}$-number [64, 41].

For any tree T of maximum degree Δ, $\Delta+h-1 \leq \lambda_{h,1}(T) \leq \min\{\Delta+2h-2, 2\Delta+h-2\}$ [64]; the lower and the upper bounds are both attainable. Lower bounds on the $\lambda_{h,1}$-number can be given also as a function of other parameters of the tree (the *big-degree* and the *neighbour-degree*) [73].

While the generalization of this algorithm is quite easy when $k = 1$, the case $k > 1$ has kept resisting all attempts up to when Fiala, Golovach and Kratochvil [118] solved the problem, as highlighted below. Before this paper, well known researchers had conjectured both the polynomiality and the NP-hardness of the $L(h,k)$-labelling problem on trees. Namely, from the one hand Welsh [119] suggested that, by an algorithm similar to Chang and Kuo's, it should have been possible to determine $\lambda_{h,k}(T)$ for a tree T and for arbitrary h and k, hence conjecturing that the general case is also polynomial for trees. From the other hand, based on some considerations concerning the crucial step of Chang and Kuo's algorithm, Fiala, Kratochvíl and Proskurowski conjecture that determining $\lambda_{h,k}(T)$ is NP-hard for trees, when $k > 1$ [26]. The feeling that $k > 1$ identified a more difficult problem seemed to be justified from the fact the problem becomes NP-complete if some nodes of the input tree are precolored, whereas for $k = 1$ the precolored version remains anyway polynomially solvable [26]. Furthermore, the difference between $k = 1$ and $k > 1$ could be put into relationship with the difference between systems of *distinct* and *distant* representative [120]. Another result going toward the same direction states that the decisional version of the $L(h,k)$-labelling problem is NP-complete for trees if h is part of the input and $k \geq 2$ is fixed [121]. The resolutive step remains to study the computational complexity of the problem when both h and k are fixed.

The paper [118] definitively resolves this question proving that for positive integers h and k, the $L(h,k)$-labelling problem restricted to trees is solvable in polynomial time only if k divides h, otherwise it is NP-complete. In particular, in the first case the $L(h,k)$-labelling problem is equivalent to the $L(h/k,1)$-labelling problem, and hence is solvable in polynomial time by the modification of the Chang and Kuo's algorithm presented in [64]. In the case of mutually prime h and k, the NP-hardness is proved by a reduction from the problem of deciding the existence of a system of distant representatives in systems of symmetric sets. The main idea is a construction of trees that allow only specific labels on their roots; the main difficulty is to keep the size of such trees polynomial.

Georges and Mauro provide bounds on the $\lambda_{h,k}$-number for general h and k for trees of maximum degree $\Delta \leq h/k$ [65] and then for trees with $h \geq k$ and $\Delta \geq 3$ [122]. For these parameters they obtain tight upper and lower bounds on $\lambda_{h,k}$ for infinite trees. In [123], the authors present results that are complementary, investigating $L(h,k)$-labellings of trees, for arbitrary positive integers $h < k$, seeking such labellings with small span. The relatively large values of $\lambda_{h,k}(T)$ achieved are witnessed by trees of large height. This fact is not accidental: for trees of height 1, i.e., for stars, the span of $L(h,k)$-labellings is in fact smaller.

Finally, an upper bound on $\lambda_{h,k}$ is given in terms of a new parameter for trees, the *maximum ordering-degree* [124].

4.5 Bounded Width Graphs

4.5.1 Bounded Clique-Width Graphs

The *clique-width* of a graph G, denoted by $cwd(G)$, is the minimum number of labels needed to construct graph G using four operators: $\cdot$, $+$, ρ and η. The operation $\cdot_i$ creates a graph with a single node labelled i. The binary operator $+$ constructs the union of two disjoint graphs. The operation $\rho_{i \to j}$ renames all nodes labelled i with label j. The unary operator $\eta_{i,j}$, with $i \neq j$, adds all the edges between every node labelled i and every node labeled j. The sequence of operations produces an expression of the graph.

Many NP-hard problems become polynomially solvable for graphs of bounded clique-width, if an expression of the graph is part of the input; the classical vertex coloring problem is one of them. Observe also that several classes of graphs (partial t-trees – that will be extensively treated in Subsection 4.5.2, distance-hereditary graphs, P_4-sparse graphs, P_4-tidy graphs, etc.) are known to have bounded clique-width.

$L(1,1)$**-labelling.** The problem is polynomial when restricted to graphs with bounded clique-width. Indeed, for any graph G of clique-width t, the clique-width of G^2 is at most $t \cdot 2^{t+1}$. So, an approach consists in using the vertex coloring algorithm for graphs of bounded clique-width proposed in [125] on G^2. Suchan and Todinca [126] propose an alternative algorithm, that improves the computational complexity from $O(n^{2^{4\cdot 2^{t\log 3}+1}})$ to $O(n^{3\cdot 2^{t\log 3}} n^4)$. Although the complexity remains high, it is considerably lower than the previous one.

$L(2,1)$**-labelling.** The decisional version of the problem is NP-complete even for graphs of clique-width at most 3 [127].

As multiplying the labels of an $L(1,1)$-labelling by 2 we obtain an $L(2,1)$-labelling, the exact algorithm provided in [126] is a 2-approximate algorithm for the $L(2,1)$-labelling problem.

Open Problem: To the best of the author's knowledge, there are no results concerning the $L(2,1)$-labelling on graphs of bounded clique-width greater than two, so any bound on $\lambda_{2,1}$ for these graphs is welcome.

The graphs of clique-width 2 coincide with the class of cographs, that can be defined alternatively as follows.

Let G and H be two graphs with disjoint node sets. The *union* of G and H, $G \cup H$, is the graph whose node set is $V(G) \cup V(H)$ and edge set is $E(G) \cup E(H)$. The *join* of G and H, $G + H$, is the graph obtained from $G \cup H$ by adding all edges between nodes in $V(G)$ and nodes in $V(H)$.

Cographs are defined recursively by the following rules:

1. A node is a cograph;
2. if G and H are cographs, then so is their join $G + H$;
3. if G and H are cographs, then so is their union $G \cup H$.

Chang and Kuo [27], as a consequence of their more general result concerning the $L(2,1)$-labelling problem on union and join of graphs and exploiting the linear time algorithm to identify whether a graph is a cograph [128], prove that there is a linear time algorithm to compute $\lambda_{2,1}(G)$ for a cograph G.

Open Problem: Of course, the polynomiality of the $L(2,1)$-labelling problem on cographs does not implies anything for the $L(h,k)$-labelling problem. In fact, it is still unknown the computational complexity of the $L(h,k)$-labelling problem on cographs, and no algorithms are known.

$L(h,k)$**-labelling.** It is known [129] that all problems expressible in MS_1-logic are fixed parameter tractable (FPT), when parameterized by the clique-width of the input graph. Hence deciding whether $\lambda_{h,k}(G) \leq \sigma$ for a fixed value of σ is polynomial for graphs of bounded clique-width.

In the following, we discuss the results dealing with a subclass of bounded clique-width graphs.

4.5.2 Bounded Treewidth Graphs

The class of *t-trees* is recursively defined as follows:

1. K_t is a t-tree;
2. if H is a t-tree, then the graph obtained from H by adding a new node joining to a t-clique (i.e. K_t) of H is a t-tree;
3. all t-trees can be formed with rules 1 and 2.

Any tree is a 1-tree. t-trees are also a subclass of chordal graphs.

Any subgraph of a t-tree is called *partial t-tree*. The partial t-trees are a particular case of graphs of bounded clique-width, more precisely, if G is a partial t-tree then $cwd(G) \leq 2^{t+1}+1$ [130]. The minimum value of t for which a graph G is a subgraph of a t-tree is called the *treewidth* $tw(G)$ of the graph. See [131, 132] for surveys on treewidth. Many NP-hard problems have been shown to be solvable in polynomial time on graphs with bounded treewidth.

This class is interesting in the wireless networks context, since pairs of antennas have no interference if their distance is far enough. Furthermore, concentrations of antennas are found in densely populated areas. These areas are connected with one another with a limited number of edges. Such networks can be represented by a constraint graph with a tree-like structure [133].

Fiala and Kratochvíl [134] observe that, given a fixed span σ, the question "Is $\lambda_{h,k}(G) \leq \sigma$?" can be expressed in MSOL (Monadic Second Order Logic), so this decision problem can be decided in polynomial time for graphs of bounded treewidth. On the contrary, if the span is part of the input, the $L(2,1)$-labelling problem is NP-complete for graphs of treewidth at most two [135]. This result adds a natural and well studied problem to the short list of problems whose computational complexity separates treewidth one from treewidth two. Indeed, usually, the problems solvable in polynomial time for trees are also polynomially solvable for graphs of bounded treewidth, though sometimes the extension to bounded treewidth is not straightforward.

$L(0,1)$**-,** $L(1,1)$**- and** $L(2,1)$**-labelling.** Bodlaender et al. [42] compute upper bounds for graphs of treewidth bounded by t proving that $\lambda_{0,1}(G) \leq t\Delta - t$, $\lambda_{1,1}(G) \leq t\Delta$ and $\lambda_{2,1}(G) \leq t\Delta + 2t$. They give also approximation algorithms for the $L(0,1)$-, $L(1,1)$- and $L(2,1)$-labellings running in $O(tn\Delta)$ time. Nevertheless, two of these three problems can be optimally solved: in [136] a polynomial time algorithm to optimally $L(1,1)$-label graphs with constant treewidth t is presented, but it applies dynamic programming and the required time is very high: $O(n) \cdot O(\Delta^{2^{8(t+1)+1}}) + O(n^3)$. A similar argument would yield the same result for the $L(0,1)$-labelling problem. If the graph with constant treewidth has also constant maximum degree, than $O(n)$ time is sufficient to optimally solve the $L(1,1)$-labelling problem [137].

Parameterized complexity of these problems was considered in [138]. It is proved that $L(0,1)$- and $L(1,1)$-labellings are $W[1]$-hard when parameterized by the treewidth of the input graph.

$L(h,k)$**-labelling.** In [64], the authors give an upper bound on the $L(h,1)$-numbers of t-trees, proving that $\lambda_{h,1}(G) \leq (2h-1+\Delta-t)t$.

Open Problem: Upper and lower bounds on $\lambda_{h,k}(G)$ for graphs of bounded treewidth are completely unexplored. The relevance of this class of graphs makes this open problem an interesting issue.

4.6 Planar Graphs

A graph G is *planar* if and only if it can be drawn on a plane so that there are no edge crossings, i.e. edges intersect only at their common extremes.

In many real cases the actual network topologies are planar, since they consist of communication stations located in a geographical area with non-intersecting communication channels [133].

The decision version of the $L(h,k)$-labelling problem is NP-complete for planar graphs [37] and even for planar graphs of bounded degree [38].

$L(1,1)$**-labelling.** The first reference concerning the $L(1,1)$-labelling problem on planar graphs seen as the problem of coloring the square of graphs, is by Wegner [11], who gives bounds on the clique number of the square of planar graphs. In particular, he gives an instance for which the clique number is at least $\lfloor 3/2\Delta \rfloor + 1$ (which is largest possible), and conjectures this to be an upper bound on $\chi(G^2)$ (i.e. on $\lambda_{1,1}(G)+1$), for $\Delta \geq 8$. He conjectures also that $\lambda_{1,1}(G) \leq \Delta + 4$ for $4 \leq \Delta \leq 7$. Moreover, Wegner proves that $\lambda_{1,1}(G) \leq 7$ for every planar graph G of maximum degree 3, and conjectures that this upper bound could be reduced to 6.

Open Problem: All three Wegner's conjectures remain open, although Thomassen [139] thought to have established the latter conjecture.

Given a planar graph G, Jonas [55] proves that $\lambda_{1,1}(G) \leq 8\Delta - 21$, $\Delta > 3$. This bound is later improved in [140] to $\lambda_{1,1}(G) \leq 3\Delta + 4$ and in [141] to $\lambda_{1,1}(G) \leq 2\Delta + 24$. The better asymptotic bound $\lambda_{1,1}(G) \leq \lfloor \frac{9}{5}\Delta \rfloor + 1$ holds for very large values of Δ ($\Delta \geq 749$) [142]. Borodin et al. [143] show that we only need $\Delta \geq 47$ for the bound $\lceil \frac{9}{5}\Delta \rceil$ to hold. Furthermore, they prove that $\lambda_{1,1}(G) \leq 58$ if $\Delta \leq 20$ and $\lambda_{1,1}(G) \leq \Delta + 38$ if $21 \leq \Delta \leq 46$. But the better asymptotic result is the following: $\lambda_{1,1}(G) \leq \lceil \frac{5}{3}\Delta \rceil + 77$ for any planar graph G and $\lambda_{1,1}(G) \leq \lceil \frac{5}{3}\Delta \rceil + 24$ if $\Delta \geq 241$ [144]. Some of the above results were obtained by identifying so-called *light structures* in planar graphs. The interested reader can see the survey [145].

For planar graphs of large girth, better upper bounds for $\lambda_{1,1}(G)$ are obtained by the results on general h and k in [146], listed below. In [147] the authors show that $\lambda_{1,1}(G) \leq 5\Delta$ if G is planar. Comparing this result with the best known one $\lambda_{1,1}(G) \leq \lceil 5/3\Delta \rceil + 77$, we get that 5Δ is better for any $\Delta \leq 24$.

Wang and Lih [146] conjecture that for any integer $g \geq 5$, there exists an integer $M(g)$ such that if G is planar of girth g and maximum degree $\Delta \geq M(g)$, then $\lambda_{1,1} \leq \Delta$. This conjecture is known to be false for $g = 5, 6$ and true for $g \geq 7$ [148, 149]. Nevertheless, the conjecture is "almost" true for $g = 6$, in the sense that for Δ large enough ($\Delta \geq 8821$), $\lambda_{1,1} \leq \Delta + 1$ if $g \leq 6$ [150].

Open Problem: It is not known whether an analogous statement can hold for planar graphs of girth 5.

Open Problem: The problem of tightly bounding $\lambda_{1,1}$ for planar graphs with relatively small Δ is far to be solved, and it would be a very interesting result.

An approximation algorithm for the $L(1, 1)$-labelling problem with a performance guarantee of at most 9 for all planar graphs is given in [40]. For planar graphs of bounded degree, in [137] there is a 2-approximation algorithm. For planar graphs of large degree ($\Delta \geq 749$) an 1.8-approximation algorithm is presented in [142]. The results of [144], given for any h and k, apply here resulting to an 1.66-approximation algorithm.

$L(2, 1)$-labelling. It is NP-complete to decide whether $\lambda_{2,1}(G) \leq r$ for a planar bipartite graph of degree $r - 1$ [42]. Nevertheless, it seems that the technique used cannot help to show the NP-completeness of the problem of deciding whether a given planar graph G has $\lambda_{2,1}(G) \leq r$ for any odd values of r. The authors leave this as an open problem. This problem has been investigated in [34] and then definitively closed in [43], where Eggemann, Havet and Noble provide a proof of NP-completeness for planar graphs of any degree.

Jonas [55] proves that $\lambda_{2,1}(G) \leq 8\Delta - 13$ when G is planar. This bound has been the best one until recently, when the general bounds for $\lambda_{h,k}$ found by Molloy and Salavatipour and discussed in the following have been derived.

Open Problem: The Griggs and Yeh's conjecture is still open for planar graphs with $\Delta = 3$, while is known to be true for the other values of Δ. Namely, for $\Delta \geq 7$ it follows from [141], while Bella et al. [151] prove it for $4 \leq \Delta \leq 6$. Further details on these results are given below.

$L(h, k)$-labelling. Van den Heuvel and McGuinnes [141] show that $\lambda_{h,k}$ is bounded above by $(4k - 2)\Delta + 10h + 38k - 23$ for planar graphs for any positive integers h and k, such that $h \geq k$. This bound implies $\lambda_{2,1}(G) \leq 2\Delta + 35$ and $\lambda_{1,1}(G) \leq 2\Delta + 24$. Then, in [144], the result is improved to $k\lceil \frac{5}{3}\Delta \rceil + 18h + 77k - 18$ for any positive integers h and k. Observe that this latter value is asymptotically better than the previous ones and leads to the best known bounds on $\lambda_{2,1}$ and $\lambda_{1,1}$, of $\lceil \frac{5}{3}\Delta \rceil + 95$ and $\lceil \frac{5}{3}\Delta \rceil + 77$, respectively.

