# On the $L(h,k)$-Labeling of Co-Comparability Graphs and Circular-Arc Graphs *

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**Abstract.** Given two nonnegative integers $h$ and $k$, an $L(h,k)$-*labeling* of a graph $G=(V,E)$ is a map from $V$ to a set of integer labels such that adjacent vertices receive labels at least $h$ apart, while vertices at distance at most 2 receive labels at least $k$ apart. The goal of the $L(h,k)$-labeling problem is to produce a legal labeling that minimizes the largest label used. Since the decision version of the $L(h,k)$-labeling problem is NP-

* This research was supported, in part, by the European Research Project *Algorithmic Principles for Building Efficient Overlay Computers* (AEOLUS). Most of the work reported here was performed while Professor Olariu visited with the Department of Computer Science, University of Rome "La Sapienza". Support through a Visiting Fellowship from the University of Rome "La Sapienza" is gratefully acknowledged.

complete, it is important to investigate classes of graphs for which the problem can be solved efficiently.

Along this line of thought, in this paper we deal with co-comparability graphs, its subclass of interval graphs, and circular-arc graphs. To the best of our knowledge, ours is the first reported result concerning the $L(h, k)$-labeling of co-comparability and circular-arc graphs. In particular, we provide the first algorithm to $L(h, k)$-label co-comparability, interval, and circular-arc graphs with a bounded number of colors. Finally, in the special case where $k = 1$ and $G$ is an interval graph, our algorithm improves on the best previously-known ones using a number of colors that is at most twice the optimum.

**Keywords:** $L(h, k)$-Labeling, co-comparability graphs, interval graphs, circular-arc graphs.

## 1 Introduction

Graph coloring is, without doubt, one of the most fertile and widely studied areas in graph theory, as evidenced by the list of solved and unsolved problems in Jensen and Toft's comprehensive book on graph coloring [25]. The classic problem of (vertex) coloring, asks for an assignment of non-negative integers (colors) to the vertices of a graph in such a way that adjacent vertices receive distinct colors. Of interest, of course, are assignments (colorings) that minimize the number of colors used.

In this paper we focus on a generalization of the classic vertex coloring problem – the so-called $L(h, k)$-*labeling problem* – that asks for the smallest $\lambda$ for which it is possible to assign integer labels $\{0, \ldots, \lambda\}$ to the vertices of a graph in such a way that vertices at distance at most two receive colors at least $k$ apart, while adjacent vertices receive labels at least $h$ apart. In the remainder of this work we shall follow established practice and refer to the largest label in an optimal $L(h, k)$-labeling for graph $G$ as $\lambda_{h,k}(G)$. Independently from the optimality of the $L(h, k)$-labeling, we call its *span* the difference between the maximum and the minimum label used. Of course, when the $L(h, k)$-labeling of a graph $G$ is optimum its span coincides with $\lambda_{h,k}(G)$.

We note that for $k = 0$, the $L(h, k)$-labeling problem coincides with the usual vertex coloring; for $h = k$, we obtain the well-known 2-distance coloring, which is equivalent to the vertex coloring of the square of a graph.

The $L(h, k)$-labeling problem arises in many applications, including the design of wireless communication systems [24], radio channel assignment [8, 30], data distribution in multiprocessor parallel memory systems [4, 15, 39], and scalability of optical networks [1, 41], among many others.

The decision version of the vertex coloring problem is NP-complete in general [17], and it remains so for most of its variations and generalizations. In particular, it has been shown that the decision version of the $L(h, k)$-labeling problem is NP-complete even for $h = k = 1$ [24, 29]. Therefore, the problem has been widely studied for many particular classes of graphs. For a survey of recent results we refer the interested reader to [9].

In this paper we deal with co-comparability graphs and its subclass of interval graphs. The literature contains a plethora of papers describing applications of these graphs to such diverse areas as archaeology, biology, psychology, management and many others (see [20–22, 31, 33, 34]).

In the light of their relevance to practical problems, it is somewhat surprising to note the dearth of results pertaining to the $L(h, k)$-labeling of these graph classes. For example, a fairly involved web search has turned up no results on the $L(h, k)$-labeling of co-comparability and circular-arc graphs and, as listed below, only two results on the $L(h, k)$-labeling of interval graphs and its subclass of unit-interval graphs.

- In [38] the special case $h = 2$ and $k = 1$ is studied; the author proves that $2\chi(G) - 2 \leq \lambda_{2,1}(G) \leq 2\chi(G)$ for unit-interval graphs, where $\chi(G)$ is the chromatic number of $G$. In terms of the maximum degree $\Delta$, as $\chi(G) \leq \Delta+1$, the upper bound becomes $\lambda_{2,1}(G) \leq 2(\Delta + 1)$, and this value is very close to being tight, as the clique $K_n$, that is an interval graph, has $\lambda_{2,1}(K_n) = 2(n - 1) = 2\Delta$.
- In [3] the authors present a 3-approximate algorithm for $L(h, 1)$-labeling interval graphs, i.e. they present an algorithm guaranteeing a number of labels at most three times larger than the optimum. Furthermore, the authors show that the same approximation ratio holds for the $L(h, k)$-labeling problem of unit-interval graphs.

