A General Approach to $L(h,k)$-Label Interconnection networks (Extended Abstract)

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Abstract

Given two non negative integers h and k, an $L(h,k)$-*labeling* of a graph $G=(V,E)$ is a function from the set V to a set of colors, such that adjacent nodes take colors at distance at least h and nodes at distance 2 take colors at distance at least k. The aim of the $L(h,k)$-labeling problem is to minimize the greatest used color. Since the decisional version of this problem is NP-complete, it is important to investigate particular classes of graphs for which the problem can be efficiently solved.

In this work the $L(h,k)$-labeling problem for a new class of graphs, called $(l \times n)$-*multistage graphs*, is studied. These graphs are characterized to have nodes organized as a matrix and they include the most common interconnection topologies, such as Butterfly-like, Beneš, CCC, Trivalent Cayley networks. The general algorithm presented in this paper leads to an efficient $L(2,1)$-labeling scheme for Butterfly and CCC networks.

Keywords: $L(h,k)$-Labeling, Interconnection Topologies, CCC Networks, Butterfly-like Networks, Trivalent Cayley Networks.

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1 Introduction

Graph coloring is one of the main topics in graph theory. Many generalizations of the notion of graph coloring are motivated by problems of channel assignment in wireless communications, traffic phasing, fleet maintenance, task assignment, and other applications. (See [7] for a survey.)

Although in classical vertex coloring of graphs [4] a condition is imposed only on colors of adjacent nodes, many generalizations require colors to respect stronger conditions, e.g. restrictions are imposed on colors both of adjacent nodes and of nodes at distance 2 in the graph.
This paper will focus on a specific graph coloring generalization that arose first from a channel assignment problem in radio networks: the *$L(h, k)$-labeling problem*. This notion was introduced by Griggs and Yeh in the special case $h = 2$ and $k = 1$ in connection with the problem of assigning frequencies in a multihop radio network [3]. Formally:

Definition 1.1 Given two non negative integers h and k, an *$L(h, k)$-labeling* of a graph $G = (V, E)$, is a function C from the node set V to a set of nonnegative integers, called colors, such that:

(i) $|C(x) - C(y)| \geq h$ if $\{x, y\} \in E$ (i.e. $\text{dist}(x, y) = 1$) and

(ii) $|C(x) - C(y)| \geq k$ if $\text{dist}(x, y) = 2$.

The aim of the $L(h, k)$-labeling problem is to minimize the greatest used color.

The *$L(h, k)$-labeling* is a special case of the following notion of $L(m_1, \ldots, m_p)$-labeling [3]:

Definition 1.2 Given a positive integer p, and p non negative integers $m_1, m_2, \ldots, m_p$, an *$L(m_1, \ldots, m_p)$-labeling* of a graph $G = (V, E)$, is a function C from the node set V to a set of nonnegative integers such that $|C(x) - C(y)| \geq m_i$ if x and y are at distance i. The aim is to minimize the greatest used color.

The decisional version of the $L(h, k)$-labeling problem is NP-complete even for small values of h and k [3]. Therefore, since the seminal work of Griggs and Yeh, researchers have produced a wide literature studying the problem for special values of h and k on special classes of graphs (for a survey see [1]).

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In this work we study the $L(h,k)$-labeling problem on a class of graphs, called $(l \times n)$-*multistage graphs*, containing the most common interconnection topologies such as Butterfly-like networks, Beneš networks, CCC networks, and Trivalent Cayley networks. A general approach for $L(h,k)$-labeling these topologies is presented, and from this method an efficient $L(2,1)$-labeling for Butterfly and CCC networks is derived.

2 $L(h,k)$-Labeling Multistage Rows-Bipartite Graphs

In this section we define the $(l \times n)$-multistage graph and its quotient graph, then we focus our attention on the subclass of bipartite multistage graphs. Working on the partition of nodes in rows, we present a method for $L(h,k)$-labeling this class of graphs. Our approach reduces the problem of labeling a given multistage row-bipartite graph of l rows to the problem of labeling a multistage row-bipartite graph of only two rows, implying a strong reduction of complexity.