Open Problem: Since for a planar graph G it holds the trivial lower bound $\lambda_{h,k}(G) \geq \Delta k + h - k$, it remains an open problem to understand which is the tight constant multiplying Δ in the value of $\lambda_{h,k}(G)$ or, at least, of $\lambda_{2,1}(G)$. This is a very discussed problem, as the large amount of produced literature proves. Furthermore, this question gives sense to all the results cited above, holding only for very large values of Δ.

In the case $h = 2$ and $k = 1$, the bound of Van den Heuvel and McGuinnes implies that the conjecture of Griggs and Yeh (i.e. $\lambda_{2,1} \leq \Delta^2$) holds for planar graphs with maximum degree $\Delta \geq 7$. In [151] it is shown that the conjecture holds for planar graphs with maximum degree $\Delta \neq 3$. Namely, the authors prove that every planar graph with maximum degree 4, 5, 6, respectively, has an $L(2,1)$-labelling with span at most 16, 25, 32, respectively, and the proof in the case of maximum degree 4 is computer-assisted.

The $L(h,k)$-labelling problem is also studied on planar graphs with conditions on their girth. More precisely, in [146] the following bounds are proven:

- if $g(G) \geq 7$, then $\lambda_{h,k}(G) \leq (2k-1)\Delta + 4h + 4k - 4$;
- if $g(G) \geq 6$, then $\lambda_{h,k}(G) \leq (2k-1)\Delta + 6h + 12k - 9$;
- if $g(G) \geq 5$, then $\lambda_{h,k}(G) \leq (2k-1)\Delta + 6h + 24k - 15$.

Observe that, since $\Delta + 8 \leq \Delta^2$ when $\Delta \geq 4$, $\Delta + 15 \leq \Delta^2$ when $\Delta \geq 5$, and $\Delta + 21 \leq \Delta^2$ when $\Delta \geq 6$ the conjecture by Griggs and Yeh holds for planar graphs with $g(G) \geq 7$ and $\Delta \geq 4$ or $g(G) = 6$ and $\Delta \geq 5$, or $g(G) = 5$ and $\Delta \geq 6$.

Furthermore, if the degree is sufficiently large (very large, indeed), a better bound can be provided: every planar graph of girth $g \geq 7$ has an $L(h,k)$-labelling of span at most $2h + \Delta k - 2$ if $\Delta \geq 190 + 2\lceil \frac{h}{k} \rceil$ [152]. Since the optimal span of an $L(h,1)$-labelling of an infinite Δ-regular tree is $2h + \Delta - 2$, the obtained bound is the best possible for any $h \geq 1$ and $k = 1$. In [153], the authors study $L(h,k)$-labellings for planar graphs without 4-cycles, as such graphs possess some interesting properties. The authors prove the following bound for planar graphs without 4-cycles: $\lambda_{h,k} \leq \min\{(8k-4)\Delta + 8h - 6k - 1, (2k-1)\Delta + 10h + 84k - 47\}$ for any $h, k \geq 1$ that is asymptotycally better than any previous bound on this subclass of planar graphs. As an immediate consequence, it follows that $\lambda_{2,1} \leq \min\{4\Delta + 9, \Delta + 57\}$ and $\lambda_{1,1} \leq \min\{4\Delta + 1, \Delta + 47\}$. Hence, the Griggs and Yeh conjecture on $\lambda_{2,1}$ and the Wegner conjecture on $\lambda_{1,1}$ hold for planar graphs without 4-cycles having $\Delta \geq 9$ and $\Delta \geq 96$, respectively.

Open Problem: For any graph G, it is well known that $\lambda_{1,1} \geq \Delta$ and $\lambda_{2,1} \geq \Delta + 1$. From the results in [153] it follows that there exist two constants c_1 and c_2 such that for all planar graphs G without 4-cycles $\lambda_{h,1}(G)$ lies in the interval $[\Delta + h - 1, \Delta + c_h]$, $h = 1, 2$. It is not known which are the precise values of c_1 and c_2. We remark that c_1 and c_2 are not bounded when planar graphs G are allowed to have 4-cycles.

We introduce a graph invariant that is an indicator of the sparseness of a graph: the *maximum average degree*, denoted $\mathrm{Mad}(G)$ is the maximum among the average degrees of its subgraphs, i.e. $\mathrm{Mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|}\}$; $H \subseteq G$. There is a well known relation between the Mad and the girth of a planar graph, indeed for every planar graph G with girth at least g, $\mathrm{Mad}(G) < \frac{2g}{g-2}$. From this, it follows that planar graphs with girth 5, 6 and 7 have Mad ¡ 10/3, 3 and 14/5, respectively. In [154], the $L(h,k)$-labelling of graphs with Mad ¡ 10/3, 3 and 14/5 is considered, proving some bounds improving the previous results in [146].

4.6.1 Outerplanar and l-Outerplanar Graphs

A graph is *outerplanar* if it can be embedded in the plane so that every node lies on the boundary of the outer face.

A graph G is *l-outerplanar* if for $l = 1$, G is outerplanar and for $l > 1$ G has a planar embedding such that if all nodes on the exterior face are deleted, the connected components of the remaining graph are all $(l-1)$-outerplanar.

It can be determined in polynomial time whether G is outerplanar and whether it is l-outerplanar.

$L(1,1)$**-labelling.** Any l-outerplanar graph has a treewidth of at most $3l-1$ [155]. Moreover, outerplanar graphs are series-parallel graphs, and series-parallel graphs are exactly partial 2-trees. Thus, applying the result of [136] for bounded treewidth graphs, we get that any l-outerplanar and outerplanar graph can be optimally $L(1,1)$-labelled in $O(n^3)$ time.

In [77] a linear time algorithm for optimally $L(1,1)$-labelling any outerplanar graphs of degree $\Delta \geq 7$ with at most $\Delta + 1$ colors is presented. When $\Delta \geq 6$ the required number of colors is anyway 11. This algorithm first executes a special traversal of an embedding of the graph (*ordered breadth first search*), then it labels the nodes in a greedy fashion starting from the root of the resulting spanning tree and following a level by level order. The optimality proof exploits some strong properties of the non-tree edges. Later, Agnarsson and Halldórsson [156, 157] derive optimal upper bounds on $\lambda_{1,1}$ for outerplanar graphs of small degree ($\Delta < 7$) proving that $\lambda_{1,1} \leq \Delta + 2$ if $\Delta = 2$ and $\lambda_{1,1} \leq \Delta + 1$ if $\Delta = 3, 4, 5$ and $\lambda_{1,1} \leq \Delta$ if $\Delta = 6$.

$L(2,1)$**-labelling.** As outerplanar graphs are graphs of treewidth 2, from the result in [42] for bounded treewidth graphs, we have an immediate upper bound, i.e. $\lambda_{2,1}(G) \leq 2\Delta + 4$. Jonas [55] proves the slightly better bound $\lambda_{2,1}(G) \leq 2\Delta + 2$. Providing a coloring algorithm, in [42], the authors improve this bound to $\lambda_{2,1}(G) \leq \Delta + 8$ for any outerplanar graph G, but they conjecture that the tightest bound could be $\Delta + 2$. Calamoneri and Petreschi [77] prove this conjecture when the input outerplanar graph has maximum degree $\Delta \geq 8$, and they provide a linear time algorithm that guarantees an $L(2,1)$-labelling of G with a number of colors far at most one from optimum. The algorithm is analogous to that one presented for the $L(1,1)$-labelling.

Open Problem: The authors of [77] conjecture that this algorithm is optimal; if this is true, the $L(2,1)$-labelling problem on outerplanar graphs would be polynomially solvable. The question is still open, and particularly interesting because outerplanar graphs are perfect and such a result would help to understand the relationship between the hardness of the vertex coloring and of the $L(2,1)$-labelling problem.

For outerplanar graphs with smaller values of Δ, in [77] it is guaranteed $\lambda_{2,1}(G) \leq 10$, improving anyway the bound of $\Delta+8$, but the authors conjecture that the bound $\Delta+2$ holds for any outerplanar graph of degree $\Delta \geq 4$. Differently, in the special case $\Delta = 3$, it is shown that there exists an infinite class of outerplanar graphs needing $\Delta+4$ colors and an algorithm using at most $\Delta + 6$ colors for any degree 3 outerplanar graph is presented.

Wang and Luo [158] prove some bounds on small degree outerplanar graphs: $\lambda_{2,1}(G) \leq \Delta + 4$ if $\Delta = 3$ and $\lambda_{2,1}(G) \leq \Delta + 5$ if $\Delta = 4$, so improving the previous bounds in these two cases. Furthermore, these author provide a graph with $\Delta = 4$ disproving the previous conjecture as it needs 8 colors (see Figure 8).

Some of the previous bounds for small values of Δ are empirically improved in [159] for some sample outerplanar graphs. In particular, the authors prove that $\lambda_{2,1}(G) \leq 6 = \Delta + 3$ for each degree 3 outerplanar graph. Furthermore, by experimental techniques, they improve the lower bounds for $4 \leq \Delta \leq 6$ showing some graphs requiring $\Delta + 3$ colors.

Li and Zhou [160] close the problem for degree three outerplanar graphs, proving that $\lambda_{2,1} \leq \Delta + 3$ for these graphs.

Open Problem: It remains an open problem to close the gap between upper and lower bounds for $4 \leq \Delta \leq 7$.

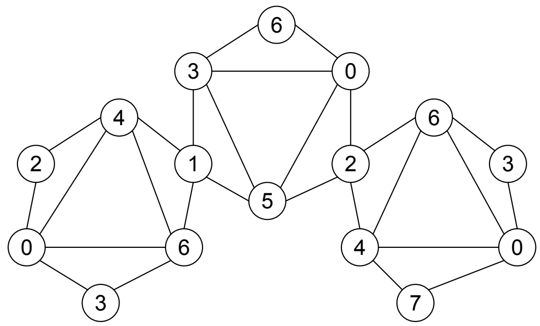

Figure 8: A counterexample outerplanar graph with $\Delta = 4$ and $\lambda_{2,1} = \Delta + 3$.

The upper bounds on $\lambda_{2,1}(G)$ can be slightly improved if the outerplanar graph G is triangulated [42, 77].

$L(h,k)$**-labelling.** To the best of the author's knowledge, the sole paper handling the general $L(h,k)$-labeling problem on outerplanar graphs is [161], proving that $\lambda_{h,k}(G) \leq (2k-1)\Delta + 2(2h-1)$ if G is outerplanar of degree Δ.

Open Problem: Observe that, when $h = 1, 2$ and $k = 2$ the previous bound does not match the tighter results achieved when directly studying these special cases. Hence, it is an open problem to understand how the bound $(2k-1)\Delta+2(2h-1)$ is far from the optimal value.

4.6.2 K_4-Minor Free Graphs

A graph G has a graph H as a *minor* if H can be obtained from a subgraph of G by contracting edges, and G is called H*-minor free* if G does not have H as a minor. A graph is called *series-parallel* graph if G can be obtained from K_2 by applying a sequence of operations, where each operation is either to duplicate an edge (i.e. replace an edge with two parallel edges) or to subdivide an edge (i.e. replace an edge with a path of length 2). A graph G is K_4-minor free if and only if each block of G is a series-parallel graph. It is well known [162] that a graph G is outerplanar if and only if G is K_4-minor free and $K_{2,3}$-minor free. Thus, the class of K_4-minor free graphs is a class of planar graphs that contains both outerplanar and series-parallel graphs.

$L(1,1)$**-labelling.** If G is a K_4-minor free graph, then $\lambda_{1,1}(G) \leq \Delta + 2$ if $2 \leq \Delta \leq 3$ and $\lambda_{1,1}(G) \leq \lfloor \frac{3}{2}\Delta \rfloor$ otherwise [163]. The result is best possible, as witnessed by the following graphs:

- for $\Delta = 2$, cycle C_5, having $\lambda_{1,1} = 4$;
- for $\Delta = 3$ let G be the graph consisting of three internally disjoint paths joining two nodes x and y, where two of the paths are of length 2 and the third one is of length 3; this graph has $\lambda_{1,1} = 5$ (see Fig. 9.a);
- for $\Delta = 2d \geq 4$ let G_{2d} be the graph consisting of d internally disjoint paths joining x and y, d internally disjoint paths joining x and z, and d internally disjoint paths joining y and z; all these paths are of length 2, except one path joining x and y of length 1, and one path joining x and z of length 1. The degree of such a graph is $2d$ and its $\lambda_{1,1}$ is 3d (see Fig. 9.b where d is 3);

– for $\Delta = 2d+1 \geq 5$, let G_{2d+1} be obtained from G_{2d} by adding a new path of length 2 joining y and z; this graph has $\lambda_{1,1} = 3d+1$.

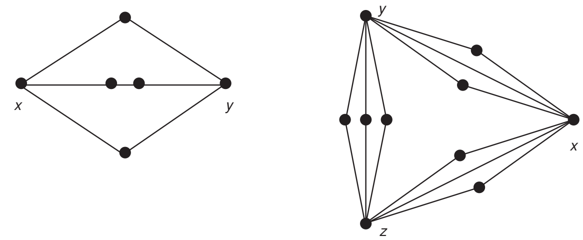

Figure 9: K_4-minor free graphs having $\lambda_{1,1} = \lfloor \frac{3}{2}\Delta \rfloor = n$.

$L(2,1)$**-labelling.** It is NP-complete to decide whether $\lambda_{h,k}(G) \leq \sigma$, where σ is part of the input, if G is series-parallel [135], as a consequence of the results on bounded treewidth graphs, as series-parallel graphs are exactly partial 2-trees. Of course, the same result holds for K_4-minor free graphs, a superclass of series-parallel graphs.

$L(h,k)$**-labelling.** Wand and Wang [164] show that every K_4-minor free graph G with maximum degree Δ has an $L(h,k)$-labelling, $h+k \geq 3$, with span at most $2(2h-1)+(2k-1)\lfloor \frac{3}{2}\Delta \rfloor$, generalizing the previous result on the $L(1,1)$-labelling. It is natural to wonder whether the bound on $\lambda_{h,k}$ is optimal, as the one on $\lambda_{1,1}$. This is not the case, as proved in [165], where it is shown that for every $h \geq 1$ there exist Δ_0 such that every K_4-minor free graph with maximum degree $\Delta \geq \Delta_0$ has an $L(h,1)$-labelling with span at most $\lfloor \frac{3}{2}\Delta \rfloor$ and this bound cannot be further decreased. This result translates to $L(h,k)$-labelling, with $k > 1$ providing an upper bound for $\lambda_{h,k}$ of $k\lfloor \frac{3}{2}\Delta \rfloor$.

4.7 Graphs with an Intersection Model

For a given set M of objects (for which intersection makes sense), the corresponding *intersection graph* G is the undirected graph whose nodes are objects and an edge connects two nodes if the corresponding objects intersect. M is called the *model* of G with respect to intersection.

Depending on the nature of the object, many interesting classes can be defined.

4.7.1 Disk Graphs and (r,s)-Civilized Graphs

A *disk graph* is the intersection graph of a set of disks in the plane, where each disk is uniquely determined by its center and its diameter. The class of disk graphs is very wide and interesting, and it includes classes as, for instance, planar graphs. When all disks are of the same diameter the graph is called *unit disk graph.*

The disk graph and unit disk graph recognition problem is NP-hard [166, 167]. Hence, labelling algorithms that require the corresponding disk graph representation are substantially weaker than those which work only with graphs.

For each fixed pair of real values $r > 0$ and $s > 0$, a graph G belongs to the class of the (r,s)*-civilized graphs* if there exists a positive integer $d \geq 2$ such that the intersection model is a set of spheres of R^d, the centers of intersecting spheres are at distance $\leq r$ and the distance between any two centers is $\geq s$ [168]. In the following, planar (r,s)-civilized graphs (i.e. with

$d = 2$) will be treated; however, all the results can be extended directly to civilized graphs of higher dimension.

Note that the class of (r, s)-civilized graphs includes disk graphs whenever there is a (fixed) minimum separation between the centers of any pairs of circles.