These bounds on $\lambda_{2,1}$ are of interest as it is neither known the complexity of the $L(h,k)$-labeling problem on co-comparability graphs and their subclasses.

deal with co-comparability graphs, its subclass of interval graphs, and circular-arc graphs.

One of our main contributions is to provide the first algorithm to $L(h,k)$-label co-comparability, interval, and circular-arc graphs with a bounded number of colors. To the best of our knowledge, ours is the first reported result concerning the $L(h,k)$-labeling of co-comparability and circular-arc graphs.

Finally, in the special case where $k = 1$ and $G$ is an interval graph, our algorithm improves on the best previously-known ones using a number of colors that is at most twice the optimum.

The remainder of this paper is organized as follows: Section 2 is devoted to definitions and a review of preliminary results; in particular we observe that the $L(1,1)$-labeling problem is polynomially solvable for co-comparability graphs. Sections 3 and 4 focus, respectively, on the $L(h,k)$-labeling problem on co-comparability and interval graphs. In Section 5 we establish an upper bound on the $L(h,k)$-labeling of circular-arc graphs. Finally, Section 6 offers concluding remarks and open problems.

## 2 Preliminaries

The graphs in the work are simple, with no self-loops or multiple edges. We follow standard graph-theoretic terminology compatible with [20] and [5].

(a) (b)

**Fig. 1.** *Illustrating two forbidden configurations.*

Vertex orderings have proved to be useful tools for studying structural and algorithmic properties of various graph classes. For example, Rose, Tarjan and Lueker [37] and Tarjan and Yannakakis [40] have used the well known simplicial ordering of the vertices of a chordal graph to obtain simple recognition and optimization algorithms for this class of graphs. To make this work as self-contained as possible, suffice it to say that a graph $G = (V, E)$ has a *simplicial ordering* if its vertices can be enumerated as $v_1, v_2, \ldots, v_n$ in such a way that for all subscripts $i,\ j,\ k$, with $1 \leq i < j < k \leq n$, the presence of the edges $v_iv_k$ and $v_jv_k$ implies the existence of the edge $v_iv_j$. Refer to Figure 1(a) for a forbidden configuration for a simplicial order.

Kratsch and Stewart [26] have shown that a graph is a *co-comparability graph* if and only if its vertices can be enumerated as $v_1, v_2, \ldots, v_n$ in such a way that for all subscripts $i,\ j,\ k$, with $1 \leq i < j < k \leq n$, the presence of the edges $v_iv_k$

implies the presence of at least one of the edges and $v_i v_k$ or $v_i v_j$. For alternate definitions of co-comparability graphs we refer to [23].

Later, Olariu [32] proved that a graph is an *interval graph* if and only if its vertices can be ordered as $v_1, v_2, \ldots, v_n$ in such a way that for all subscripts $i,\ j,\ k$, with $1 \leq i < j < k \leq n$, the presence of the edge $v_i v_k$ implies the presence of the edge $v_i v_j$.

Finally, Looges and Olariu [28] showed that a graph is a *unit interval graph* if its vertices can be ordered as $v_1, v_2, \ldots, v_n$ in such a way that for all subscripts $i,\ j,\ k$, with $1 \leq i < j < k \leq n$, the presence of the edge $v_i v_k$ implies the presence of the edges $v_i v_j$ and $v_j v_k$.

The next proposition summarizes the previous discussion.

**Proposition 1.** *Let $G = (V, E)$ be a graph.*

1. *$G$ is a co-comparability graph if and only if there exists an ordering of its vertices $v_0 < \ldots < v_{n-1}$ such that if $v_i < v_j < v_l$ and $(v_i, v_l) \in E$ then either $(v_i, v_j) \in E$ or $(v_j, v_l) \in E$ [26];*
2. *$G$ is an interval graph if and only if there exists an ordering of its vertices $v_0 < \ldots < v_{n-1}$ such that if $v_i < v_j < v_l$ and $(v_i, v_l) \in E$ then $(v_i, v_j) \in E$ [32];*
3. *$G$ is a unit-interval graph if and only if there exists an ordering of its vertices $v_0 < \ldots < v_{n-1}$ such that if $v_i < v_j < v_l$ and $(v_i, v_l) \in E$ then $(v_i, v_j) \in E$ and $(v_j, v_l) \in E$ [28].*

In the remainder of this work we shall refer to a linear order satisfying the above proposition as *canonical* and to the property that characterizes which edges must exist in a certain class as the *umbrella property* of that class (see Figure 1(b)). Also, Figure 2 summarizes the umbrella properties for co-comparability, interval, and unit-interval graphs. Observe that Proposition 1 confirms the well-known fact that unit-interval graphs $\subseteq$ interval graphs $\subseteq$ co-comparability graphs.

