# Constructing bipartite Ramsey graphs from affine extractors

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A bipartite graph over vertex sets $L$ and $R$, $|L| = |R| = N$, is said to be an $M$-Ramsey graph if for every $S \subset L$ and $T \subset R$ satisfying $|S| = |T| = M$, the bipartite graph induced by $S$ and $T$ is neither empty, nor complete. A celebrated result of Erdős from 1947 shows that a random bipartite graph is, with high probability, a $(2 \log N)$-Ramsey graph but constructing explicit Ramsey graphs for small values of $M$ is a notoriously hard problem. In fact, even going below $M = N^{\frac{1}{2}}$ was only recently obtained by [Pudlak, Rődl] and by [Barak, Kindler, Rao, Shaltiel, Sudakov, Wigderson].

We give a new explicit construction of bipartite Ramsey graphs for values $M = N^{\frac{1}{2}-\epsilon}$ for some $\epsilon > 0$. Assuming a conjecture from additive combinatorics known as the "polynomial Freiman-Ruzsa conjecture" we show that our construction is in fact a "two-source extractor" which means that any graph induced by two vertex sets of size $M$ each has close to $M/2$ induced edges, i.e., it looks like a random bipartite graph.

Our construction essentially converts any "affine extractor" (see below) with sufficiently good parameters into a bipartite $N^{\frac{1}{2}-\epsilon}$–Ramsey graph. We point out that explicit constructions of such extractors were recently obtained by Bourgain. Given an affine extractor, the Ramsey graph obtained from it is fairly easy to describe and we give this description after defining formally the notion of an affine extractor.

An $(d, n, m, \delta)$-affine extractor is a "pseudorandom function" for affine spaces. Formally, it is a function $f : F_2^n \to F_2^m$, where $F_2$ denotes the two-element field, such that for any nontrivial additive character $\chi : F_2^m \to \{-1, 1\}$ and any $d$-dimensional affine subspace $A$ of $F_2^n$ we have

$$|E_{a \in A} [\chi(a)]| \leq \delta.$$

Given a $(d, n, m, \delta)$–affine extractor, our bipartite graph over vertex sets $L = R = F_2^n$ is as follows. Connect $x \in L$ to $y \in R$ if and only if

$$\sum_{i=1}^{n} x_i \cdot y_i + \sum_{j=1}^{m} (f(x))_j \cdot (f(y))_j \equiv 0 \mod 2.$$