Let us introduce some definitions:

Definition 2.1 Let V be a set of $l \times n$ nodes organized as a matrix; let $\mathcal{R} = \{R_1, R_2, \ldots, R_l\}$ be the set of the matrix rows and let $r_{i,j}$ be the j-th node of row R_i. An $(l \times n)$-*multistage graph* G is a simple loopless graph whose node set is V and such that the subgraph induced by each R_i ($i = 1, \ldots, l$) is either a simple ordered path $r_{i,1}, r_{i,2}, \ldots, r_{i,n}$ or a simple ordered cycle $r_{i,1}, r_{i,2}, \ldots, r_{i,n}, r_{i,1}$.

Given an $(l \times n)$-multistage graph G, let us consider the *quotient graph* $G/\mathcal{R}$ resulting from the following contracting operation:

- each row R_i of G is a node i of $G/\mathcal{R}$;
- an edge connects nodes i and j of $G/\mathcal{R}$ if and only if there exists an edge (u,v) in G such that $u \in R_i$ and $v \in R_j$.

Definition 2.2 An $(l \times n)$-multistage graph G is *rows-bipartite* if and only if its quotient graph $G/\mathcal{R}$ is bipartite. We call *Upper* and *Lower* the sets of rows of G corresponding to the two classes in which nodes of $G/\mathcal{R}$ are bipartitioned.

Definition 2.3 The *reduced graph* $R(G)$ of an $(l \times n)$-multistage rows-bipartite graph G is a simple graph with $2n$ nodes partitioned into two rows U and L containing nodes $u_1, u_2, \ldots, u_n$ and $l_1, l_2, \ldots, l_n$, respectively. The edges of $R(G)$ are:

(i) all edges (u_i, u_{i+1}) and (l_i, l_{i+1}), $i = 1, \ldots, n-1$ *(straight edges)*;

(ii) edge (u_n, u_1) if and only if at least one row in *Upper* induces a cycle in G and edge (l_n, l_1) if and only if at least one row in *Lower* induces a cycle in G *(closing edges)*;

(iii) edge (u_i, l_j) if and only if there exist two rows $R_a \in$ *Upper* and $R_b \in$ *Lower* such that $(r_{a,i}, r_{b,j})$ is an edge in G *(cross edges)*.

An example of a multistage graph is depicted in Figure 1a). Figures 1b) and 1c) represent its quotient graph and its reduced graph respectively.

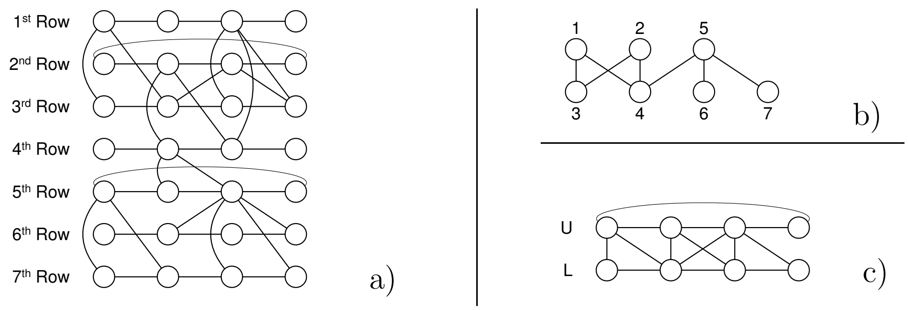

Fig. 1. a) A (7×4)-multistage graph G. b) The quotient graph G/R. c) The reduced graph $R(G)$.

Remark 2.4 Definition 2.3 becomes much shorter if we define the mapping μ from nodes of G to nodes of $R(G)$ as follows:

$$\mu(r_{i,j}) = \begin{cases} u_j \text{ if } R_i \in Upper \\ l_j \text{ if } R_i \in Lower \end{cases}$$

In this way the edge set of $R(G)$ is $\{(\mu(x), \mu(y))$ s.t. (x, y) is an edge of $G\}$.