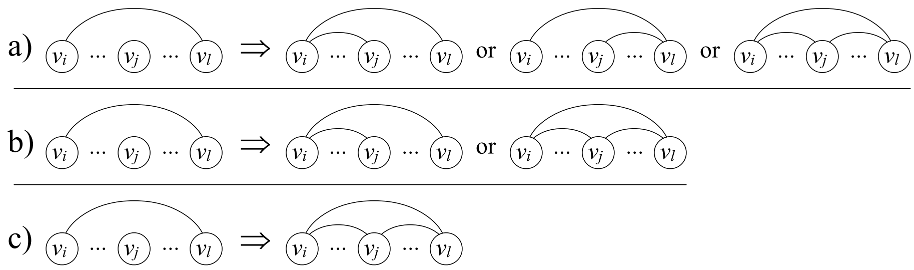


**Fig. 2.** *Illustrating the umbrella properties for a) co-comparability, b) interval, and c) unit-interval graphs.*

Before proving general results concerning the $L(h,k)$-labeling of the above classes of graphs, we make a few observations about the corresponding $L(1,1)$-labelings. To begin, we observe that unit-interval, interval and co-comparability graphs are all perfect graphs and hence the vertex-coloring problem is polynomially solvable [20]. As already mentioned, the $L(1,1)$-labeling problem for a graph $G$ is exactly the vertex-coloring problem for its square graph $G^2$ (i.e. the graph having the same vertex set as $G$ and having an edge connecting $u$ to $v$ if and only if $u$ and $v$ are at distance at most 2 in $G$). Since all these classes are closed under powers [14, 36], the following theorem holds:

**Theorem 1.** *The $L(1,1)$-labeling problem is polynomially solvable for unit-interval, interval and co-comparability graphs.*

## 3 The $L(h, k)$-Labeling of Co-Comparability Graphs

Given a co-comparability graph $G = (V, E)$ of maximum degree $\Delta$, in view of the umbrella property (Proposition 1 item 1), if $(v_i, v_l) \in E$ and $v_i < v_l$, then any $v_j$, $v_i < v_j < v_l$, must be adjacent to either $v_i$ or $v_l$ or both. Let the number of vertices between $v_i$ and $v_l$ that are adjacent to $v_i$ and $v_l$ respectively be $d'$ and $d''$. Since there are $l-i-1$ vertices between $v_i$ and $v_l$, we have $l-i-1 \leq d'+d''$.

The degree, $d(v_i)$, of $v_i$ is at least $d'+1$, analogously $d(v_l) \geq d''+1$. Since the maximum degree is $\Delta$ we have $2\Delta \geq d' + d'' + 2 \geq l - i + 1$. Analogous reasonings can be done when $v_i$ and $v_l$ are at distance two, considering also the vertex in between. Let us formalize this fact in the following proposition:

**Proposition 2.** *Given a co-comparability graph of maximum degree $\Delta$, if $(v_i, v_l) \in E$ and $v_i < v_l$ then $l - i < 2\Delta$; if $v_i$ and $v_l$ are at distance 2 and $v_i < v_l$ then $l - i < 3\Delta$.*

**Lemma 1.** *A co-comparability graph $G$ can be $L(h, k)$-labeled with span at most $2\Delta h + k$ if $k \leq \frac{h}{2}$.*

*Proof.* Let us consider the following ordered set of labels: $0, h, 2h, \ldots, 2\Delta h, k, h+k, 2h+k, \ldots, 2\Delta h+k$.

Let us label all vertices of $G$ with labels in the given order following a canonical order of $G$'s vertices; once the labels have been finished, we start again from label 0.

We will now prove that such a labeling is a feasible $L(h, k)$-labeling by showing that adjacent vertices are labeled with colors at least $h$ apart and that vertices at distance 2 are labeled with colors at least $k$ apart. The proofs are by contradiction and $v_i$ and $v_l$ are any two vertices with $i < l$.

**Distance 1.** Let $v_i$ and $v_l$ be adjacent vertices, assume by contradiction that their labels $l(v_i)$ and $l(v_l)$ differ by less than $h$. Then only two cases are possible:

(1.1) $l(v_i) = sh$, for some $s$ such that $0 \leq s \leq 2\Delta$. Then $l(v_l) - l(v_i)$ can be smaller than $h$ only if either $l(v_l) = sh+k$ or $l(v_l) = (s-1)h+k$. In both cases $l-i \geq (2\Delta - s) + (s-1) + 1 = 2\Delta$ as illustrated in Figure 3. This is impossible, for otherwise either $v_i$ or $v_l$ would have degree greater than $\Delta$ (see Proposition 2.) Notice that $l(v_l)$ cannot be $sh$ because there are $4\Delta$ distinct labels and $l - i$ is bounded by $2\Delta$.

(1.2) $l(v_i) = sh+k$, with $0 \leq s \leq 2\Delta$. Then $l(v_l)$ must be either $sh$ or $(s+1)h$. In both cases $l-i \geq (2\Delta - s - 1) + s + 1 = 2\Delta$. Again, this is impossible. As in the previous case, $l(v_l)$ cannot be equal to $l(v_i)$.

**Distance 2.** Let $v_i$ and $v_l$ be at distance two with labels $l(v_i)$ and $l(v_l)$ that differ by less than $k$. Since $k \leq \frac{h}{2}$ it must be $l(v_i) = l(v_l)$, and therefore that $l - i = 4\Delta + 2$, i.e. the number of the different labels. This contradicts Proposition 2.