The previously described classes of graphs are considered as reasonable models for several classes of packet radio networks. To see this, consider packet radio networks in which the range of any transmitter can be considered as a circular region with the transmitter at the center of the circle; let r be the radius of the region corresponding to a transmitter's maximum range. Further, it is natural to assume a minimum separation s between any pair of transmitters, otherwise the equipment carrying the transmitters cannot be placed. Clearly, the graphs that model such networks belong to the class of (r, s)-civilized graphs. In many other realistic situations, the ratio of maximum to the minimum transmitter range is not fixed; in such cases disk graphs are more realistic [137].

$L(1, 1)$**-labelling.** The decision version of the problem is NP-complete for unit disk graphs. Krumke, Marathe and Ravi [137] give a 2-approximation algorithm for the $L(1, 1)$-labelling problem for (r, s)-civilized graphs. The performance guarantee of the algorithm is independent of the values of r and s. An approximation algorithm with a performance guarantee of 14 for disk graphs is given in [169]. It has been later shown [170] that the performance guarantee of 14 can also be achieved using the greedy paradigm. The performance ratio has been improved to 13 (to 12 if the radii are quasi-uniform) [171], by means of FIRST-FIT algorithms. In [169, 171] two approximation algorithms for unit disk graphs with a performance guarantee of 7 are presented. Finally, in [171] the authors prove that for unit disk graphs whose nodes lie in a horizontal strip of heigth $\frac{\sqrt{3}}{2}$ the $L(1, 1)$-labelling problem can be solved in polynomial time. The reason is that such graphs are *co-comparability graphs*, that are perfect graphs with the property that their square graphs are still co-comparability graphs, and hence optimally $L(1, 1)$-labellable in polynomial time (see Subsection 4.9.8).

Open Problem: Is it possible to improve the performance ratios of 13 and 7 for disk and unit disk graphs, respectively?

$L(2, 1)$**-labelling.** Fiala, Fishkin and Fomin [172] explore the $L(2, 1)$-labelling problem on disk and unit disk graphs. For the first class of graphs they provide an approximation algorithm having performance ratio bounded by 12. For the second class, they present a robust labelling algorithm, i.e. an algorithm that does not require the disk representation and either outputs a feasible labelling, or answers the input is not a unit disk graph. Its performance ratio is constant and bounded by 32/3. In both cases, $\lambda_{2,1}(G)$ is bounded by a linear function of the size of the maximum clique in G.

In [173] the first known upper bound for unit disk graphs in terms of Δ is shown: $\lambda_{2,1}(G) \leq \frac{4}{5}\Delta^2 + 2\Delta$.

Open Problem: This latter upper bound is far from being tight, indeed consider as an example the triangular grid G_6: it is a unit disk graph of maximum degree 6, its span is 8, but the value of this upper bound is 40.

$L(h, k)$**-labelling.** In [172] it is also studied also the more general $L(h, k)$-labelling problem on disk graphs (in fact the even more general $L(p_1, p_2, \ldots p_r)$-labelling problem) and it is presented an approximation algorithm whose performance depends on the *diameter ratio* σ, i.e. the ratio between the biggest and the smallest diameters of the set of disks.

4.7.2 Chordal Graphs

A graph is *chordal* (or *triangulated*) if and only if it is the intersection graph of subtrees of a tree. An equivalent definition is the following: a graph is *chordal* if every cycle of length greater than three has a chord. Chordal graphs have been extensively studied as a subclass of perfect graphs [2].

An *n-sun* is a chordal graph with a Hamiltonian cycle $x_1, y_1, x_2, y_2, \dots x_n, y_n, x_1$ in which each x_i is of degree exactly 2. A *sun-free-chordal* (respectively *odd-sun-free-chordal*) graph is a chordal graph which contains no n-sun with $n \geq 3$ (respectively odd $n \geq 3$) as an induced subgraph. Sun-free-chordal graphs are also called *strongly chordal graphs* and are particularly interesting as they include directed path graphs, interval graphs, unit interval graphs and trees. Also strongly chordal graphs can be defined as intersection graphs of subtrees with certain properties of a tree. A graph is *weakly chordal* if it has no induced cycle of length at least five. A graph G is called a *block graph* if each block of G is a complete graph. The class of block graphs includes trees and is a subclass of strongly chordal graphs.

$L(1,1)$**-labelling.** In [174] it is proven that the $L(1,1)$-labelling problem on chordal graphs is hard to approximate within a factor of $n^{\frac{1}{2}-\epsilon}$, for any $\epsilon > 0$, unless NP-problems have randomized polynomial time algorithms. The authors match this result with a simple $O(\sqrt{n})$-approximation algorithm for $L(1,1)$-labelling chordal graphs.

$L(2,1)$**-labelling.** To decide whether $\lambda_{2,1}(G) \leq n$ is NP-complete if G is a chordal graph [42]; this can be proven by means of a reduction from HAMILTONIAN PATH and exploiting the notion of *complement* of a graph.

The $L(2,1)$-labelling for chordal graphs has been first investigated by Sakai [175] in order to approach the general conjecture $\lambda_{2,1}(G) \leq \Delta^2$. The author proves that chordal graphs satisfy the conjecture and more precisely that $\lambda_{2,1}(G) \leq \frac{1}{4}(\Delta+3)^2$. More recently, this bound has been improved by 1 in [75].

Chang and Kuo [27] study upper bounds on $\lambda_{2,1}(G)$ for odd-sun-free-chordal graphs and strongly chordal graphs and prove that $\lambda_{2,1}(G) \leq 2\Delta$ if G is odd-sun-free-chordal and $\lambda_{2,1}(G) \leq \Delta + 2\chi(G) - 2$ if G is strongly chordal. Although a strongly chordal graph is odd-sun-free-chordal, the upper bounds are incomparable. The result on strongly chordal graphs is a generalization of the result that $\lambda_{2,1}(T) \leq \Delta + 2$ for any non trivial tree T. The authors conjecture that $\lambda_{2,1}(G) \leq \Delta + \chi(G)$ for any strongly chordal graph G. All the previous results for chordal graphs have been improved by the result in [176], stating that the $\lambda_{1,1}$- and $\lambda_{2,1}$-numbers are both $O(\Delta^{3/2})$ for this class of graphs, and that there exists a chordal graph G such that $\lambda_{2,1}(G) = \Omega(\Delta^{3/2})$.

Some classes related to chordal graphs have been investigated. In particular, for chordal bipartite graphs it has been proved that $\lambda_{2,1} \leq \Delta^2 - \Delta + 2$ [177], later improved to $\lambda_{2,1} \leq 4\Delta - 1$ [178], hence the Griggs and Yeh's conjecture is true for chordal bipartite graphs with $\Delta \neq 3$. Dually chordal graphs are a superclass of strongly chordal graphs and strongly orderable graphs are a superclass of both strongly chordal and chordal bipartite graphs. In [179], it is shown that $\lambda_{2,1} \leq 2\Delta$ for dually chordal graphs and that $\lambda_{2,1} \leq 4\Delta - 1$ for strongly orderable graphs. Furthermore, as block graphs are strongly chordal, all the results for this latter class holds for the former one, i.e. $\lambda_{2,1} \leq 2\Delta$ and $\lambda_{2,1} \leq \Delta + 2\chi - 2 = \Delta + 2\omega - 2$, as $\chi = \omega$ for a block graph. In [180] these results are improved to $\lambda_{2,1} \leq \Delta + \omega$. Nevertheless, if $\Delta \leq 4$, $\omega = 3$ and G does not contain a certain subgraph with 7 nodes and 9 edges, then $\lambda_{2,1} \leq 6$ [181]. In [177] weakly chordal graphs are proved to respect the Griggs and Yeh's conjecture.

$L(h,k)$**-labelling.** As a generalization of the result known for $h=2$, if G is a chordal graph with maximum degree Δ, then $\lambda_{h,1}(G) \leq \frac{1}{4}(2h+\Delta-1)^2$; if G is an odd-sun-free chordal graph, then $\lambda_{h,1}(G) \leq h\Delta$ and if G is strongly chordal then $\lambda_{h,1}(G) \leq \Delta + (2h-2)(\chi(G)-1)$ [64]. Also for this problem it is proven that $\lambda_{h,k}(G) = O(\Delta^{3/2}(2k-1))$ [176].

4.7.3 Interval Graphs

An *interval graph* is an intersection graph whose model is a set of intervals of the real line.

The class of *unit interval graphs* is a subclass of interval graphs for which all the intervals are of the same length, or equivalently, for which no interval is properly contained within another.

Interval graphs are used to model wireless networks serving narrow surfaces, like highways or valleys confined by natural barriers (e.g. mountains or lakes).

$L(0,1)$**-labelling.** The computation of $\lambda_{0,1}(G)$ is NP-hard for general graphs and also for some special classes of graphs, but it can be computed in polynomial time for interval graphs [182].

$L(1,1)$**-labelling.** Interval and unit-interval graphs are perfect; furthermore, interval graphs are closed under powers and the square of a unit-interval graph is still a unit-interval graph [183]. It follows that the $L(1,1)$-labelling problem on interval and unit-interval graphs is polynomially solvable. A linear time algorithm for finding an optimal $L(1,1)$-labelling of interval graphs is presented in [184].

$L(2,1)$**-labelling.** Sakai [175] proves that $2\chi(G) - 2 \leq \lambda_{2,1}(G) \leq 2\chi(G)$ for unit interval graphs. In [185] the authors discuss some necessary and sufficient conditions for unit interval graphs G to have $\lambda_{2,1}(G) = 2\chi(G) - 2$ and obtain some sufficient conditions for unit interval graphs to have $\lambda_{2,1}(G) = 2\chi(G)$.

In terms of Δ, as $\chi(G) \leq \Delta + 1$, the upper bound becomes $\lambda_{2,1}(G) \leq 2(\Delta + 1)$, and this value is very close to be tight, as the clique K_n, that is an interval graph, has $\lambda_{2,1}(K_n) = 2(n-1) = 2\Delta$.

Finally, in terms of Δ and ω, where ω is the dimension of the larger clique in the graph, $\lambda_{2,1}(G) \leq \Delta + \omega$ [186].

$L(h,k)$**-labelling.** In [184] the authors present a 3-approximate algorithm for $L(h,1)$-labelling interval graphs. In the special case of unit interval graphs, the same approximation ratio holds for the $L(h,k)$-labelling problem.

An $L(h,k)$-labelling algorithm for interval graphs with span at most $\max(h, 2k)\Delta$ is provided in [187]; this span can be slightly improved under some constraints that the graph has to respect. In the same paper, it is proved that the classical greedy algorithm guarantees a span never larger than $\min((2h+2k-2)(\omega-1), \Delta(2k-1) + (\omega-1)(2h-2k))$, where ω is the dimension of the larger clique in the graph.

Open Problem: It is still not known whether the decisional version of the $L(h,k)$-labelling problem is NP-complete on interval graphs or not. Concerning this problem, the feeling of the author is that it is NP-complete, even for unit-interval graphs.

Open Problem: From the results for interval graphs, the authors of [187] deduce a result on *circular arc graphs*, i.e. intersection graphs whose model is a set of intervals in a circle. The approach they follow is: i. to consider a clique in the graph whose remotion gives an interval graph, ii. to label the interval graph, iii. to insert again the clique labeling it with further labels. In this way, it is possible to guarantee $\lambda_{h,k} \leq \min((3h+2k-2)\omega-(2h+2k-2), \Delta(2k-1)+\omega(3h-2k)-(2h-2k)$. The interest of this result is that it is the first one dealing with circular arc graphs, but it should be possible to improve it. An interesting open problem is to provide tight upper and lower bound on $\lambda_{h,k}$ for circular arc graphs. It is worth to be mentioned that Paul, Pal and Pal [186] improve the upper bound to $\Delta+3\omega$ in the special case of $h=2$ and $k=1$.

4.7.4 Permutation Graphs

An intersection model of straight lines between two parallel lines describes *permutation graphs* as follows: let $\mathcal{L}_1, \mathcal{L}_2$ be two parallel lines in the plane and label n points by 1,2, $\dots, n$ (not necessarily in this order) on $\mathcal{L}_1$ as well as on $\mathcal{L}_2$. The straight lines L_i connect i on $\mathcal{L}_1$ with i on $\mathcal{L}_2$. $\mathcal{L}=\{L_1,\dots,L_n\}$ is the intersection model of the corresponding permutation graph.

The name permutation graph comes from the fact that the points on $\mathcal{L}_1, \mathcal{L}_2$ can be seen as a permutation $\pi=\{\pi_1,\dots\pi_n\}$ and $(i,j)\in E(G)$ if and only if i and j form an inversion in π.

$L(0,1)-, L(1,1)$**- and** $L(2,1)$**-labelling.** In [42] it is described an approximation algorithm for $L(h,1)$-labelling ($h=0,1,2$) a permutation graph in $O(n\Delta)$ time; it guarantees the following bounds: $\lambda_{0,1}(G)\leq 2\Delta-2$, $\lambda_{1,1}(G)\leq 3\Delta-2$ and $\lambda_{2,1}(G)\leq 5\Delta-2$. In [188] the result concerning the $L(2,1)$-labelling is improved to $\lambda_{2,1}(G)\leq max\{4\Delta-2, 5\Delta-8\}$ by doing a detailed analysis of Chang and Kuo's heuristic for $L(2,1)$-labelling of general graphs applied to the particualr case of permutation graphs.

$L(h,k)$**-labelling.** For those permutation graphs that are also bipartite, there exists a polynomial $L(h,k)$-labelling approximation algorithm [189] that guarantees to use at most $2h-1+k(\text{bc}(G)-2)$ colors, where $\text{bc}(G)$ is the biclique number of G. (In a bipartite graph, a subset of nodes is a *biclique* if it induces a complete bipartite graph. The *biclique number* of a bipartite graph is the number of nodes in a maximum biclique.) Since $\lambda_{h,k}(G)\geq h+k(\text{bc}(G)-2)$ for any bipartite graph, this algorithm guarantees a number of colors that is at most $h-1$ far from optimal.

4.7.5 Split Graphs

A graph G is a *split graph* if and only if G is the intersection graph of a set of distinct substars of a star. Alternatively, a *split graph* is a graph G of which node set can be split into two sets K and S, such that K induces a clique and S an independent set in G. All split graphs are chordal.

$L(0,1)$**-and** $L(1,1)$**-labelling.** Bodlaender et al. [42] prove that it is NP-complete to decide both whether $\lambda_{0,1}(G)\leq 3$ and whether $\lambda_{1,1}(G)\leq r$ when r is given in input and G is a split graph. This also implies NP-completeness of the problems to decide the $\lambda_{0,1}$- and $\lambda_{1,1}$-numbers for chordal graphs. In [174] it is proven that the $L(1,1)$-labelling problem on split graphs is hard to approximate within a factor of $n^{1/2-\epsilon}$, for any $\epsilon>0$, unless NP-problems have randomized polynomial time algorithms.

$L(2,1)$**-labelling.** As split graphs are chordal, the results stated for chordal graphs hold for these graphs. Moreover, split graphs are the first known class of graph for which $\lambda_{2,1}$ is neither linear nor quadratic in Δ. Namely, in [42] it is presented an algorithm $L(2,1)$-labelling

G with at most $\Delta^{1.5} + 2\Delta + 3$ colors, and it is shown that there exist split graphs for which this bound is tight. Similar bounds are obtained also for $\lambda_{0,1}$ and $\lambda_{1,1}$. The value of $\lambda_{2,1}$ has been improved to $\frac{2\sqrt{3}}{9}\Delta^{1.5} + \Theta(\Delta)$ in [177].

Open Problem: Split graphs and chordal graphs (see Subsection 4.7.2) represent the only known class for which $\lambda_{2,1}$ is neither linear nor quadratic in Δ. It remains an open problem to understand if there exist other classes of graphs whose $\lambda_{2,1}$-number has this property. A characterization of the class of graphs having $\lambda_{2,1} = \Theta(\Delta^{1.5})$ would be a probably hard but very interesting target.

4.8 Hypercubes and Related Networks

The *n-dimensional hypercube* Q_n is an n-regular graph with 2^n nodes, each having a binary label of n bits (from 0 to $2^n - 1$). Two nodes in Q_n are adjacent if and only if their binary labels differ in exactly one position. The n-dimensional hypercube Q_n can also be defined as the cartesian product of n K_2 graphs. The more general cartesian product $K_{n_1} \square K_{n_2} \square \ldots \square K_{n_d}$ of complete graphs is called a *Hamming graph*, where $n_i \geq 2$ for each $i = 1, \ldots, d$.