Theorem 2.5 *Let G be an $(l \times n)$-multistage rows-bipartite graph and $R(G)$ be its reduced graph. If $R(G)$ admits an $L(h, k)$-labeling using colors from 0 to c then G admits an $L(h, k)$-labeling using colors from 0 to c.*

Proof. Let C_R be a feasible $L(h, k)$-labeling for $R(G)$ using colors from 0 to c. A labeling C_G for G can be derived from C_R using the mapping μ: $C_G(r_{i,j}) = C_R(\mu(r_{i,j}))$. By definition of $R(G)$, if x, y are adjacent nodes in G then $\mu(x), \mu(y)$ are adjacent nodes in $R(G)$, and if x, y are at distance 2 in G then $\mu(x), \mu(y)$ are at distance 2 in $R(G)$. As a consequence, C_G is a feasible $L(h, k)$-labeling for G. □

The previous theorem provides a simple algorithmic technique for computing an $L(h, k)$-labeling of any $(l \times n)$-multistage rows-bipartite graph. The computational complexity of this technique depends on the time necessary to compute the $L(h,k)$-labeling of $R(G)$: the worst case happens when we use an exhaustive approach, and this requires $O(2^n)$ time. It is to notice that if $l = O(2^n)$, this time complexity is linear in the number of nodes of G.

Remark 2.6 For any graph G, with maximum degree $\Delta(G)$, the number of necessary colors to $L(h,k)$-label G is never less than $h + (\Delta(G) - 1)k$ when $h \geq k$. Since graph $R(G)$ may be dense even if graph G is sparse, this implies that $\Delta(R(G))$ may be much bigger than $\Delta(G)$ and then the greatest used color is not necessarily minimized.

3 Interconnection Topologies

An *interconnection network* consists of a collection of processors or routing-switches with direct connections between them. An *interconnection topology* is the undirect graph underlying an interconnection network: it has a node for each processor/switch and an edge between each pair of connected processors/switches. A large variety of topologies have been investigated: the interested reader can refer to [5] for a wide survey.

In this section we show how to exploit Theorem 2.5 to $L(h,k)$-label interconnection topologies. Indeed, looking at the classical representation of most of the common interconnection topologies, it is quite simple to see that most of them belong to the class of row-bipartite multistage graphs. See for example the classical bidimensional layout of the Cube-Connected-Cycles (CCC) network [6] and the classical representation of the Butterfly network [5].

Remark 3.1 In Definition 2.1 the nodes of the graph are organized as a matrix: in the following we choose the matrix naturally induced by the classical representation of an interconnection topology.

Theorem 2.5 allows us to $L(h,k)$-label each interconnection topology G that is $(l \times n)$-multistage rows-bipartite, in the following simple way:

(i) compute the reduced graph $R(G)$ associated to G;

(ii) compute an $L(h,k)$-labeling of $R(G)$;

(iii) assign to each node $r_{i,j}$ of G the color of its corresponding node $\mu(r_{i,j})$ of $R(G)$.

It is not difficult to see that steps 1 and 3 work in linear time with respect to the size of G. We remark that we are dealing with interconnection topologies whose maximum degree is bounded by a constant, then the size of G is linear with respect to the number of its nodes.

The complexity of step 2 is related to how the algorithm for $L(h,k)$-labeling $R(G)$ works; however, even if the algorithm searches the solution in an exhaustive way, this step does not require more then $O(2^n)$ time, where n is the number of columns of G. Since we are considering interconnection topologies whose number l of rows is exponential with respect to the number n of the columns, this step is linear with respect to the size of G.