**Lemma 2.** *A co-comparability graph $G$ can be $L(h,k)$-labeled with span at most $4k\Delta + k$, if $k \geq \frac{h}{2}$.*

*Proof.* The proof is analogous to the one of Lemma 1. The only difference is the ordered set of labels used: $0, 2k, 4k, \ldots, 4k\Delta, k, 3k, 5k, \ldots, 4k\Delta + k$.

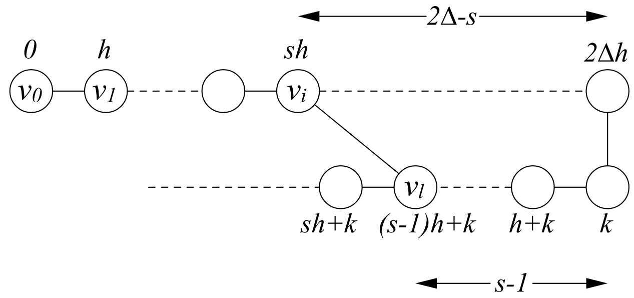


**Fig. 3.** Scheme for labeling vertices of a co-comparability graph.

We can summarize both previous results in the following theorem:

**Theorem 2.** *A co-comparability graph $G$ can be $L(h,k)$-labeled with span at most* $2\Delta \max\{h, 2k\} + k$.

## 4 The $L(h,k)$-Labeling of Interval Graphs

If the graph $G$ is an interval graph, we can exploit its particular umbrella property to derive better bounds on $\lambda_{h,k}(G)$.

First observe that the degree of any vertex $v_i$ adjacent to a vertex $v_l$, $v_i < v_l$, is at least $l - i$; furthermore, if $i \neq 0$ then the degree of $v_i$ is at least $l - i + 1$ if $G$ is connected, because at least one edge must reach $v_i$ from vertices preceding it in the ordering.

**Proposition 3.** *Given a connected interval graph of maximum degree $\Delta$, if $(v_i, v_l) \in E$ and $v_i < v_l$ then $l - i \leq \Delta$ and, if $i \neq 0$ then $l - i < \Delta$; if $v_i$ and $v_l$ are at distance 2 and $v_i < v_l$ then $l - i \leq 2\Delta - 1$.*

**Lemma 3.** *An interval graph $G$ can be $L(h,k)$-labeled with span at most $\Delta h$, if $k \leq \frac{h}{2}$.*

*Proof.* Without loss of generality, we focus on connected graphs. We proceed as in Lemma 1 with the difference that the set of labels is $0, h, 2h, \ldots, \Delta h, k, h+k, 2h+k, \ldots, (\Delta-1)h+k$.

**Distance 1.** Let $v_i$ and $v_l$ be adjacent vertices, assume by contradiction that their labels $l(v_i)$ and $l(v_l)$ differ by less than $h$. Then only two cases are possible:

(1.1) $l(v_i) = sh$, for some $s$, and $l(v_l)$ is either $sh+k$ or $(s-1)h+k$. Then $l-i \geq (\Delta-s)+(s-1)+1 = \Delta$. In view of Proposition 3 this is impossible because $G$ has maximum degree $\Delta$. If $i=0$ then $l$ can be at most $\Delta$; hence, $l(v_l) - l(v_i)$ is never smaller than $h$.

(1.2) $l(v_i) = sh+k$, for some $s$, and $l(v_l)$ is either $sh$ or $(s+1)h$. Then $i$ cannot be 0 and $l-i \geq (\Delta-1-s)+s+1 = \Delta$. This is in contradiction with Proposition 3.

**Distance 2.** Let $v_i$ and $v_l$ be vertices at distance two with labels $l(v_i)$ and $l(v_l)$ that differ by less than $k$. Since $k \leq \frac{h}{2}$ it must be $l(v_i) = l(v_l)$, and therefore $l - i = 2\Delta + 1$, i.e. the number of the different labels. This contradicts Proposition 3.

From the previous proof it easily follows:

**Corollary 1.** *If an interval graph $G$ has a canonical order such that the degree of $v_0$ is strictly less than $\Delta$, $G$ can be $L(h,k)$-labeled with span at most $(\Delta-1)h+k$, if $k \leq \frac{h}{2}$.*

The bound stated in the previous lemma is the best possible, as shown by the following:

**Theorem 3.** *There exists an interval graph requiring at least span $\Delta h$ to be $L(h,k)$-labeled.*

*Proof.* Consider $K_{\Delta+1}$, the clique on $\Delta+1$ vertices. As all vertices are adjacent a span of $\Delta h$ is necessary.

**Lemma 4.** *An interval graph $G$ can be $L(h,k)$-labeled with span at most $2k\Delta$, if $k \geq \frac{h}{2}$.*

*Proof.* The proof is analogous to the one of Lemma 3. The only difference is the ordered set of labels used: $0, 2k, 4k, \dots, 2k\Delta, k, 3k, 5k, \dots, 2k(\Delta-1)+k$.

Again, it easily follows:

**Corollary 2.** *If the canonical order of an interval graph $G$ is such that the degree of $v_0$ is strictly less than $\Delta$, $G$ can be $L(h,k)$-labeled with span at most $2k(\Delta-1)+k$, if $k \geq \frac{h}{2}$.*

Unfortunately, we are not able to exhibit an interval graph requiring at least span $2k\Delta$, if $k \geq \frac{h}{2}$, so it remains an open problem to understand if this result is tight or not.