The *N-input Butterfly network* B_N (with N power of 2) has $N(\log_2 N + 1)$ nodes. The nodes correspond to pairs (i, j), where i $(0 \leq i < N)$ is a binary number and denotes the row of the node, and j $(0 \leq j \leq \log_2 N + 1)$ denotes its column. Two nodes (i, j) and (i', j') are connected by an edge if and only if $j' = j + 1$ and either i and i' are identical (*straight edge*) or i and i' differ in precisely the j'-th bit (*cross edge*). A 3-input butterfly is depicted in Figure 10.a. The N-input butterfly network is strictly related to the hypercube, as its quotient graph, obtained by shrinking each row in a unique node, is exactly the $(\log_2 N)$-dimensional hypercube.

The *n-dimensional Cube-Connected-Cycles* network, CCC_n, is constructed from the n-dimensional hypercube by replacing each node of the hypercube with a cycle of n nodes. The i-th dimension edge incident to a node of the hypercube is connected to the i-th node of the corresponding cycle of the CCC. In Figure 10.b the classical representation of a 3-dimensional CCC network is depicted.

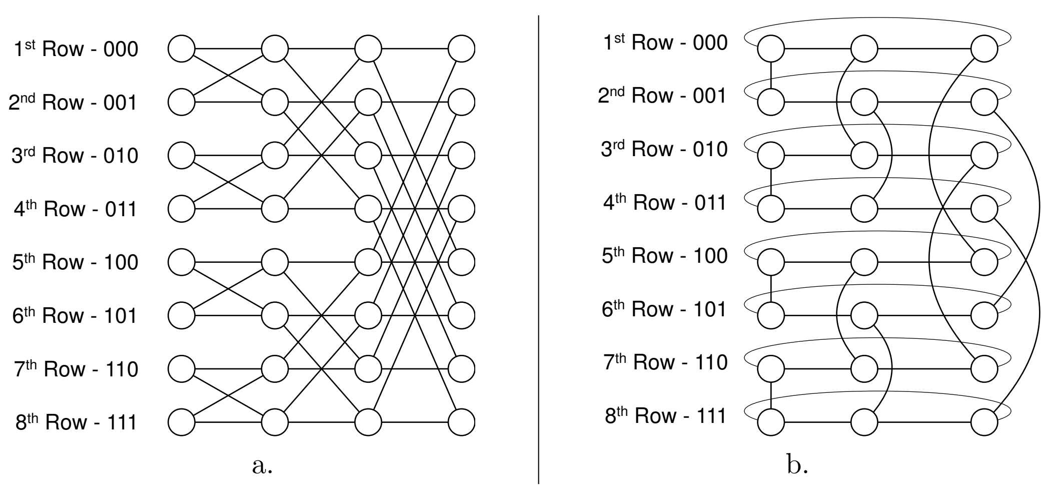

Figure 10: a. The classical representation of an 8-input Butterfly network; b. The bidimensional layout of a 3-dimensional CCC network.)

$L(0,1)$**-labelling.** $\lambda_{0,1}(Q_n) \leq 2^{\lceil \log n \rceil}$ and there exists a labelling scheme using such a number of colors. This labelling is optimal when $n = 2^k$ for some k, and it is a 2-approximation otherwise [140].

$L(1,1)$**-labelling.** In [140] an $L(1,1)$-labelling scheme of the hypercube Q_n is described in the context of optical cluster based networks. A different approach is used in [78] with respect to the problem of data distribution in parallel memory systems: it is presented an algorithm that uses $2^{\lfloor \log n \rfloor + 1}$ colors, and requires $O(n)$ time and space, improving the previously known results. In both papers, the upper bound on $\lambda_{1,1}(Q_n)$ is a 2-approximation, that is conjectured to be the best possible. Finally, in [190], the $\lambda_{1,1}$-number is proved to be exactly n when $n = 2^k - 1$.

$L(2,1)$**-labelling.** For the n-dimensional hypercube Q_n, $\lambda_{2,1}(Q_n) \geq n+3$ and $\lambda_{2,1}(Q_n) \geq n+4$ for $n = 8$ and $n = 16$, respectively [191]. Furthermore, $\lambda_{2,1}(Q_n) \leq 2n+1$ for $n \geq 5$ [4]. The same authors also determine $\lambda_{2,1}(Q_n)$ for $n \leq 5$ and conjecture that the lower bound $n+3$ is the actual value of $\lambda_{2,1}(Q_n)$ for $n \geq 3$. Using a coding theory method, the upper bound is improved by 1 in [85], where it is proven that it ranges from $\lfloor n + 1 + \log_2 n \rfloor$ to $2n$, depending on the value of n. Furthermore, lim inf $\lambda_{2,1}(Q_n)/n = 1$.

In [192] an $L(2,1)$-labelling algorithm of Q_n is described: exploiting a coding theoretic approach, each color is assigned with a $f(n)$-bit binary number, where $f(n)$ =min $\{r$ such that $n + r + 1 \leq 2^r\}$. Therefore, the labelling uses $2^{f(n)}$ colors.

Georges, Mauro and Stein [90] determine the $\lambda_{2,1}$-number of Hamming graph $H(d,n) = K_n \Box K_n \Box \ldots \Box K_n$ (d factors) where $n = p^r$, p prime and either $d \leq p$ and $r \geq 2$, or $d < p$ and $r = 1$. They prove that, under these conditions, $\lambda_{2,1}(H(d,p^r)) = p^{2r} - 1$. Chang, Lu and Zhou [193] extend this result proving that $\lambda_{2,1}(G) = \lambda_{1,1}(G) = n_1 n_2 - 1$ if $G = K_{n_1} \Box K_{n_2}$ and n_1 sufficiently large.

In [194] a constructive algorithm to $L(2,1)$-label multistage interconnection networks in general is presented, then butterflies and CCCs are particularly considered. The authors observe that $\lambda_{2,1}(B_N) \geq 6$ and $\lambda_{2,1}(\mathrm{CCC}_n) \geq 5$ in view of their degree, and they $L(2,1)$-label these networks almost optimally. More precisely, if N is either 2^2 or 2^3, they provide a labelling for B_N using 7 colors, that is optimal; for all greater values of N their method requires 8 colors. For what concerns CCC_n, the authors provide a labelling ensuring $\lambda_{2,1}(\mathrm{CCC}_n) \leq 6$, that is 1 far from optimal, and they experimentally verify that there exist some values of n (e.g. $n = 5$) requiring a 6 colors labelling.

Open Problem: The approach presented in [194], consisting in shrinking some cycles of the networks and in reducing to label these simpler graphs instead of the complete networks, seems to be promising: it is very general both because it can be applied to many multistage networks, and because it works for every value of h and k. Nevertheless, it needs to be refined: first of all, the authors themselves realize that sometimes the reduced graph needs more colors than the whole network, because the reduced graph typically has degree higher than the original network. Secondly, the main bottleneck of this method is that the reduced graph must be labelled using exhaustive methods; nevertheless, it should be relatively easy to design algorithms to efficiently label the reduced graphs, that are constituted by two parallel cycles joined by a very regular set of edges.

$L(h,k)$**-labelling.** Zhou [195] proves that $\lambda_{h,k}(Q_n) \leq 2^s \cdot \max\{k, \lceil h/2 \rceil\} + 2^{s-t} \cdot \min\{h - k, \lfloor h/2 \rfloor\} - h$, where $h \geq k \geq 1$, $s = 1 + \lfloor \log_2 n \rfloor$ and t =min$\{2^s - n - 1, s\}$. In particular, if

$2k \geq h$, then $\lambda_{h,k}(Q_n) \leq 2^s k + 2^{s-t}(h-k) - h$ leading to $\lambda_{2,1}(Q_n) \leq 2^s + 2^{s-t} - 2$. The proof of this theorem gives rise to a systematic way of generating $L(h,k)$-labellings of Q_n which use 2^s labels and have span equal to the right-side of the previous formula. For Hamming graphs, the same authors show an upper bound on $\lambda_{h,k}$ for special values of n_i, $i = 1, \ldots, d$, and this bound is optimal when $h \leq 2k$.

The paper [196] is a survey on the $L(h,k)$-labelling problem with focus on hypercubes and Hamming graphs.

4.9 Other Graphs

In this section there are collected some classes of graphs for which very few results appear in the literature, and they are not enough to justify a devoted section. It is amazing that the $L(h,k)$-labelling problem appears more tricky just for some very studied classes of graphs, included in this collection, that are very relevant from the theoretical point of view and that have many interesting properties.

4.9.1 Diameter 2 Graphs

A *diameter 2 graph* is a graph where all nodes have either distance 1 or 2 each other.

$L(1,1)$- and $L(2,1)$-labelling. Intuitively, diameter 2 graphs seem to be a particularly feasible class to efficiently solve the $L(h,k)$-labelling problem. On the contrary, while it is easy to see that the $\lambda_{1,1}$-number for these graphs is $n-1$, Griggs and Yeh [4] prove that the $L(2,1)$-labelling problem is NP-hard. They prove also that $\lambda_{2,1} \leq \Delta^2$ and state that this upper bound is sharp only when $\Delta = 2, 3, 7$ and, possibly 57 because a diameter 2 graph with $n = \Delta^2 + 1$ can exist only if Δ is one of these numbers (see more details on these graphs below). Since the diameter is 2, all labels in V must be distinct. Hence, $\lambda(G) \geq n - 1 = \Delta^2$ and therefore the equality holds.

$L(h,k)$-labelling. In [73] bounds for the $\lambda_{h,1}$-number are presented, for all $h \geq 2$. In particular, it is proven that $\lambda_{h,1}(G) \leq \Delta^2 + (h-2)\Delta - 1$ if G is a diameter 2 graph with maximum degree $\Delta \geq 3$ and $n \leq \Delta^2 - 1$. Since a diameter 2 graph can have at most $\Delta^2 + 1$ nodes, and, with the exception of C_4, there are no diameter 2 graphs with maximum degree Δ and Δ^2 nodes [200], it just remains to investigate the graphs with $n = \Delta^2 + 1$. There are merely four such graphs; they are regular and have $\Delta = 2, 3, 7$ and 57, respectively:

- $\Delta = 57$: it is neither known whether such a graph exists; this hypothetical graph is called *Aschbacher graph*. Junker [201] strongly conjectures that there is no such a graph, and we will not consider it for obvious reasons;
- $\Delta = 2$: the cycle of length 5: $\lambda_{h,1}(C_5) = 2h$ [65];
- $\Delta = 3$: the Petersen graph P: $\lambda_{2,1}(P) = 9$ [4]; for $h \geq 3$ $\lambda_{h,1}(P) = 3 + 2h$ (see Figure 11) [73];
- $\Delta = 7$: the Hoffman-Singleton graph HS: $\max\{49, 3h\} \leq \lambda_{h,1}(HS) \leq 19 + 3h$ [73] for $h \geq 10$; when $h = 10$ it holds $\lambda_{h,1}(HS) = 49$ that of course is optimal (as $n = 50$ and HS is a diameter 2 graph).

Open Problem: Since for $h \geq h'$ every $L(h,1)$-labelling is also a proper $L(h',1)$-labelling, it holds $\lambda_{h,1}(HS) = 49$ for $\Delta \leq 10$. An interesting open problem is to improve the upper bound for $h > 10$. Another question is whether 10 is the highest value for h such that $\lambda_{h,1} = 49$.

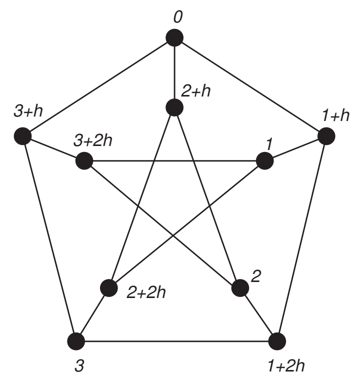

Figure 11: An $L(h,1)$-labelling of Petersen graph.

4.9.2 Regular Graphs

An *r-regular graph* is a graph in which all nodes have degree exactly r. Although they constitute an interesting class of graphs, they have not been very studied from the $L(h,k)$-labelling point of view. Indeed, to the best of the author's knowledge, only few papers deal with regular graphs.

Among all r-regular graphs, particularly important are *snarks*, i.e. 3-regular graphs with chromatic index 4. (By Vizing's theorem, the edge chromatic number of every 3-regular graph is either three or four.) They are investigated since Tait [197] prove that the Four Color theorem is equivalent to the statement that every planar bridgeless 3-regular graph is 3-colorable. The smallest snark is the Petersen graph; the next smallest snark has 18 nodes.

$L(2,1)$**-labelling.** For every $r \geq 3$, it is NP-complete to decide whether an r-regular graph admits an $L(2,1)$-labelling of span (at most) $\lambda_{2,1} = r+2$ [44]. The result is best possible, since no r-regular graph (for $r \geq 2$) allows an $L(2,1)$-labelling of span $r+1$.

In the special case of 3-regular Hamiltonian graphs (consisting of a spanning cycle and a perfect matching), the Griggs and Yeh conjecture ($\lambda_{2,1} \leq \Delta^2$) has been proved [198]. The proof is rather intricate, and requires the study of structural properties of the involved graphs.

Ma, Zhu and He [199] prove that the $\lambda_{2,1}$-number of some families of snarks is 6.

$L(h,k)$**-labelling.** The $\lambda_{h,1}$-number of an r-regular graph is at least $2h+r-2$ [65].

In [202], Georges and Mauro prove that the $\lambda_{h,k}$-number of any r-regular graph G is no less than the $\lambda_{h,k}$-number of the infinite r-regular tree (see Section 4.4). Then, they define a graph G to be (h,k,r)-*optimal* if and only if the equality holds, they consider the structure of (h,k,r)-optimal graphs for $h/k > r$ and show that (h,k,r)-optimal graph are bipartite with a certain edge-coloring property. Finally, the same authors determine the exact $\lambda_{1,1}$- and $\lambda_{2,1}$-numbers of prisms. More precisely, for $n \geq 3$, the *n-prism* $P_r(n)$ is the graph consisting of two disjoint n-cycles $v_0, v_1, \ldots, v_{n-1}$ and $w_0, w_1, \ldots, w_{n-1}$ and edges $\{v_i, w_i\}$ for $0 \leq i \leq n-1$.

Observe that $P_r(n)$ is isomorphic to $C_n \Box P_2$. In [122] it is proven that $\lambda_{2,1}(P_r(n))$ is equal to 5 if $n \equiv 0 \bmod 3$ and to 6 otherwise, improving the result in [86], and that

$$\lambda_{1,1}(P_r(n)) = \begin{cases} 3 & \text{if } n \equiv 0 \bmod 4 \\ 5 & \text{if } n = 3, 6 \\ 4 & \text{otherwise.} \end{cases}$$

Open Problem: Regular graphs seem to be particularly relevant for the $L(h,k)$-labelling questions (notice that most of the graphs shown as extremal cases are regular graphs), so they are worth being studied more deeply.

4.9.3 Bipartite Graphs

Bipartite graphs are graphs with $\chi(G) \leq 2$. Nevertheless, their $\lambda_{h,k}$-number can be very large, as shown in the following.

Before detailing the known results for bipartite graphs, we recall some definitions. A bipartite graph $G = (U \cup V, E)$, with $|U| = n_1$ and $|V| = n_2$, is called a *chain graph* if there exists an ordering $u_1, \ldots, u_{n_1}$ of U and an ordering $v_1, \ldots, v_{n_2}$ of V such that $N(u_1) \subseteq \ldots \subseteq N(u_{n_1})$ and $N(v_1) \subseteq \ldots \subseteq N(v_{n_2})$, where $N(x)$ is the set of all adjacent nodes of x. A subset of nodes of a bipartite graph is a *biclique* if it induces a complete bipartite subgraph; the maximum order of a biclique of G is denoted by $bc(G)$.

$L(0,1)$**-labelling.** Bipartite graphs may require $\lambda_{0,1} = \Omega(\Delta^2)$, indeed there exist bipartite graphs with $\lambda_{0,1} \geq \frac{\Delta^2}{4}$ [42]. Of course the same bound holds for each $\lambda_{h,1}$, $h \geq 1$. Later this lower bound has been improved by a constant factor of $\frac{1}{4}$ in [53].