The worth of our approach consists in coloring a graph of only $2n$ nodes instead of a graph of $O(n2^n)$ nodes and it allows us to $L(h,k)$-label an interconnection topology in linear time with respect to its size.
We have to highlight that the considered interconnection topologies are often characterized by a high symmetry. This implies that the density of their reduced graphs is of the same order of their density and then, despite the Remark 2.6, the size of our solution is nearly optimal. The results stated in the following in the special case $h=2$ and $k=1$ for two well known interconnection topologies will confirm this statement (due to the lack of space, we omit proofs.).

Theorem 3.2 *Given an n-dimensional Cube-Connected-Cycles network G, there exists an $L(2,1)$-labeling of G that is optimal if n is multiple of 3, and that uses a number of colors 1 far from optimal otherwise.*

Theorem 3.3 *Given an N-input Butterfly network G, there exists an $L(2,1)$-labeling of G that is optimal if N is either 2^2 or 2^3, and that uses a number of colors 1 far from optimal otherwise.*

We have seen that most of the interconnection topologies are rows-bipartite. Only few of them – such as the 3-ary Butterfly (see [5] pp. 740) and Trivalent Cayley networks [2,8] – have not this property: indeed their reduced graphs are 3- and 4-partite, respectively. It is possible to introduce a generalization of our technique in order to $L(m_1,\ldots,m_{p'})$-label those topologies whose reduced graph is p-partite ($p' \leq p$):

Definition 3.4 An $(l \times n)$-multistage graph G is *rows-p-partite* if and only if its quotient graph $G/\mathcal{R}$ is p-partite. We call $S_1, S_2, \ldots, S_p$ the sets of rows of G corresponding to the classes in which nodes of $G/\mathcal{R}$ are partitioned.

Definition 3.5 The *reduced graph* $R(G)$ of an $(l \times n)$-multistage rows-p-partite graph G is a graph with $p \times n$ nodes partitioned into p rows containing nodes $v_1^i, v_2^i, \ldots, v_n^i$ ($1 \leq i \leq p$). The set of edges of $R(G)$ is $\{(\mu(x),\mu(y))$ s.t. (x,y) is an edge of $G\}$ where: $\mu(r_{a,b}) = v_a^i$ s.t. $R_b \in S_i$.

Theorem 3.6 *Let G be an $(l \times n)$-multistage rows-p-partite graph and $R(G)$ its reduced graph. If $R(G)$ admits an $L(m_1, \ldots, m_{p'})$-labeling ($p' \leq p$) using colors from 0 to c then G admits an $L(m_1, \ldots, m_{p'})$-labeling using colors from 0 to c.*

References

[1] T. Calamoneri. The $L(h, k)$-Labelling Problem: a Survey. *Technical Repeport 04/2004 Department of Computer Science, University of Rome "La Sapienza"*, 2004. URL: http://www.dsi.uniroma1.it/∼calamo/survey.html

[2] T. Calamoneri and R. Petreschi. A new 3D representation of Trivalent Cayley Networks. *Information Processing Letters*, 61, pp. 247–252, 1997.

[3] J.R. Griggs and R.K. Yeh. Labeling graphs with a condition at distance 2. *SIAM Journal of Discrete Mathematics* 5, pp. 586-595, 1992.

[4] T.R. Jensen and B. Toft. *Graph coloring problems*. John Wiley & Sons, New York, 1995.

[5] F.T. Leighton. *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes*. Morgan Kaufmann, San Mateo, 1992.

[6] F.P. Preparata and J. Vuillemin. The Cube-Connected Cycles: A Versatile Network for Parallel Computation. *Communications of ACM*, 24(5), pp 300–309, 1981.

[7] F.S. Roberts. From Garbage to Rainbows: Generalizations of Graph Coloring and their Applications. In *Graph Theory, Combinatorics, and Applications*, Y. Alavi, G. Chartrand, O.R. Oellermann, and A.J. Schwenk (eds.), 2, pp. 1031-1052, Wiley, New York, 1991.

[8] P. Vadapalli and P.K. Srimani. Trivalent Cayley Graphs for Interconnection Networks. *Information Processing Letters*, 54, pp 329–335,1995.