We can summarize the previous results in the following theorem:

**Theorem 4.** *An interval graph $G$ can be $L(h,k)$-labeled with span at most* $\max(h,2k)\Delta$ *and, if $G$ has a canonical order such that the degree of $v_0$ is strictly less than $\Delta$, $G$ can be $L(h,k)$-labeled with span at most max$(h,2k)(\Delta-1)+k$.*

Next theorem allows us to compute another bound for $\lambda_{h,k}(G)$, for $G$ an interval graph, that involves $\omega(G)$, the number of vertices in the largest clique of $G$.

**Theorem 5.** *An interval graph $G$ can be $L(h,k)$-labeled with span at most $min((\omega-1)(2h+2k-2), \Delta(2k-1)+(\omega-1)(2h-2k))$.*

*Proof.* Consider the greedy algorithm designed as follows:

ALGORITHM `Greedy-Interval`

consider the canonical order $v_0, v_1, \dots v_{n-1}$

FOR $i = 0$ TO $n-1$ DO

label $v_i$ with the first available label, taking into account

the labels already assigned to neighbors of $v_i$

and to vertices at distance 2 from $v_i$.

At the $i$-th step of this $O(n^2)$ time algorithm, consider the vertex set $C_i$ constituted by all the labeled neighbors of $v_i$ and the vertex set $D_i$ constituted by all the labeled vertices at distance 2 from $v_i$. It is straightforward that $C_i \cap D_i = \emptyset$. As an example consider the graph of Figure 4, when $i = 3$ we have $C_3 = \{v_1, v_2\}$ and $D_3 = \{v_0\}$.

Let $v_{min}$ be the vertex in $C_i$ with the minimum index; in view of the umbrella property for interval graphs, $v_{min}$ is connected to all vertices inside $C_i$. On the other hand, each vertex $v_k$ in $D_i$ must be adjacent to some vertex in $C_i$, as it is at distance 2 from $v_i$; therefore the umbrella property implies that all vertices in $D_i$ are connected to $v_{min}$. It follows that $\Delta \geq d(v_{min}) \geq |D_i| + (|C_i| - 1) + 1$.

Observe also that both $C_i \cup \{v_i\}$ and $D_i \cup \{v_{min}\}$ are cliques, and hence $|C_i| \leq \omega - 1$ and $|D_i| \leq \omega - 1$.

So, when vertex $v_i$ is going to be labeled, each labeled vertex in $C_i$ forbids at most $2h - 1$ labels and each labeled vertex in $D_i$ forbids at most $2k - 1$ labels. Hence the number $f$ of forbidden labels is at most $|C_i|(2h - 1) + |D_i|(2k - 1)$. About $f$ we can also say:

$f \leq (\omega-1)(2h+2k-2)$ due to the inequalities $|C_i| \leq \omega-1$ and $|D_i| \leq \omega-1$;

$f \leq (|C_i| + |D_i|)(2k - 1) + |C_i|2(h - k) \leq \Delta(2k - 1) + (\omega - 1)(2h - 2k)$ for the inequalities $\Delta \geq |D_i| + |C_i|$ and $|C_i| \leq \omega - 1$.

As the previous reasoning does not depend on $i$, the maximum span is bounded by $\min((\omega - 1)(2h + 2k - 2), \Delta(2k - 1) + (\omega - 1)(2h - 2k))$.

In Figure 4 it is shown a graph $L(2, 1)$-labeled with the greedy algorithm. It is easy to see that in this case the bounds given in the previous theorem and arguments of the min function coincide, and are exactly equal to the required span.

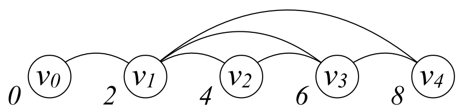


**Fig. 4.** A graph $L(2, 1)$-labeled with the greedy algorithm.

Observe that a trivial lower bound for $\lambda_{h,k}(G)$ is $(\omega - 1)h$. So, when $k = 1$ the previous theorem provides a 2-approximate algorithm for interval graphs, improving the approximation ratio of [3].

## 5 The $L(h,k)$-Labeling of Circular-Arc Graphs

Circular-arc graphs are a natural super-class of interval graphs, although they are not necessarily co-comparability graphs. In this section we exploit the results for interval graphs presented in the previous section for deriving bounds on $\lambda_{h,k}$ for circular-arc graphs.

**Definition 1.** *A circular-arc family $\mathcal{F}$ is a collection of arcs on a circle. A graph $G$ is a* circular-arc graph *if there is a circular- arc family $\mathcal{F}$ and a one-to-one mapping of the nodes in $G$ and the arcs in $\mathcal{F}$ such that two nodes in $G$ are adjacent if and only if their corresponding arcs in $\mathcal{F}$ overlap.*

Garey, Johnson, Miller and Papadimitriou [19] have shown that the minimum vertex coloring problem for circular-arc graphs is NP-hard, so even if circular-arc graphs are closed under taking powers [35] we cannot say anything about their $L(1,1)$-labeling.

In the following we show that it is possible to present results on the $L(h,k)$-labeling of a circular-arc graph by exploiting the $L(h,k)$-labeling of interval graphs.