$L(1,1)$**-labelling.** The decision version of the $L(1,1)$-labelling problem is NP-complete even for 3-regular bipartite graphs using 4 colors [203]. In [174] it is proven that the $L(1,1)$-labelling problem on bipartite graphs is hard to approximate within a factor of $n^{1/2-\epsilon}$, for any $\epsilon > 0$, unless NP-problems have randomized polynomial time algorithms.

$L(2,1)$**-labelling.** Since the general upper bound $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 2$ [57] holds also for bipartite graphs of any degree Δ and the lower bound on $\lambda_{0,1}$ holds a fortiori for $\lambda_{2,1}$, it follows that $\lambda_{2,1}(G) = \Theta(\Delta^2)$ for this class of graphs.

In [42], the authors prove that the decisional version of the $L(2,1)$-labelling problem is NP-complete for planar bipartite graphs.

For the subclass of chain graphs, the $L(2,1)$-labelling problem can be optimally solved in linear time and $\lambda_{2,1}(G) = bc(G)$ [178]. Furthermore, $\lambda_{2,1}(G) \leq 4\Delta - 1$ for a chordal bipartite graph G [178]. In the same paper, the $L(2,1)$-labelling problem of several subclasses of bipartite graphs is studied, such as *bipartite distance hereditary graphs* and *perfect elimination bipartite graphs.*

$L(h,k)$**-labelling.** In [67] the $L(h,k)$-labelling problem is considered even on bipartite graphs, and it is proven that the simplest approximation algorithm, i.e. the one based on First Fit strategy, guarantees a performance ratio of $O(\min(2\Delta, \sqrt{n}))$, and this is tight within a constant factor in view of the $n^{1/2-\epsilon}$-hardness result. On the contrary, exact results can be achieved if the bipartite graphs are complete. Indeed, in the special case of $k = 1$, given a complete bipartite graph $G = (U \cup V, E)$, where $|U| = n_1$ and $|V| = n_2$, $n_1 \geq n_2$ [204]:

$$\lambda_{h,1}(G) = \begin{cases} \max(n_1 - 1, n_2 - 1 + h) \\ \qquad\qquad \text{if } 0 \leq h \leq \frac{1}{2} \\ (2n_2 - 1)h + \max(n_1 - n_2 - 1 + h, 0) \\ \qquad\qquad \text{if } \frac{1}{2} \leq h \leq 1 \\ h + n_1 + n_2 - 2 \\ \qquad\qquad \text{if } h \geq 1 \end{cases}$$

If the complete bipartite graph is the star, the value of the minimum span is [123]:

$$\lambda_{h,k}(G) = \begin{cases} (\Delta-1)k & \text{if } h \leq \frac{k}{2} \\ (\Delta-2)k+2h & \text{if } \frac{k}{2} \leq h \leq k \\ (\Delta-1)k+h & \text{if } h \geq k \end{cases}$$

and, of course, these latter values match with the previous ones in the special case $n_1 = \Delta$, $n_2 = 1$ and $k = 1$.

4.9.4 Cayley Graphs

Cayley graphs of the group Γ relative to the finite group generating set S is the labeled directed graph $G = (V, E)$ for which $V = \Gamma$ and $E = \{(u, u_s) : u \in V, s \in S\}$, where the edge (u, u_s) is labeled s. In other words, there is an edge labeled s between two nodes of G if one is obtained from the other through right multiplication by s. Note that if $|S| = n$, then the undirected graph underlying the Cayley graph G is $2n$-regular if for all $s, s' \in S$, $ss' \neq 1$, hence the upper bounds on regular graphs hold for Cayley graphs, too.

Some authors study the $L(h, k)$-labelling problem on this class of graphs, when varying the group Γ. The reader may refer to [205, 195] for the $L(2, 1)$-labelling of Cayley graphs on abelian groups, and to [206] for Cayley graphs on more general groups; the $L(2, 1)$-labelling of cubic Cayley graphs on dihedral group is investigated in [207]. Some of these results contain as a special case the $L(2, 1)$-labelling of the square grid and of the hypercube network.

4.9.5 Unigraphs

Unigraphs are graphs uniquely determined by their own degree sequence up to isomorphism and are a superclass including *matrogenic*, *matroidal*, *split matrogenic* and *threshold graphs*. In this section we will deal with all these classes of graphs. The interested reader can find further information related to these classes of graphs in [208].

An *antimatching of dimension* h of X onto Y is a set A of edges such that $M(A) = X \times Y - A$ is a perfect matching of dimension h of X onto Y. A graph $G = (\{v_1, v_2, \ldots v_k\}, \emptyset)$ is a *null graph* if its edge set is empty, irrespective of the dimension of the node set.

A *split* graph G with clique K and stable set S is *matrogenic* (Fig. 12.a) if and only if the edges of G can be colored red and black so that [209]:

a. The red subgraph is the union of vertex-disjoint pieces, $C_i, i = 1, ..., z$. Each piece is either a null graph N_j, belonging either to K or to S; or matching M_r of dimension h_r of $K_r \subseteq V_K$ onto $S_r \subseteq V_S, r = 1, \ldots \mu$; or antimatching A_t of dimension h_t of $K_t \subseteq V_K$ onto $S_t \subseteq V_S, t = 1, \ldots \alpha$ (Fig. 12.b).
b. The linear ordering $C_1, ...C_z$ is such that each node in V_K belonging to C_i is not linked to any node in V_S belonging to C_j , $j = 1, ..., i-1$, but is linked by a black edge to every node in V_S belonging to C_j, $j = i+1, ..., z$. Furthermore, any two nodes in V_K are linked by a black edge (Fig. 12.c).

A graph is *matrogenic* [210] if and only if its node set V can be partitioned into three disjoint sets V_K, V_S, and V_C such that:

a. $V_K \cup V_S$ induces a split matrogenic graph in which K is the clique and S the stable set;
b. V_C induces a *crown*, i.e. either a *perfect matching* or a *h-hyperoctahedron* (that is the complement of a perfect matching of dimension h – or a *chordless* C_5;

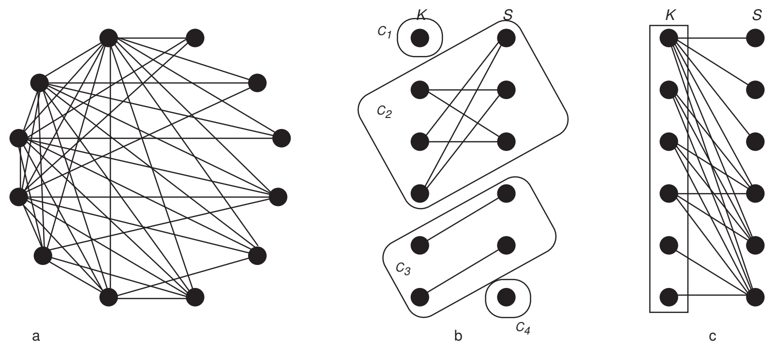

Figure 12: a. A split matrogenic graph; b. its red graph; c. its black graph.

c. every node in V_C is adjacent to every node in V_K and to no node in V_S.

Observe that *split matrogenic graphs* are matrogenic graphs in which $V_C = \emptyset$.

A result in [210] is that a graph $G = (V, E)$ is *matrogenic* if and only if it does not contain the configuration in Fig. 13.a. A graph $G = (V, E)$ is *matroidal* if and only if it contains neither the configuration in Fig. 13.a nor a chordless C_5 [211].

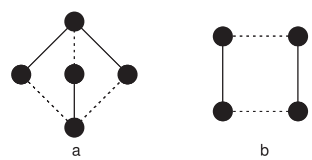

Figure 13: The forbidden configuration of: a. a matrogenic graph and b. a threshold graph: — shows a present edge, - - - shows an absent edge.

The *vicinal preorder* $\preceq$ on $V(G)$ is defined as follows: $x \preceq y$ iff $N(x) - y \subseteq N(y) - x$, where $N(x)$ is the set of x's adjacent nodes. A graph G is a *threshold graph* if and only if G is a split graph and the vicinal preorder on $V(G)$ is total, i.e. for any pair $x, y \in V(G)$, either $x \preceq y$ or $y \preceq x$. G is *threshold* if and only if it does not contains the configuration in Fig. 13.b.

$L(2,1)$**-labelling.** A linear time algorithm for $L(2,1)$-labelling matrogenic graphs is provided in [212]. Upper bounds for the specific subclasses defined above are proved. In particular, in the special case of *threshold graphs* an optimal $L(2,1)$-labelling is provided with $\lambda_{2,1} \leq 2\Delta + 1$ (the exact values depends on the graph). The optimal algorithm for threshold graphs matches the polynomiality result of Chang and Kuo on cographs [27], as threshold graphs are a subclass of cographs.

For the more general class of unigraphs, in [213] a 3/2-approximate algorithm for $L(2,1)$-labeling this class of graphs is proposed. This algorithm runs in $O(n)$ time, improving the time of the algorithm based on the greedy technique, requiring $O(m)$ time, that may be near to $\Theta(n^2)$ for unigraphs.

Open Problem: It is still not known if the $L(2,1)$-labelling problem is NP-hard for unigraphs and matrogenic graphs or not. Furthermore, the cited results are the only ones present in the literature concerning these graphs, so it is probably possible to refine the algorithm in order to improve the upper bound on $\lambda_{2,1}$.

4.9.6 q-Inductive Graphs

Let q be a positive integer. A class of graphs $\mathcal{G}$ is *q-inductive* if for every $G \in \mathcal{G}$, the nodes of G can be assigned distinct integers in such a way that each node is adjacent to at most q higher numbered nodes.

Several well known classes of graphs belong the q-inductive class for appropriate values of q. For example, trees are 1-inductive, outerplanar graphs are 2-inductive, planar graphs are 5-inductive, chordal graphs with maximum clique size ω are $(\omega - 1)$-inductive and graphs of treewidth t are t-inductive.

$L(1,1)$-labelling. In [137] it is presented an approximation algorithm for $L(1,1)$-labelling q-inductive graphs having performance ratio at most $2q-1$. The running time of this algorithm is $O(nq\Delta)$.

$L(h,k)$-labelling. Halldórson [67] applies his greedy algorithm for bipartite graphs to q-inductive graphs, achieving a performance ratio of at most $2q - 1$, hence generalizing the result for $L(1,1)$-labelling to all values of h and k.

Open Problem: Observe that for outerplanar and planar graphs the bound of $2q-1$ is rather far from optimum (see Section 4.6), so probably the cited algorithm can be improved in order to guarantee a better performance ratio for all values of q.

4.9.7 Generalized Petersen Graphs

For $n \geq 3$, a 3-regular graph G with $n = 2N$ nodes is a *generalized Petersen graph of order N* if and only if G consists of two disjoint N-cycles, called inner and outer cycles, such that each node on the outer cycle is adjacent to a node on the inner cycle (see Figure 14). In applications involving networks, one seeks to find a balance between network connectivity, efficiency, and reliability. The double-cycle structure of the generalized Petersen graphs is appealing for such applications since it is superior to a tree or cycle structure as it ensures network connectivity in case of any two independent node/connection failures while keeping the number of connections at a minimum level.

$L(2,1)$-labelling. The $\lambda_{2,1}$-number of every generalized Petersen graph is bounded from above by 8, with the exception of the Petersen graph itself, having $\lambda_{2,1}$-number equal to 9. This bound can be improved to 7 for all generalized Petersen graphs of order $N \leq 6$ [214]. The authors conjecture that the Petersen graph is the only connected 3-regular graph with $\lambda_{2,1}$-number 9 and that there are neither generalized Petersen graphs nor 3-regular graphs with $\lambda_{2,1}$-number 8, i.e. 7 is an upper bound also for generalized Petersen graphs of order greater than 6. In [215, 216] the authors prove that this conjecture is true for orders 7 and 8, and give exact $\lambda_{2,1}$-numbers for all generalized Petersen graphs of orders 5, 7 and 8, thereby closing all cases with orders up to 8. Finally, in [217] the exact $\lambda_{2,1}$-numbers for all generalized Petersen graphs of orders 9, 10, 11 and 12 are given, thereby closing all open cases up to order $N = 12$ and lowering the upper bound on $\lambda_{2,1}$ down to 6 for all but three graphs of these orders.

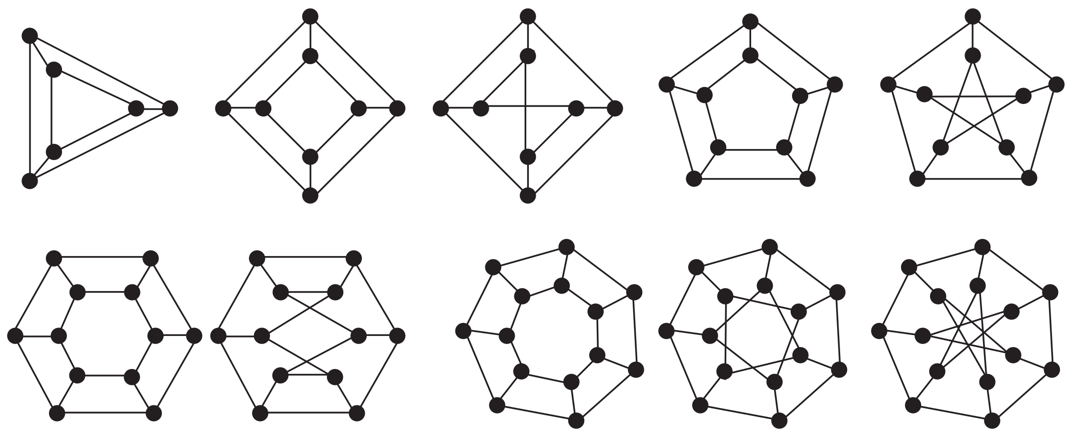

Figure 14: Some generalized Petersen graphs for all values of n from 3 to 7.

Open Problem: It is not known whether there is a generalized Petersen graph of order greater than 11 with $\lambda_{2,1} \geq 7$. In [217] seven generalized Petersen graphs of order at most 11 having $\lambda_{2,1} \geq 7$ are shown. If there is no such a graph for any order greater than 11, then $\lambda_{2,1}$ for generalized Petersen graphs of order greater than 6 would be at most 6, lower than the upper bound of 7 conjectured by Georges and Mauro [214].

Open Problem: Both to solve the mentioned conjecture and to increase the order N for which it is known $\lambda_{2,1}$would be interesting issues, although such results would have a graph theoretic flavor, more than algorithmic.

In [216, 217] some subclasses of generalized Petersen graphs, particularly symmetric, are considered. The authors provide the exact $\lambda_{2,1}$-numbers of such graphs, for any order.

4.9.8 Comparability and Co-Comparability Graphs

A graph is a *comparability graph* if and only if there exists an order of its nodes $v_0 < v_1 < \ldots < v_{n-1}$ such that for each $i < j < l$, if (v_i, v_j) is an edge and (v_j, v_l) is an edge, then (v_i, v_l) is an edge.

Comparability graphs are a very interesting and wide class: they are perfect graphs and include bipartite, chordal, permutation, threshold graphs and cographs.

The class of *co-comparability graphs* contains all graphs that are the complement of a comparability graph. From the definition of comparability graph, if G is a co-comparability graph, then there exists an ordering of the nodes set such that, if $v_i < v_j < v_l$ and $(v_i, v_l) \in E$ then either $(v_i, v_j) \in E$ or $(v_j, v_l) \in E$.

Co-comparability graphs are also perfect graphs and include interval and permutation graphs.

$L(1,1)$**-labelling.** As the square of a comparability graph G is G itself and co-comparability graphs are closed under powers [218], in view of the fact that both comparability and co-comparability graphs are perfect, it easily follows that the $L(1,1)$-labeling problem is polynomially solvable on these classes of graphs.

$L(h,k)$**-labelling.** A co-comparability graph can be $L(h,k)$-labeled with span at most $\max(h,2k)2\Delta+k$ [187]. This result is obtained exploiting the linear order of the nodes of co-comparability graphs and some considerations on the degree of nodes based on the property of their edges.

Open Problem: Comparability and co-comparability graphs are very interesting graphs containing many classes, so they deserve to be better investigated; above all, it would be interesting to understand whether the $L(2,1)$-labelling problem is still polynomially solvable or not. Observe that interval graphs lie in the intersection between comparability and co-comparability graphs, so a complexity result for this class would imply results on its superclasses.

4.9.9 Kneser Graphs

Kneser graphs are an important graph class which has been extensively studied in the context of coloring problems. In particular, they have been introduced by Lovász in 1978 to prove Kneser's conjecture [219].