Let us call a *cut* a straight line orthogonal to the circle that intersects a certain number of intervals. The removal of these intervals from the intersection model produces a new graph that is an interval graph, as illustrated in Figure 5. The set of intersected intervals corresponds to a clique.

**Theorem 6.** *A circular-arc graph $G$ can be $L(h,k)$-labeled with span at most $max(h,2k)\Delta + h\omega$.*

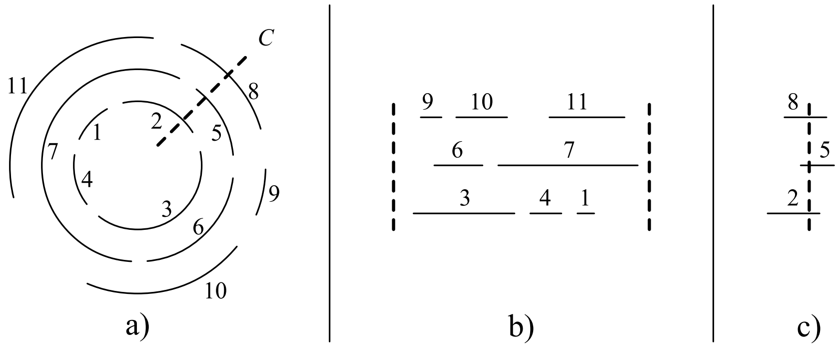


**Fig. 5.** a) A circular-arc graph $G$, a cut $C$ is highlighted; b) the interval graph obtained from $G$ removing all intervals intersected by the cut $C$; c) the clique corresponding to the cut $C$.

*Proof.* In order to prove the claim, we proceed in the following constructive way: first, we label the interval graph obtained by eliminating a cut from the circular-arc graph, where the choice of the cut is random. Then, we label the clique corresponding to the eliminated cut. Hence, the labeling algorithm is the following:

1. choose a cut $C$ of $G$;
2. eliminate all intervals of $G$ intersected by $C$; let $G'$ be the resulting interval graph;
3. label $G'$ with span $\lambda$ at most $\max(h, 2k)\Delta$;
4. label vertices of $C$ with at most $h(\omega - 1) + 1$ additional labels (this number of colors comes from the fact that a clique of dimension $k$ can be labeled with labels $0$, $h$, $\ldots$, $h(k-1)$); the $h(\omega - 1) + 1$ additional labels must be taken starting from $\lambda + h$;

5. the labeling of circular-arc graph $G$ is obtained by considering the labels of interval graph $G'$ and of clique $C$.

The second step is justified by Theorem 4. The $h\omega$ additional labels used in the third step guarantee that all the labels assigned to the nodes in the clique are at distance $h$ from each other and from the labels assigned to the interval graph.

It follows that the produced labeling is a feasible $L(h,k)$-labeling.

**Theorem 7.** *A circular-arc graph $G$ can be $L(h,k)$-labeled with span at most $min((3h+2k-2)\omega-(2h+2k-2), \Delta(2k-1)+\omega(3h-2k)-(2h-2k))$.*

*Proof.* The proof is exactly the same as the previous one, with the only difference that we label the interval graph according to Theorem 5 instead of Theorem 4.

When $k=1$ the previous theorem provides a 3-approximate algorithm for circular-arc graphs.

## 6 Concluding Remarks and Open Problems

In the literature there are no results concerning the $L(h,k)$-labeling of general co-comparability and circular-arc graphs. It is neither known whether the problem remains NP-complete when restricted to these classes or to some subclasses, as interval or unit-interval graphs.

In this paper we offered the first known algorithms to $L(h,k)$-label co-comparability, circular-arc and interval graphs with a bounded number of colors.

Namely, the following upper bounds on $\lambda_{h,k}$ are given:

$$\lambda_{h,k}(G) \leq \max(h, 2k)2\Delta + k$$

if $G$ is a co-comparability graph,

$$\lambda_{h,k}(G) \leq \max(h, 2k)\Delta + h\omega$$

if $G$ is a circular-arc graph, and

$$\lambda_{h,k}(G) \leq \max(h, 2k)\Delta$$

if $G$ is an interval graph.

Moreover, for interval graphs with certain restrictions, we have reduced this latter bound to $\max(h, 2k)(\Delta - 1) + k$. We have also shown a greedy algorithm that guarantees, for all interval graphs, a new upper bound on $\lambda_{h,k}(G)$ in terms of both $\omega$ and $\Delta$, that is:

$$\lambda_{h,k}(G) \leq \min((\omega - 1)(2h + 2k - 2), \Delta(2k - 1) + (\omega - 1)(2h - 2k))$$

This bound is provided by a 2-approximate algorithm, improving the approximation ratio in [3].

Moreover, we have exploited these results to get further upper bounds on $\lambda_{h,k}$ for circular-arc graphs. Namely, if $G$ is a circular-arc graph:

$$\lambda_{h,k}(G) \leq \min((3h+2k-2)\omega - (2h+2k-2), \Delta(2k-1) + \omega(3h-2k) - (2h-2k))$$

Finally, we have shown that the $L(1, 1)$-labeling problem is polynomially solvable for co-comparability graphs.