Given two positive integers n and k, the *Kneser graph* $K(n,k)$ is the graph whose nodes represent the k-subsets of $\{1,2,\ldots,n\}$ and where two nodes are connected if and only if they correspond to disjoint subsets. Observe that $K(5,2)$ is the Petersen graph, $K(n,1)$ is the complete graph K_n, $K(n,k)$ has no edges when $k<2n$ and $K(n,k)$ is a matching when $n=2k$. For this reason, it only makes sense to consider the case when $k>1$ and $n>2k$.

$L(2,1)$**-labelling.** Kang [220] proves that for $K(2k+1,k)$ it holds $\lambda_{2,1}\le 4k+2$ providing an $L(2,1)$-labelling obtained from a classification of structures between and within the color classes of a special node coloring. This coloring is nearly optimal for the Petersen graph.

Shao, Solis-Oba and Lin [221] study the $L(2,1)$-labelling of Kneser graphs providing an upper bound of $\binom{n}{k}-1$, that is tight for $n\ge 3k-1$.

Open Problem: From a combinatorial point of view, the $L(h,k)$-labelling on this class of graphs deserves to be further investigated.

4.9.10 Total Graphs

The total graph $T(G)$ of a graph G is the graph whose nodes correspond to the nodes and edges of G, and whose two nodes are joint if and only if the corresponding nodes are adjacent, edges are adjacent or nodes and edges are incident in G. Observe that $T(G)$ is isomorphic to the graph found by replacing each edge by a path of length 2.

$L(2,1)$**-labelling.** In [222] it is shown that $\lambda_{2,1}(T(G))\le\max\{\frac{3}{4}\Delta^2+\frac{1}{2}\Delta,\frac{1}{2}\Delta^2+2\Delta\}$. In [223] this bound is improved to $\frac{1}{2}\Delta^2+\Delta$, which shows that the conjecture of Griggs and Yeh is true for total graphs.

4.9.11 Sierpiński Graphs

The nodes of a *Sierpiński graph* $S(n,k)$, $n,k\ge 1$, are labelled with strings in $\{0,\ldots,k-1\}^n$; two different nodes $u=(u_1,\ldots,u_n)$ and $v=(v_1,\ldots v_n)$ are adjacent if and only if there exists an index h such that:
(i) $u_t=v_t$ for $t=1,\ldots,h-1$;

(ii) $u_h \neq v_h$ and
(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h+1, \ldots, n$.

The graphs $S(2,3)$ and $S(3,3)$ are shown in Figure 15.

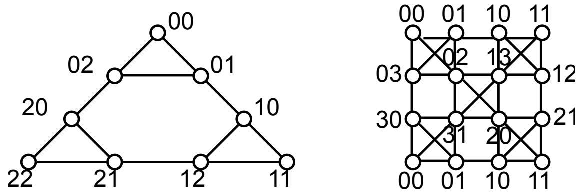

Figure 15: $S(2,3)$ and $S(3,3)$.

$L(2,1)$**-labelling.** Gravier, Klavžar and Mollard [224] prove that for any $n \geq 2$ and any $k \geq 3$, $\lambda_{2,1}(S(n,k)) = 2k$.

Acknowledgments

I am keeping a continuously updated version of this paper at the web page www.dsi.uniroma1.it/$\sim$calamo/survey.htm, so I would appreciate to receive news about new results in order to update this collection of references. Every note highlighting missing results, possible errors and further references is wellcome.

I would like to thank the anonymous referees that have carefully read both the first and the updated version of this survey and annotated bibliography: their comments and suggestions have been very useful. Furthermore, a special appreciation goes to all those people that have sent me their papers either unpublished or still in press and those that have contributed to enrich the web page associated to the paper: each of their contributions has been precious and thanks to all of them I could collect so many references and carry on this work.

References

[1] Roberts, F.S. (1991) From Garbage to Rainbows: Generalizations of Graph Coloring and their Applications. In Alavi, Y., Chartrand, G., Oellermann, O.R. , and Schwenk, A.J. (eds.), *Graph Theory, Combinatorics, and Applications*. Wiley, New York.

[2] Jensen, T.R. and Toft, B. (1995) *Graph coloring problems*. John Wiley & Sons, New York.

[3] Hale, W. K. (1980) Frequency assignment: theory and applications. *Proceedings of IEEE*, **68**, 1497–1514.

[4] Griggs, J.R. and Yeh,R.K. (1992) Labeling graphs with a Condition at Distance 2. *SIAM Journal on Discrete Mathematics*, **5**, 586–595.

[5] Yeh, R.K. (1990) *Labeling Graphs with a Condition at Distance Two.* Ph.D. Thesis, University of South Carolina, Columbia, South Carolina.

[6] Aardal, K.I. , van Hoesel, S.P.M., Koster, A.M.C.A., Mannino, C. and Sassano, A. (2001) Models and Solution Techniques for Frequency Assignment Problems. ZIB-Report 01-40, Konrad-Zuse-Zentrum fur Informationstechnik Berlin.

[7] Eisenblätter, A. , Grötschel, M. and Koster, A.M.C.A. (2000) Frequency Planning and Ramifications of Colorings. Discussione Mathematicae, Graph Theory, **22**, 51–88.

[8] Koster, A.M.C.A. (1999) *Frequency Assignment*. Ph.D. thesis, Universiteit Maastricht.

[9] Murphey, R.A., Pardalos, P.M. and Resende, M.G.C. (1999) Frequency Assignment Problems. In Du, D.-Z. and Pardalos, P.M. (eds.), *Handbook of Combinatorial Optimization*, 295–377. Kluwer Academic Publishers.

[10] Roberts, F.S. (1991) T-Colorings of graphs: recent results and open problems. *Discrete Mathematics*, **93**, 229–245.

[11] Wegner, G. (1977) Graphs with given diameter and a coloring problem. Tech. Rep. University of Dortmund, Dortmund.

[12] Bertossi, A.A. and Bonuccelli, M.A. (1995) Code Assignment for Hidden Terminal Interference Avoidance in Multihop Packet Radio Networks. *IEEE/ACM Trans. on Networking*, **3(4)**, 441–449.

[13] Aly, K.A. and Dowd, P.W. (1994) A Class of Scalable Optical Interconnection Networks through Discrete Broadcast-Select Multi-Domain WDM. *Proceedings of IEEE INFOCOM*, Toronto, Ontario, Canada, 12–16 June, pp. 392–399, IEEE Computer Society Press.

[14] Shepherd, M. (1998) *Radio Channel Assignment*. Ph.D. thesis, Merton College, Oxford.

[15] Duque-Anton, M. , Kunz, D. and Rüber, B. (1993) Channel assignment for cellular radio using simulated annealing. *IEEE Transaction on Vehicular Technology*, **42**, 14–21.

[16] Mathar, R. and Mattfeld, J. (1993) Channel assignment in cellular radio networks. *IEEE Trans. on Vehicular Technology*, **42**, 647–656.

[17] Crompton, W. , Hurley, S. and Stephen, N.M. (1994) A parallel genetic algorithm for frequency assignment problems. *Proceedings of IMACS/IEEE International Symp. on Signal Processing, Robotics and Neural Networks (SPRANN '94)*, Lille, France, April, pp. 81–84.

[18] Lai, W.K. and Coghill, G.G. (1996) Channel assignment through evolutionary optimization. *IEEE Trans. on Vehicular Technology*, **45**, 91–96.

[19] Castelino, D. , Hurley, S. and Stephens, N.M. (1996) A tabu search algorithm for frequency assignment. *Annals of Operations Research*, **63**, 301–319.

[20] Funabiki, N. and Takefuji, Y. (1992) A neural network parallel algorithm for channel assignment problem in cellular radio networks. *IEEE Transaction on Vehicular Technology*, **41**, 430–437.

[21] Kunz, D. (1991) Channel assignment for cellular radio using neural networks. *IEEE Trans. on Vehicular Technology*, **40**, 188–193.

[22] Roberts, F.S. (1993) No-hole 2-distance colorings. *Mathematical and Computer Modelling*, **17**, 139–144.

[23] van den Heuvel, J., Leese, R.A. and Shepherd, M.A. (1998) Graph Labelling and Radio Channel Assignment. *Journal of Graph Theory*, **29**, 263–283.

[24] Griggs, J.R. (2006) Real Number Channel Assignments with Distance Conditions. *SIAM Journal on Discrete Mathematics*, **20(2)**, 302–327.

[25] Fiala, J. and Kratochvíl, J. (2001) Complexity of Partial Covers of Graphs. *Proc. of Int.l Symp. on Algorithms and Computation (ISAAC '01)*, Christchurch, New Zeland, 19–21 December, pp. 537–549, Lectures Notes in Computer Science 2223, Springer Verlag, Berlin.

[26] Fiala, J., Kratochvíl, J. and Proskurowski, A. (2001) Distance Constrained Labeling of Precolored Trees. *Proceedings of 7th Italian Conf. on Theoretical Computer Science (ICTCS '01)*, Torino, Italy, 4–6 October, pp. 285–292, Lectures Notes in Computer Science 2202, Springer Verlag, Berlin.

[27] Chang, G.J. and Kuo, D. (1996) The $L(2,1)$-labeling Problem on Graphs. *SIAM Journal on Discrete Mathematics*, **9**, 309–316.

[28] Chang, G.J., Chen, J.-J. , Kuo, D. and Liaw, S.-C. (2007) Distance-two labelings of digraphs. *Discrete Applied Mathematics*, **155(8)**, 1007–1013.

[29] Yeh, R.K. and Li, C.-S. (1997) The edge span of the distance two labelling of graphs. *Taiwanese Journal of Mathematics*, **4(4)**, 675–684.

[30] Lih, K.W. (1999) The Equitable Coloring of Graphs. In In Du, D.-Z. and Pardalos, P.M. (eds.), *Handbook of Combinatorial Optimization*, 543–566.

[31] Yeh, R.K. (2006) A Survey on Labeling Graphs with a Condition at Distance Two. *Discrete Mathematics*, **306**, 1217–1231.

[32] Calamoneri, T. (2006) The $L(h,k)$-Labelling Problem: a Survey and Annotated Bibliography. *the Computer Journal*, **49(5)**, 585–608.

[33] Brandstadt, A., Le, V.B. and Spinrad, J.P. (1999) *Graph Classes: A Survey*. SIAM Monographs on Discrete Mathematics and Applications, Philadelphia.

[34] Fotakis, D.A., Nikoletseas, S.E., Papadopoulou, V.G. and Spirakis, P.G. (2005) Radiocoloring in Planar Graphs: Complexity and Approximations. *Theoretical Computer Science*, **340**, 514–538.

[35] McCormick, S.T. (1983) Optimal approximation of sparse Hessians and its equivalence to a graph coloring problem. *Mathematical Programming* ,**26**, 153–171.

[36] Sen, A. and Huson, M.L. (1997) A new model for scheduling packet radio networks. *Wireless Networks*, **3**, 71–82.

[37] Ramanathan, S. and Lloyd, E.L. (1992) The complexity of distance-2-coloring. *Proceedings of 4th International Conference of Computing and Information (ICCI '92)*, Toronto, Ontario, 28-30 May, pp. 71–74. IEEE Computer Society Press.

[38] Ramanathan, S. (1993) *Scheduling algorithms for multi-hop radio networks.* Ph.D. thesis; Dept. of Computer Science, University of Delaware, Newark.

[39] Heggernes, P. and Telle, J.A. (1998) Partitioning graphs into generalized dominating sets. *Nordic J. Computing*, **5(2)**, 128–143.

[40] Ramanathan, S. and Lloyd, E.L. (1993) Scheduling algorithms for multi-hop radio networks. *IEEE/ACM Transactions on Networking*, **1**, 166–172.

[41] Fiala, J., Kloks, T. and Kratochvíl, J. (2001) Fixed-parameter Complexity of λ-Labelings. *Discrete Applied Mathematics*, **113(1)**, 59–72.

[42] Bodlaender, H.L., Kloks, T., Tan, R.B. and van Leeuwen, J. (2004) Approximations for λ-Colorings of Graphs. *the Computer Journal*, **47**, 193-204.

[43] Eggemann, N., Havet, F. and Noble, S.D. (2010) $k - L(2,1)$-Labelling for planar graphs is NP-complete for $k \geq 4$. *Discrete Applied Mathematics*, **158(16)**, 1777–1788.

[44] Fiala, J. and Kratochvíl, J. (2005) On the computational complexity of the $L(2,1)$-labeling problem for regular graphs. *Proceedings of 11th Italian Conf. on Theoretical Computer Science (ICTCS '05)*, Siena, Italy, 12–14 October, pp. 228–236, Lectures Notes in Computer Science 3701, Springer Verlag, Berlin.

[45] Klavzar, M., Kratochvíl, J., Kratsch, D., and Liedloff, M. (2011) Exact Algorithms for $L(2,1)$-Labelings of Graphs. *Proc. of 32nd Int.l Symp. on Mathematical Foundations of Computer Science (MFCS '07)*, Cesky Krumlov, Czech Republic, 27–31 August, pp. 513–524, Lecture Notes in Computer Science 4708, Springer-Verlag, Berlin.

[46] Král', D. (2006) Channel assignment problem with variable weights. *SIAM Journal on Discrete Mathematics*, **20**, 690–704.

[47] Havet, F., Klavzar, M., Kratochvíl, J., Kratsch, D., and Liedloff, M. (2011) Exact Algorithms for $L(2,1)$-Labelings of Graphs. *Algorithmica*, **59**, pp. 169194..

[48] Junosza-Szaniawski, K., and Rzążewski, P. (2011) On the Complexity of Exact Algorithms for $L(2,1)$-Labeling of Graphs. *Information Processing Letters*, **111(14)**, 697–701.

[49] Junosza-Szaniawski, K. ,Kratochvíl, J., Liedlo, M., Rossmanith, P., and Rzążewski, P. (2013)Fast Exact Algorithm for $L(2,1)$-Labeling of Graphs. *Theoretical Computer Science*, **505**, 42–54.

[50] Junosza-Szaniawski, K., Kratochvil, J., Liedloff, M. and Rzążewski, R. (2013) Determining the $L(2,1)$-Span in Polynomial Space. *Discrete Applied Mathematics*, **161(13-14)**, 2052–2061.

[51] Janczewski, R., Kosowski, A. and Malafiejski, M. (2009) The complexity of the $L(p,q)$-labeling problem for bipartitete planar graphs of small degree. *Discrete Mathematics*, **309**, 3270–3279.

[52] Jin, X.T. and Yeh, R.K. (2004) Graph distance-dependent labeling related to code assignment in computer networks. *Naval Research Logistics*, **51**, 159–164.

[53] Alon, N. and Mohar, B. (2002) The chromatic number of graph powers. *Combinatorics, Probability and Computing*, **11**, 1–10.

[54] Nikoletseas, S., Papadopoulou, V. and Spirakis, P. (2003) Radiocoloring Graphs via the Probabilistic Method. *4th Panhellenic Logic Symp.*, Thessalonik, Greece, 7–10 July, pp. 135–140.

[55] Jonas, K. (1993) *Graph Coloring Analogues With a Condition at Distance Two: $L(2,1)$-Labelings and List λ-Labelings*. Ph.D. thesis, University of South Carolina, Columbia.

[56] Král', D. and Škrekovski, R. (2003) A Theorem about the Channel Assignment. *SIAM Journal on Discrete Mathematics*, **16(3)**, 426–437.

[57] Gonçalves, D. (2008) On the $L(p,1)$-labeling of graphs. *Discrete Mathematics*, **308**, 1405–1414.

[58] Havet, F., Reed, B. and Sereni, J.-S. (2008) $L(2,1)$-Labeling of graphs. *Proceedings of the ACM-SIAM Symposium on Discrete Algorithm (SODA '08)*, San Francisco, California, 20-22 January, 621–630.

[59] Havet, F., Reed, B. and Sereni, J.-S. (2012) Griggs and Yeh's Conjecture and $L(p,1)$-Labelings. *SIAM Journal of Discrete Mathematics*, **26**, 145–168.

[60] Georges, J.P. , Mauro, D.W. and Whittlesey, M.A. (1994) Relating path coverings to vertex labellings with a condition at distance two. *Discrete Mathematics*, **135**, 103–111.

[61] Balakrishnan, H. and Deo, N. (2003) Parallel Algorithm for Radiocoloring a Graph. *Congressus Numerantium*, **160**, 193–204.