Many open problems are connected to this research. Here we list just some of them:

- Is the $L(h,k)$-labeling (or even only the $L(2,1)$-labeling) polynomially solvable on co-comparability graphs and on the other graph classes mentioned in the paper?
- Is it possible to find some lower bounds to understand how much our results are tight?
- What about the complexity of the $L(1,1)$-labeling on circular-arc graphs?
- What can we say about the $L(h,k)$-labeling of comparability graphs? It is easy to see that their $L(1,1)$-labeling is polynomially solvable as they are perfect and the square of a comparability graph is still a comparability graph; does the $L(2,1)$-labeling remain polynomially solvable?

Last, but not least, we wish to point out the connection between the linear orderings of co-comparability, interval and unit-interval graphs with a more general concept, namely that of a *dominating pair*, introduced by Corneil, Olariu and Stewart [11]. Considerable attention has been paid to exploiting the linear structure exhibited by various graph families. Examples include interval graphs [27], permutation graphs [16], trapezoid graphs [10, 13], and co-comparability graphs [23].

The linearity of these four classes is usually described in terms of ad-hoc properties of each of these classes of graphs. For example, in the case of interval graphs, the linearity property is traditionally expressed in terms of a linear order on the set of maximal cliques [6, 7]. For permutation graphs the linear behavior is explained in terms of the underlying partial order of dimension two [2], for co-comparability graphs the linear behavior is expressed in terms of the well-known linear structure of comparability graphs [26], and so on.

As it turns out, the classes mentioned above are all subfamilies of a class of graphs called the asteroidal triple-free graphs (AT-free graphs, for short). An independent set of three vertices is called an *asteroidal triple* if between any pair in the triple there exists a path that avoids the neighborhood of the third. AT-free graphs were introduced over three decades ago by Lekkerkerker and Boland [27] who showed that a graph is an interval graph if and only if it is chordal and AT-free. Thus, Lekkerkerker and Boland's result may be viewed as showing that the absence of asteroidal triples imposes the linear structure on chordal graphs that results in interval graphs. Recently, the authors [11] have studied AT-free graphs with the stated goal of identifying the "agent" responsible for the linear behavior observed in the four subfamilies. Specifically, in [11] the authors presented evidence that the property of being asteroidal triple-free is what is enforcing the linear behavior of these classes.

One strong "certificate" of linearity is the existence of a *dominating pair* of vertices, that is, a pair of vertices with the property that every path connecting them is a dominating set. In [11], the authors gave an existential proof of the fact that every connected AT-free graph contains a dominating pair.

In an attempt to generalize the co-comparability ordering while retaining the AT-free property, Corneil, Koehler, Olariu and Stewart [12] introduced the concept of *path orderable graphs*. Specifically, a graph $G = (V, E)$ is path orderable if there is an ordering $v_1, \ldots, v_n$ of its vertices such that for each triple $v_i, v_j, v_k$ with $i < j < k$ and $v_i v_k \notin E$, vertex $v_j$ intercepts each $v_i, v_k$-path of $G$; such an ordering is called a *path ordering*.

It is easy to confirm that co-comparability graphs are path orderable. It is also clear that path orderable graphs must be AT-free. It is a very interesting open question whether the results in this paper about the $L(h, k)$-labeling of co-comparability graphs can be extended to

– graphs that have an induced dominating pair, and/or
– graphs that are path orderable.

This promises to be an exciting area for further investigation.

## References


1. K. A. Aly and P. W. Dowd, A class of scalable optical interconnection networks through discrete broadcast-select multi-domain WDM, Proc. IEEE INFOCOM, 1994, pp. 392–399.
2. K. A. Baker, P. C. Fishburn, and F. S. Roberts, Partial orders of dimension two, Networks 2, (1971), 11–28.
3. A.A. Bertossi, C.M. Pinotti and R. Rizzi, Channel assignment on strongly-simplicial graphs, Proc. Int.l Parallel and Distributed Processing Symposium (IPDPS '03), 222b, 2003.
4. G.E. Blelloch, P.B. Gibbons, Y. Mattias, M. Zagha, Accounting for memory bank contentions and delay in high-bandwidth multiprocessors, IEEE Trans. on Parallel and Distributed Systems 8 (1997), 943–958.

5. J. A. Bondy, and U. S. R. Murty, Graph Theory with Applications, North-Holland, Amsterdam, 1976.

6. K. S. Booth and G. S. Lueker, Testing for the consecutive ones property, interval graphs and graph planarity using PQ-tree algorithms, Journal of Comput. Syst. Sci. 13 (1976), 335–379.

7. K. S. Booth and G. S. Lueker, A linear time algorithm for deciding interval graph isomorphism, Journal of the ACM 26 (1979), 183–195.

8. T. Calamoneri, Exact Solution of a Class of Frequency Assignment Problems in Cellular Networks and Other Regular Grids, Discrete Mathematics & Theoretical Computer Science 8 (2006), 141-158.

9. T. Calamoneri, The $L(h,k)$-labelling problem: a survey,The Computer Journal 49(5) (2006), 585–608. A continuously updated version is available on `http://www.dsi.uniroma1.it/~calamo/survey.html`

10. D.G. Corneil and P.A. Kamula, Extensions of permutation and interval graphs, Proc. 18th Southeastern Conference on Combinatorics, Graph Theory and Computing, 1987, pp. 267–276.

11. D. G. Corneil, S. Olariu, and L. Stewart, Asteroidal triple-free graphs, SIAM Journal on Discrete Mathematics 10 (1997), 399–430.

12. D. G. Corneil, E. Koehler, S. Olariu and L. Stewart, On Subfamilies of AT-Free Graphs, SIAM Journal on Discrete Mathematics 20(1) (2006), 241–253.

13. I. Degan, M.C. Golumbic, and R.Y. Pinter, Trapezoid graphs and their coloring, Discrete Applied Mathematics 21 (1988), 35–46.

14. P. Damaschke, Distance in cocomparability graphs and their powers, Disc. Applied Math 35 (1992), 67–72.

15. S.K. Das, I. Finocchi, and R. Petreschi, Star-coloring of graphs for conflict-free access to parallel memory systems, Proc. 18th International Parallel & Distributed Processing Symposium, 2004.

16. S. Even, A. Pnueli, and A. Lempel, Permutation graphs and transitive graphs, Journal of the ACM 19 (1972), 400–410.

17. M.R. Garey and D.S. Johnson, Computers and Intractability - A Guide to the Theory of NP-Completeness, W.H. Freeman and Co., 1979.

18. T. Gallai, Transitiv orientierbare Graphen, Acta Math. Acad. Sci. Hungar. 18 (1967), 25–66.

19. M.R. Garey, D.S. Johnson, G.L. Miller, and C.H. Papadimitriou, The complexity of coloring circular arcs and chords, SIAM Jornal of Algebraic and Discrete Methods 1 (1980), 185–200.

20. M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Academic Press, New York, 1980.

21. M.C. Golumbic, H. Kaplan, and R. Shamir, Graph sandwich problems, Journal of Algorithms 19 (1995), 449–473.

22. P.W. Goldberg, M.C. Golumbic, H. Kaplan, and R. Shamir, Four strikes against physical mapping of DNA, Journal of Computational Biology 2 (1995), 139–152.

23. M. C. Golumbic, C. L. Monma, and W. T. Trotter Jr., Tolerance graphs, Discrete Applied Mathematics 9 (1984), 157–170.

24. J.R. Griggs and R.K. Yeh, Labeling graphs with a condition at distance 2, SIAM Journal of Discrete Mathematics 5 (1992), 586–595.

25. T.R. Jensen and B.Toft, Graph Coloring Problems, John Wiley & Sons, New York, 1995.

26. D. Kratsch and L. Stewart, Domination on cocomparability graphs, SIAM Journal on Discrete Mathematics 6 (1993), 400–417.

27. C. G. Lekkerkerker and J. C. Boland, Representation of a finite graph by a set of intervals on the real line, Fundamenta Mathematicae 51, (1962), 45–64.
28. P. Looges and S. Olariu, Optimal Greedy Algorithms for Indifference Graphs, Computers and Mathematics with Application 25 (1993), 15–25.
29. S.T. McCormick, Optimal approximation of sparse Hessians and its equivalence to a graph coloring problem, Mathematical Programming 26 (1983), 153–171.
30. J. van den Heuvel, R.A. Leese, and M.A. Shepherd, Graph Labelling and Radio Channel Assignment, Journal of Graph Theory 29 (1998), 263–283.
31. R.M. Karp, Mapping the genome: some combinatorial problems arising in molecular biology, Proc. 25th Ann. ACM Symp. on Theory of Comp. (STOC '93), 1993, pp. 278–285.
32. S. Olariu, An optimal greedy heuristic to color interval graphs, Information Processing Letters 37(1) (1991), 21–25.
33. I. Pe'er and R. Shamir, Interval graphs with side (and size) constraints, Proc. European Symp. on Algorithms (ESA '95), Lecture Notes in Computer Science 979, 1995, pp. 142–154.
34. I. Pe'er and R. Shamir, Realizing interval graphs with side and distance constraints, SIAM Journal of Discrete Mathematics 10 (1997), 662–687.
35. A. Raychauduri, On Powers of Strongly Chordal and Circular Arc Graphs, Ars Combinatoria 34 (1992), 147–160.
36. A. Raychauduri, On powers of interval and unit interval graphs, Conressus Numererantium 59 (1987), 235–242.
37. D. J. Rose, R. E. Tarjan, G. S. Lueker, Algorithmic aspects of vertex elimination on graphs, SIAM Journal on Computing 5 (1976), 266–283.
38. D. Sakai, Labeling chordal graphs: distance two condition, SIAM Journal of Discrete Mathematics 7 (1994), 133–140.

39. H. D. Shapiro, Theoretical limitations on the efficient use of parallel memories, IEEE Transactions on Computers 17 (1978), 421–428.

40. R. E. Tarjan and M. Yannakakis, Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs and selectively reduce acyclic hypergraphs, SIAM Journal on Computing 13 (1984), 566–579.

41. P. J. Wan, Near-optimal conflict-free channel set assignments for an optical cluster-based hypercube network, Journal of Combinatorial Optimization 1 (1997), 179–186.