[62] Nordhaus, E.A. and Gaddun, J.W. (1956) On Complementary Graphs. *American Mathematical Monthly*, **63**, 175–177.

[63] Chang, G.J. and Lu, C. (2003) Distance-two labelings of Graphs. *European Journal of Combinatorics*, **24(1)**, 53–58.

[64] Chang, G.J., Ke, W.-T., Kuo, D., Liu, D.D.-F. and Yeh, R.K. (2000) On $L(d,1)$-labelings of graphs. *Discrete Mathematics*, **220(1-3)**, 57–66.

[65] Georges, J.P. and Mauro, D.W. (1995) Generalized vertex labeling with a condition at distance two. *Congressus Numerantium*, **109**, 141–159.

[66] Calamoneri, T. and Vocca, P. (2004) Approximability of the $L(h,k)$-Labelling Problem. *Proceedings of 12th Colloquium on Structural Information and Communication Complexity (SIROCCO 2005)*, Le Mont Saint-Michel, France, 24–26 May, pp. 65–77, Lecture Notes in Computer Science 3499, Springer Verlag, Berlin.

[67] Halldórsson, M.M. (2006) Approximating the $L(h,k)$-labelling problem. *International Journal of Mobile Network Design and Innovation*, **1(2)**, 113–117.

[68] Halldórsson, M.M. (1993) A still better performance guarantee for approximate graph coloring. *Information Processing Letters*, bf 45, 19-23.

[69] Makansi, T. (1987) Transmitter-Oriented Code Assignment for Multihop Packet Radio. *IEEE Transaction on Communication*, **35(12)**, 1379–1382.

[70] Khan, N, Pal, M. and Pal, A. (2012) *Communications and Network*, **4**, 18–29.

[71] Battiti, R., Bertossi, A.A. and Bonuccelli, M.A. (1999) Assigning Codes in Wireless Networks: Bounds and Scaling Properties. *Wireless Networks*, **5**, 195–209.

[72] Griggs, J.R. and Jin, X.T. (2007) Real number labellings for paths and cycles. *Internet Math.*, **4**, 65–86.

[73] Kohl, A. (2006) Bounds for the $L(d,1)$-number of diameter 2 graphs, trees and cacti. *International Journal of Mobile Network Design and Innovation*, **1(2)**, 124–135.

[74] Georges, J.P., Mauro, D.W. and Wang, Y. (2009) Labeling the r-path with a condition at distance two. *Discrete Applied Mathematics*, **157**, 3203–3215.

[75] Lam,P.C.B., Gu, G., Lin, W. and Chung, P.-T. (2006) Bounds on the labelling numbers of chordal graphs. *Proceedings of the 2006 International Conference on Foundations of Computer Science*, Las Vegas, Nevada, USA, June 26–29.

[76] Sen, A. , Roxborough, T. and Medidi, S. (1998) Upper and Lower Bounds of a Class of Channel Assignmet Problems in Cellular Networks. *Proceedings of IEEE INFOCOM 1998*, San Francisco, CA, 29 March – 2 April, Volume 3, pp. 1273-1283, IEEE Computer Society Press.

[77] Calamoneri, T. and Petreschi, R. (2004) $L(h,1)$-Labeling Subclasses of Planar Graphs. *Journal on Parallel and Distributed Computing*, **64(3)**, 414-426.

[78] Das, S.K., Finocchi, I. and Petreschi, R. (2006) Conflict-Free Star-Access in Parallel Memory Systems. *Journal on Parallel and Distributed Systems*, **66(11)**, 1431–1441.

[79] Calamoneri, T. (2006) Exact Solution of a Class of Frequency Assignment Problems in Cellular Networks and Other Regular Grids. *Discrete Mathematics & Theoretical Computer Science*, **8**, 141–158.

[80] Calamoneri, T., Caminiti, S. and Fertin, G. (2006) $L(h,k)$-Labelling of Regular Grids. *International Journal of Mobile Network Design and Innovation*, **1(2)**, 92–101.

[81] Griggs, J.R. and Jin, X. (2005) Optimal Channel Assignments for Lattices.SIAM J. Discrete Mathematics, **22(3)**, 996–1021.

[82] Kim, B.M., Rho, Y. and Song, B.C. (2014) $L(h,k)$-Labeling for Octogonal Grid. *International Journal of Computer Mathematics*. To appear.

[83] Dubhashi, A.N., Shashanka, M.V.S., S., Pati, A., Shashank R. and Shende, A.M. (2002) Channel assignment for wireless networks modelled as d-dimensional square grids. In Proc. *International Workshop on Distributed Computing (IWDC '02)*, Calcutta, India, 28–31 December, pp. 130–141, Lecture Notes in Computer Science 2571, Springer Verlag, Berlin.

[84] Fertin, G. and Raspaud, A. (2007) $L(p,q)$ Labeling of d-Dimensional Grids. *Discrete Mathematics*, **307**, 2132–2140.

[85] Whittlesey, M.A. , Georges, J.P. and Mauro, D.W. (1995) On the λ number of Q_n and related graphs. *SIAM Journal on Discrete Mathematics*, **8**, 499-506.

[86] Jha, P.K.. Narayanan, A., Sood, P., Sundaran, K. and Sunder, V. (2000) On $L(2,1)$-labeling of the Cartesian product of a cycle and a path. *Ars Combinatoria*, **55**, 81–89.

[87] Klavžar, S. and Vesel, A. (2003) Computing graph invariants on rotographs using dynamic algorithm approach: the case of $(2,1)$-colorings and independence. *Discrete Applied Mathematics*, **129**, 449–460.

[88] Kuo, D. and Yan, J.-H. (2004) On $L(2,1)$-labelings of Cartesian products of paths and cycles. *Discrete Mathematics* ,**283**, 137–144.

[89] Schwarz, C. and Sakai Troxell, D. (2006) $L(2,1)$-Labelings of Products of Two Cycles. *Discrete Applied Mathematics*, **154**, 1522-1540.

[90] Georges, J.P., Mauro, D.W. and Stein, M.I. (2000) Labeling products of complete graphs with a condition at distance two. *SIAM Journal on Discrete Mathematics*, **14**, 28-35.

[91] Jha, P.K. (2000) Optimal $L(2,1)$-labeling of Cartesian products of cycles, with an application to independent domination. *IEEE Trans. Circ. Syst.-I: Fund. Theory and Appl.*, **47**, 1531–1534.

[92] Jha, P.K. (2001) Optimal $L(2,1)$-labeling of strong Cartesian products of cycles. *IEEE Trans. Circ. Syst.-I: Fund. Theory and Appl.*, **48**, 498–500.

[93] Korže, D. and Vesel, A (2005) $L(2,1)$-labeling of strong products of cycles. *Information Processing Letters*, **94**, 183–190.

[94] Kim, B.M., Song, B.C. and Rho, Y. (2014) $L(2,1)$-labellings for direct products of a triangle and a cycle. *International Journal of Computer Mathematics*, **90(3)**, 475–482.

[95] Jha, P.K., Klavžar, S. and Vesel,A. (2005) $L(2,1)$-labeling of direct product of paths and cycles. *Discrete Applied Mathematics*, **145(2)**, 317–325.

[96] Klavžar, S. and Špacapan, S. (2006) The Δ^2-conjecture for $L(2,1)$-labelings is true for directed and strong products of graphs. *IEEE Trans. Circuits and Systems II*, **53**, 274–277.

[97] Shao, Z. and Yeh, R.K. (2005) The $L(2,1)$-labeling and operations of graphs. *IEEE Trans. Circuits Syst. I Fund. Theory Appl.*,**52**, 668–671.

[98] Shao, Z., Klavžar, S., Shiu, W.C. and Zhang, D. (2008) Improved Bounds on the $L(2,1)$-Number of Direct and Strong Products of Graphs. *IEEE Transactions on Circuits and Systems II*, **55(7)**, 685–689.

[99] Shao, Z. and Solis-Oba, R. (2010) $L(2,1)$-Labelings on the composition of n graphs. *Theoretical Computer Science*, **411**, 3287–3292.

[100] Chudá, K. and Škoviera, M. (2012) $L(2,1)$-labelling of generalized prisms. *Discrete Applied Mathematics*, **160**, 755–763.

[101] Jha, P.K., Klavžar, S. and Vesel, A. (2005) Optimal $L(d,1)$-labelings of directed products of cycles and Cartesian products of cycles. *Discrete Applied Mathematics*, **152**, 257–265.

[102] Georges, J.P. and Mauro, D.W. (1999) Some results on λ_k^j-numbers of the products of complete graphs. *Congressus Numerantium*, **140**, 141–160.

[103] Erwin, D.J. , Georges, J.P. and Mauro, D.W. (2003) On Labeling the Vertices of Products of Complete Graphs with distance constraints. *Naval Research Logistics*, **52**, 138–141.

[104] Huang, L.-H. and Chang, G.J. (2009) $L(h,k)$-labelings of Hamming graphs. *Discrete Mathematics*, **309(8)**, 2197–2201.

[105] Chiang, S.-H., Yan, J.-H. (2008) On $L(d,1)$-Labeling of Cartesian Products of a cycle and a path. *Discrete Applied Mathematics*, **156(15)**, 2867–2881.

[106] Haque, E., and Jha, P.K. (2008) $L(j,k)$-Labelings of Kronecker Products of Complete Graphs. *IEEE Transactions on Circuits and Systems-II*, **55(1)**, 70–73.

[107] Kohl, A. (2009) The $L(d,1)$-number of Powers of Paths. *Discrete Mathematics*, **309**, 3427–3430.

[108] Adams, S.S., Howell, N., Karst, N., Sakai Troxell, D. and Zhu, J. (2013) The $L(2,1)$-labelings of amalgamations of graphs. *Discrete Applied Mathematics*, **161**, 881–888.

[109] Karst, N., Oehrlein, J., Sakai Troxell, D. and Zhu, J. (2014) The Minimum Span of $L(2,1)$-Labelings of Generalized Flowers. *Discrete Applied Mathematics*. To appear.

[110] Karst, N., Oehrlein, J., Sakai Troxell, D. and Zhu, J. (2014) Labeling Amalgamations of Cartesian Products of Complete Graphs with a Condition at Distance Two. *Discrete Applied Mathematics*, *178*, 101–108.

[111] Shao, Z. and Solis-Oba, R. (2013) $L(2,1)$-labelings on the modular product of two graphs. *Theoretical Computer Science*, *487*, 74–81.

[112] Shao, Z., Yeh, R.K., and Zhang, D. (2007) The $L(2,1)$-labeling on the skew and converse skew product of graphs. *Applied Mathematics Letters*, **20**, 59–64.

[113] Shao, Z. and Zhang, D. (2008) Improved upper bounds on the $L(2,1)$-labeling of the skew and converse skew product graphs (Note). *Theoretical Computer Science*, **400**, 230–233.

[114] Duan, Z., Lv, P., Miao, L., Miao, Z. and Wang, C. (2011) New upper bounds on the $L(2,1)$-labeling of the skew and converse skew product graphs. *Theoretical Computer Science*, 412, 2393–2397.

[115] Hasunama, T., Ishii, T., Ono, H., and Uno, Y. (2009) An $O(n^{1.75})$ algorithm for $L(2,1)$-labeling of trees. *Theoretical Computer Science*, **410 (38-40**, 3702–3710.

[116] Hasunuma, T., Ishii, T., Ono, H., and Uno, Y. (2009) A linear time algorithm for $L(2,1)$-labeling of trees. *Proc. 17th Annual European Symp. on Algorithms (ESA '09)*, Copenhagen, Denmark, pp. 35–46, Lecture Notes in Computer Science 5757, Springer-Verlag, Berlin.

[117] Wang, W.-F. (2007) The $L(2,1)$-Labelling of trees. *Discrete Applied Mathematics*, **154**, 598–603.

[118] Fiala, J., Golovach, P.A., and Kratochvíl, J. (2008) Computational Complexity of the Distance Constrained Labeling Problem for Trees. *35th Int.l Colloquium on Automata, Languages and Programming (ICALP '08)*, Reykjavik, Iceland, 6–13 July, pp. 294–305, Lectures Notes in Computer Science 5125, Springer Verlag, Berlin..

[119] Welsh, D. (1999) Private communication with J. Fiala, J. Kratochvíl and A. Proskurowski.

[120] Fiala, J., Kratochvíl, J. and Proskurowski, A. (2005) Systems of Distant Representatives. *Discrete Applied Mathematics*, **145(2)**, 306–316.

[121] Fiala, J., Golovach, P.A., and Kratochvíl, J. (2008) Distance Constrained Labeling of Trees. *5th International Conference on Theory and Applications of Models of Computation (TAMC '08)*, Xian, China, 25–29 April8, pp. 125–135, Lectures Notes in Computer Science 4978, Springer Verlag, Berlin..

[122] Georges, J.P. and Mauro, D.W. (2003) Labeling trees with a condition at distance two. *Discrete Mathematics*, **269**, 127–148.

[123] Calamoneri, T., Pelc, A. and Petreschi, R. (2006) Labeling trees with a condition at distance two. *Discrete Mathematics*, **306**, 1534–1539.

[124] Juan, J.S.-T., Liu, D.D.-F. and Chen, L.-Y. (2010) $L(j,k)$-labelling and maximum ordering-degrees for trees. *Discrete Applied Mathematics*, **158**, 692–698.

[125] Kobler, D. and Rotics, U. (2003) Polynomial algorithms for partitioning problems on graphs with fixed clique-width. *Theoretical Computer Science*, **299(1-3)**, 719–734.

[126] Suchan, K. and Todinca, I. (2007) On powers of graphs of bounded NLC-width (clique-width). *Discrete Applied Mathematics*,**155(14)**, 1885-1893.

[127] Bodlaender, H.L. and Kratsch, D. Private communication with Todinca, I.

[128] Corneil, D.G., Perl, Y. and Stewart, L.K. (1985) A linear recognition algorithm for cographs. *SIAM Journal on Computing*, **14**, 926–934.

[129] Courcelle, B., Makowsky, J. A., and Rotics, U. (2000) Linear time solvable optimization problems on graphs of bounded clique-width. *Theory on Computing Systems*, **33**, pp. 125-150.

[130] Courcelle, B. and Olariu, S. (2000) Upper bounds to the clique-width of graphs. *Discrete Applied Mathematics*, **101**, pp. 77-114.

[131] Bodlaender, H.L. (1998) A partial k-arboretum of graphs with bounded treewidth. *Theoretical Computer Science*, **209(1-2)**, 1–45.

[132] Reed, B. A. (2003) Algorithmic aspects of treewidth. In Reed, B. A. and Sales, C. L. (eds.), *Recent Advances in Algorithms and Combinatorics*, Springer, Berlin.

[133] Report of the Project *Critical Resource Sharing for Cooperation in Complex Systems (Crescco)* funded by the European Community under the "Information Society Technologies" Programme (1998–2002), Technical Report IST-2001-33135.

[134] Fiala, J. and Kratochvíl, J. (2005) Private communication.

[135] Fiala, J., Golovach, P.A., and Kratochvíl, J. (2005) Distance Constrained Labeling of Graphs of Bounded Treewidth. *32th Int.l Colloquium on Automata, Languages and Programming (ICALP '05)*, Lisboa, Portugal, 1115 July, pp. 360–372, Lectures Notes in Computer Science 3580, Springer Verlag, Berlin.

[136] Zhou, X., Kanari,Y. and Nishizeki,T. (2000) Generalized vertex-coloring of partial k-trees. *IEICE Trans. Fundamentals of Electronics, Communication and Computer Sciences*, **E83-A**, 671-678.

[137] Krumke, S.O. , Marathe, M.V. and Ravi, S.S. (2001) Models and Approximation Algorithms for Channel Assignment in Radio Networks. *Wireless Networks*, **7**, 575–584.

[138] Fiala, J., Golovach, P.A., and Kratochvíl, J. (2011) Parameterized complexity of coloring problems: Treewidth versus vertex cover. *Theoretical Computer Science*, **412**, 2513–2523.

[139] Thomassen, C. (2001) *Applications of Tutte Cycles.* Technical Report MAT 2001-16, Technical University of Denmark, Copenhagen.

[140] Wan, P. J. (1997) Near-Optimal Conflict-Free Channel Set Assignments for an Optical Cluster-Based Hypercube Network. *Journal of Combinatorial Optimization*, 1, 179–186.

[141] van den Heuvel, J. and McGuinnes, S. (2003) Colouring the square of a planar graphs. *J. of Graph Theory*, **42**, 110–124.

[142] Agnarsson, G. and Halldórsson, M. (2003) Coloring Powers of planar graphs. *SIAM Journal on Discrete Mathematics*, **16(4)**, 654–662.

[143] Borodin, O.V., Broersma, H.J., Glebov, A. and van den Heuvel, J. (2001) Stars and Bunches in Planar Graphs. Part II: General Planar Graphs and Colourings. Tech. Rep. London School of Economics, London, 2002. Translated and adapted from *Diskretn. Anal. Issled. Oper. Ser.*, **1(8)**, 9-33 (in russian).

[144] Molloy, M. and Salavatipour, M.R. (2005) A Bound on the Chromatic Number of the Square of a Planar Graph. *Journal of Combinatorial Theory (Series B)*, **94(2)**, 189-213.

[145] Jendrol', S and Voss, H.-J. (2004) Light subgraphs of graphs embedded in the plane and in the projective plane – a survey. *IM Preprint series A, No 1/2004, Pavol Jozef Šafárik Univ., Slovakia.*

[146] Wang, W.-F. and Lih, K.-W. (2003) Labeling planar graphs with conditions on girth and distance two. *SIAM Journal on Discrete Mathematics*, **17(2)**, 264–275.

[147] Shao, Z. and Yeh, R.K. (2007) The $L(2,1)$-labeling on planar graphs. *Applied Mathematics Letters*,**20**, 222–226.

[148] Borodin, O.V., Glebow, A.N., Ivanova, A.O., Neustroeva, T.K. and Taskinov, V.A. (2004) Sufficient conditions for planar graphs to be 2-distance $(\Delta+1)$-colorable. *Sib. Elektron. Mat. Izv*, **1**, 129-141 (in russian).

[149] Borodin, O.V., Ivanova, A.O., Neustroeva, T.K. (2004) 2-distance coloring of sparse planar graphs. *Sib. Elektron. Mat. Izv*, **1**, 76-90 (in russian).

[150] Dvořák, Z., Král', D., Nejedlý, P. and Škrekovski, R. (2008) Coloring squares of planar graphs with girth 6. *European Journal of Combinatorics*, **29**, 838–849.

[151] Bella, P., Král, D., Mohar, B. and Quittnerová, K. (2007) Labeling planar graphs with a condition at distance two. *European Journal on Combinatorics*, **28**, 2201–2239.

[152] Dvořák, Z., Král', D., Nejedlý, P. and Škrekovski, R. (2009) Distance constrained labeling of planar graphs with no short cycles. *Discrete Applied Mathematics*, **157**, 2634–2645.

[153] Wang, W.-F. and Cai, L. (2008) Labelling planar graphs without 4-cycles with a condition at distance two. *Discrete Applied Mathematics*, **156**, 2241–2249.

[154] Charpentier, C., Montassier, M. and Raspaud, A. (2013) $L(p,q)$-labeling of sparse graphs. *Journal on Combinatorial Optimization*, **25**, 646–660.

[155] Bodlaender, H.L. (1988) Dynamic programming on graphs of bounded treewidth. *Proceedings of 15th International Colloquium on Automata, Languages and Programming (ICALP'88)*, Tampere, Finland, 11–15 July, pp. 105–118, Lectures Notes in Computer Science 317, Springer Verlag, Berlin.

[156] Agnarsson, G. and Halldórsson, M.M. (2004) On Colorings of Squares of Outerplanar Graphs. $arXiv : 0706_1526v1$. An earlier version appeared in *Proceedings of 15th annual ACM-SIAM Symp. on Discrete Algorithms (SODA)*, New Orleans, LA, 11–13 January, pp. 242-253.

[157] Agnarsson, G. and Halldórsson, M.M. (2004) Vertex Coloring the Square of Outerplanar Graphs of Low Degree. *Discussiones Mathematicae*, **30**, 619–636.

[158] Wang, W.-F. and Luo, X.-F. (2009) Some Results on Distance Two Labellings of Outerplanar Graphs. *Acta Mathematicae Applicatae Sinica*, English Series, **25(1)**, 21–32.

[159] Bruce, R.J. and Hoffmann, M. (2003) $L(p,q)$-labeling of outerplanar graphs. Tech. Rep. No. 2003/9, Department of Mathematics and Computer Science, University of Leicester, England.

[160] Li, X. and Zhou, S. (2013) Labeling outer planar graphs with maximum degree three. *Discrete Applied Mathematics*, **161**, 200–211.

[161] Zhang, S. and Ma, Q. (2007) Labelling of Some Planar Graphs with a Condition at Distance Two. *Journal on Applied Mathematics & Computing*, **24**, 421–426.

[162] Chartrand, G. and Harary, F. (1967) Planar Permutation graphs. *Ann. Inst. H. Poincaré*, sect. B(N.S.), **3**, 433–438.

[163] Lih, K.-W., Wang, W.-F. and Zhu, X. (2003) Coloring the square of K_4-minor Free Graph, *Discrete Mathematics*, **269**, 303–309.

[164] Wang, W.-F. and Wang, Y. (2006) $L(p,q)$-Labelling of K_4-minor free graphs. *Information Processing Letters*, **98**, 169–173.

[165] Král', D. and Nejedlý, P. (2009) Distance Constrained Labelings of K_4-minor free graphs. *Discrete Mathematics*, **309**, 5745–5756.

[166] Breu, H. and Kirkpatrick, D.G. (1998) Unit disk graph recognition is NP-hard. *Computational Geometry: Theory and Applications*, **9**, 3–24.

[167] Hliněný, P. and Kratochvíl, J. (2001) Representing graphs by disks and balls (a survey of recognition complexity results). *Discrete Mathematics*, **229(1-3)**, 101–124.

[168] Teng, S.H. (1991) *Points, spheres and separators, A unified geometric approach to graph separators*. Ph.D. thesis, School of Computer Science, Carnegie Mellon University, CMU-CS-91-184, Pittsburgh.

[169] Sen, A. and Malesinska, E. (1997) Approximation algorithms for radio network schedulings. *Proceedings 35th Allerton Conf. on Communication, Control and Computing*, Allerton, IL, pp. 573–582.

[170] Ma, X. and Loyd, E.L. (1999) Private communication with Krumke, S.O., Marathe, M.V. and Ravi, S.S..

[171] Wan, P.-J. Conflict-Free Channel Assignment in Wireless Ad Hoc Networks. *lecture 9 of course "Wireless Networking" at Illinois Institute of technology.*

[172] Fiala, J. , Fishkin, A.V., and Fomin, F.V. (2004) On distance constrained labeling of disk graphs. *Theoretical Computer Science*, **326(1–3)**, 261–292.

[173] Shao, Z., Yeh, R.K., Shiu, W.C. (2008) The $L(2,1)$-Labeling of $K_{1,n}$-free graphs and its applications. *Applied Mathematics Letters*, **21**, 1188–1193.

[174] Agnarsson, G. , Greenlaw, R. and Halldórsson, M.M. (2000) On Powers of Chordal Graphs And Their Colorings. *Congressus Numerantium*, **144**, 41–65.

[175] Sakai, D. (1994) Labeling Chordal Graphs: Distance Two Condition. *SIAM Journal on Discrete Mathematics*, **7** , 133–140.

[176] Král', D. (2004) Coloring Powers of Chordal Graphs. *SIAM Journal on Discrete Mathematics*, **18(3)**, 451–461.

[177] Cerioli, M.R. and Posner, D.F.D. (2012) On $L(2,1)$-coloring split, chordal bipartite, and weakly chordal graphs. *Discrete Applied Mathematics*, **160**, 2655–2661.

[178] Panda, B.S. and Goel, P. (2011) $L(2,1)$-labeling of perfect elimination bipartite graphs. *Discrete Applied Mathematics*, **159(16)**, 1878–1888.

[179] Panda, B.S. and Goel, P. (2012) $L(2,1)$-labeling of dually chordal graphs and strongly orderable graphs. *Information Processing Letters*, **112(13)**, 552–556.

[180] Panda, B.S. and Goel, P. (2009) $L(2,1)$-labeling of block graphs. *Ars Combinatoria*, to appear.

[181] F. Bonomo and M.R. Cerioli. On the $L(2,1)$-labeling of block graphs. *International Journal on Computer Mathematics.* To appear.

[182] Paul, S., Pal, M. and Pal, A. (2013) An Efficient Algorithm to Solve $L(0,1)$-Labelling Problem on Interval Graphs. Advanced Modeling and Optimization, **15(1)**, 31–43.

[183] Raychauduri, A. (1987) On Powers of Interval and Unit Interval Graphs. *Congressus Numerantium*, **59**, 235–242.

[184] Bertossi, A.A., Pinotti, C.M. and Rizzi, R. (2003) Channel assignment on Strongly-Simplicial Graphs. *Proceedings of 3rd Int.l Workshop on Wireless, Mobile and Ad Hoc Networks, (W-MAN 2003)*, Nice, France, 22–26 April, IEEE Computer Society Press.

[185] Lam, P.C.B., Wang, T.-M., Shiu, W.C. and Gu, G. (2009) On Distance Two Labelling of Unit Interval Graphs. *Taiwanese Journal of Mathematics*, **13**, 1167–1179.

[186] Paul, S., Pal, M. and Pal, A. (2014) $L(2,1)$-labeling of interval and circular-arc graphs. *Journal of Applied Mathematics and Computing.* To appear.

[187] Calamoneri, T., Caminiti, S., Olariu, S. and Petreschi, R. (2009) On the $L(h,k)$-labeling of Co-Comparability Graphs and Circular-Arc Graphs. *Networks*, **53(1)**, 27–34.

[188] Paul, S., Pal, M. and Pal, A. (2014) $L(2,1)$-Labeling of Permutation and Bipartite Permutation Graphs. *Mathematics in Computer Science*, DOI 10.1007/s11786-014-0180-2

[189] Araki, T. (2009) Labeling bipartite permutation graphs with a condition at distance two. *Discrete Applied Mathematics*, **157**, 1677–1686.

[190] Zhou, H., Shiu, W.C., and Lam, P.C.B. (2014) Notes on $L(1,1)$ and $L(2,1)$ labelings for n-cubes. *Journal of Combinatorial Optimization*, **28**, 626–638.

[191] Jonas, K. Private communications to J.R. Griggs.

[192] Frieder, O., Harary, F. and Wan, P.-J. (2002) A radio coloring of a hypercube. *International Journal of Computer Mathematics*, **79(6)**, 665–670.

[193] Chang, G.J., Lu, C. and Zhou, S. (2009) Distance-two labellings of Hamming graphs. *Discrete Applied Mathematics*, **157(8)**, 1896–1904.

[194] Calamoneri, T., Caminiti, S. and Petreschi, R. (2008) A General Approach to $L(h,k)$-Label Interconnection Networks. *Journal of Computer Science & Technology*, **23(4)**, 652–659.

[195] Zhou, S. (2006) Labelling Cayley Graphs on Abelian Groups. *SIAM Journal on Discrete Mathematics*, **19(4)**, 985–1003.

[196] Zhou, S. (2007) Distance Labelling Problems for Hypercubes and Hamming Graphs – A Survey. *Electronic Notes in Discrete Mathematics*, **28**, 527–534.

[197] Tait, P.G. (1880) Remarks on the colourings of maps. Proc. of the *Royal Society of Edinbourgh*, **10**, 728–729.

[198] Kang, J.-H. (2008) $L(2,1)$-labelling for Hamiltonian graphs of maximum degree 3. *SIAM J. on Discrete Math*, **22(1)**, 213–230.

[199] Ma, D., Zhu, H. and He, J. (2014) λ-numbers of several classes of snarks. *Journal of Combinatorial Optimization*, **28**, 787–799.

[200] Erdös, P. , Fajtlowicz, S. and Hoffman, A. J. (1980) Maximum degree in graphs of diameter 2. *Networks*, **10**, 87–90.

[201] Junker, M. (2003) Moore graphs, latin squares and outer automorphisms. Tech. Rep. Albert Ludwigs Univ. Freiburg, Germany.

[202] Georges, J.P. and Mauro., D.W. (2003) On regular graphs optimally labeled with a condition at distance two. *SIAM Journal on Discrete Mathematics*, **17(2)**, 320–331.

[203] Costa, M.-C., de Werra, D., Picouleau, C. and Ries, B. (2009) Graph coloring with cardinality constraints on the neighborhood. *Discrete Optimization*, **6(4)**, 362–369.

[204] Griggs, J.R. and Jin, X. (2007) Recent Progress in Mathematics and Engineering on Optimal Graph Labellings with Distance Conditions. *Journal of Combinatorial Optimization*, **14**, 249–257.

[205] Zhou, S. (2004) A channel assignment problem for optical networks modelled by Cayley graphs. *Theoretical Computer Science*, **310**, 501–511.

[206] Bahls, P. (2011) Channel Assignment on Cayley Graphs. *Journal of Graph Theory*, 67(3), 169–177.

[207] Li, X., Mak-Hau, V. and Zhou, S. (2013) The $L(2,1)$-labeling problem for cubic Cayley graphs on dihedral groups. *Journal of Combinatorial Optimization*, 25(4), 716–736.

[208] Mahadev, N.V.R. and Peled,U.N. (1995) *Threshold Graphs and Related Topics.* Annales on Discrete Mathematics 56, North-Holland, Amsterdam.

[209] Marchioro, P., Morgana, A. , Petreschi, R. and Simeone, B. (1984) Degree sequences of matrogenic graphs. *Discrete Mathematics*, **51**, 47–61.

[210] Foldes, S. and Hammer, P. (1978) On a class of matroid producing graphs. *Colloq. Math.Soc. J. Bolyai (Combinatorics)*, **18**, 331–352.

[211] Peled, U.N. (1977) Matroidal graphs. *Discrete Mathematics*, **20**, 263–286.

[212] Calamoneri, T. and Petreschi, R. (2006) *λ-Coloring Matrogenic Graphs. Discrete Applied Mathematics*, **154**, 2445–2457.

[213] Calamoneri, T. and Petreschi, R. (2011) $L(2,1)$-labeling of unigraphs. *Discrete Applied Mathematics*, to appear.

[214] Georges, J.P. and Mauro, D.W. (2002) On generalized Petersen graphs labeled with a condition at distance two, *Discrete Mathematics*, **259**, 311–318.

[215] Adams, S.S., Cass, J. and Sakai Troxell, D. (2006) An extension of the channel-assignment problem: $L(2,1)$-labelings of generalized Petersen graphs. *IEEE Transaction on Circuits and Systems I*, **53**, 1101-1107.

[216] Adams, S.S., Cass, J., Tesch, M., Sakai Troxell, D. and Wheeland, C. (2007) The minimum span of $L(2,1)$-Labelings of Certain Generalized Petersen Graphs. *Discrete Applied Mathematics*, **155**, 1314–1325.

[217] Adams, S.S., Booth, P., Jaffe, H., Sakai Troxell, D. and Zinnen, S.L. (2010) Exact λ-numbers of generalized Petersen graphs of certain higher-orders and on Möbius strips. *Manuscript*.

[218] Damaschke, P. (1992) Distance in cocomparability graphs and their powers. *Discrete Applied Mathematics*, **35**, 67–72.

[219] Lovász, L. (1978) Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory, Ser. A* **25**, 319–324.

[220] Kang, J.-H. (2005) $L(2,1)$-labelling for Kneser graphs. Manuscript.

[221] Shao, Z., Solis-Oba, R. and Lin, G (2014) $L(2,1)$-Labelings of Kneser Graphs. Manuscript.

[222] Shao, Z., Yeh, R.K., and Zhang, D. (2008) The $L(2,1)$-labeling on graphs and the frequency assignment problem. *Applied Mathematics Letters*,**21**, 37–41.

[223] Duan, Z., Lv, P., Miao, L., Miao, Z. and Wang, C. (2011) The Δ^2-conjecture for $L(2,1)$-labelings is true for total graphs. *Applied Mathematics Letters*, **24**, 1491–1494.

[224] Gravier, S., Klavžar, S. and Mollard, M. (2005) Codes and $L(2,1)$-labelings in Sierpiński graphs. *Taiwanese Journal on Mathematics*, **9(4)**, 671-681.