

Rectangle Free Coloring of Grids

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Credit Where Credit is Due

This Work Grew Out of a Project In the UMCP SPIRAL (Summer Program in Research and Learning) Program. Program was for College Math Majors at HBCU's.

One of the students, **Brett Jefferson** has his own paper on this subject.

ALSO: Multidim version has been worked on by Cooper, Fenner, Purewal (submitted)

Square Theorem:

Theorem

For all c , there exists G such that for every c -coloring of $G \times G$ there exists a monochromatic square.

...
...	R	...	R	...
...	\vdots	...	\vdots	...
...	R	...	R	...
...

Square Theorem:

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...
...	R	...	R	...
...	\vdots	...	\vdots	...
...	R	...	R	...
...

How to prove?

1. Corollary of Gallai's theorem [3,4,6]. Bounds on G HUGE!
2. From VDW directly (folklore). Bounds on G HUGE!
3. Directly (folklore?). Bounds on G HUGE!
4. Graham and Solymosi [2]. Bounds on G huge! But smaller.

What If We Only Care About Rectangles?

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $G_{n,m}$ is **c-colorable** if there is a c-colorings of $G_{n,m}$ such that no rectangle has all four corners the same color.
2. $\chi(G_{n,m})$ is the least c such that $G_{n,m}$ is c-colorable.

Our Main Question

Fix c

Exactly which $G_{n,m}$ are c -colorable?

Two Motivations!

1. Relaxed version of Square Theorem- hope for better bounds.
2. Coloring $G_{n,m}$ without rectangles is equivalent to coloring edges of $K_{n,m}$ without getting monochromatic $K_{2,2}$.

Our results yield **Bipartite Ramsey Numbers!**

Theorem

For all c there exists a unique finite set of grids OBS_c such that

$G_{n,m}$ is c -colorable *iff*

$G_{n,m}$ does not contain any element of OBS_c .

1. Can prove using well-quasi-orderings. No bound on $|\text{OBS}_c|$.
2. Our tools yield alternative proof and show

$$2\sqrt{c}(1 - o(1)) \leq |\text{OBS}_c| \leq 2c^2.$$

Rephrase Main Question

Fix c

What is OBS_c

Rectangle Free Sets and Density

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $X \subseteq G_{n,m}$ is **Rectangle Free** if there are NOT four vertices in X that form a rectangle.
2. $\text{rfree}(G_{n,m})$ is the size of the largest Rect Free subset of $G_{n,m}$.

Rectangle Free subset of $G_{21,12}$ of size $63 = \left\lceil \frac{21 \cdot 12}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10	11	12
1	•	•										
2	•		•									
3		•	•									
4			•	•	•							
5		•		•		•						
6	•				•	•						
7						•	•	•				
8					•		•		•			
9				•				•	•			
10						•				•	•	
11					•					•		•
12				•							•	•
13			•			•			•			•
14			•					•		•		
15			•				•				•	
16		•							•	•		
17		•			•			•			•	
18		•					•					•
19	•								•		•	
20	•							•				•
21	•			•			•			•		

Colorings Imply Rectangle Free Sets

Lemma

Let $n, m, c \in \mathbb{N}$. If $\chi(G_{n,m}) \leq c$ then $\text{rfree}(G_{n,m}) \geq \lceil mn/c \rceil$.

Note: We use to get non-col results as density results!!

Zarankiewics's Problem

Definition

$Z_{a,b}(m, n)$ is the largest subset of $G_{n,m}$ that has no $[a] \times [b]$ submatrix.

Zarankiewics [7] asked for exact values for $Z_{a,b}(m, n)$.
We care about $Z_{2,2}(m, n)$.

We will **EXACTLY** Characterize which $G_{n,m}$ are 2-colorable!

$G_{5,5}$ IS NOT 2-Colorable!

Theorem

$G_{5,5}$ *is not* 2-Colorable.

Proof:

$$\begin{aligned}\chi(G_{5,5}) = 2 &\implies \text{rfree}(G_{5,5}) \geq \lceil 25/2 \rceil = 13 \\ &\implies \text{there exists a column with } \geq \lceil 13/5 \rceil = 3 \text{ } R\text{'s}\end{aligned}$$

$G_{5,5}$ IS NOT 2-Colorable (Continued)

Case 1: Max in a column is 3 R 's.

Case 1a: There are ≤ 2 columns with 3 R 's.

Number of R 's $\leq 3 + 3 + 2 + 2 + 2 \leq 12 < 13$.

Case 1b: There are ≥ 3 columns with 3 R 's.

R	○	○	○	○
R	○	○	○	○
R	R	○	○	○
○	R	○	○	○
○	R	○	○	○

Can't put in a third column with 3 R 's!

$G_{5,5}$ IS NOT 2-Colorable (Continued)

Case 2: There is a column with ≥ 4 .
Easy exercise to show can't have 13.

Theorem

$G_{4,6}$ *is* 2-Colorable.

Proof.

<i>R</i>	<i>R</i>	<i>R</i>	<i>B</i>	<i>B</i>	<i>B</i>
<i>R</i>	<i>B</i>	<i>B</i>	<i>R</i>	<i>R</i>	<i>B</i>
<i>B</i>	<i>R</i>	<i>B</i>	<i>R</i>	<i>B</i>	<i>R</i>
<i>B</i>	<i>B</i>	<i>R</i>	<i>B</i>	<i>R</i>	<i>R</i>



$G_{3,7}$ IS NOT 2-Colorable

Theorem

$G_{3,7}$ *is not* 2-Colorable.

Proof.

$$\begin{aligned}\chi(G_{3,7}) = 2 &\implies \text{rfree}(G_{3,7}) \geq (\lceil 21/2 \rceil = 11 \\ &\implies \text{there is a row with } \geq \lceil 11/3 \rceil = 4 \text{ } R\text{'s}\end{aligned}$$

Proof similar to $G_{5,5}$ — lots of cases.



Complete Char of 2-Colorability

Theorem

$$\text{OBS}_2 = \{G_{3,7}, G_{5,5}, G_{7,3}\}.$$

Proof.

Follows from results $G_{5,5}$, $G_{7,3}$ not 2-colorable and $G_{4,6}$ is 2-colorable. □

We show that if A is a Rectangle Free subset of $G_{n,m}$ then there is a relation between $|A|$ and n and m .

Bound on Size of Rectangle Free Sets

Theorem

Let $n, m \in \mathbb{N}$. If there exists rectangle-free $A \subseteq G_{n,m}$ then

$$|A| \leq \frac{m + \sqrt{m^2 + 4m(n^2 - n)}}{2}$$

Note: Proved by Reiman [5] while working on Zarankiewicz's problem.

Bound on Size of Rectangle Free Sets (new)

Theorem

Let $a, n, m \in \mathbb{N}$. Let q, r be such that $a = qn + r$ with $0 \leq r \leq n$. Assume that there exists $A \subseteq G_{m,n}$ such that $|A| = a$ and A is rectangle-free.

1. If $q \geq 2$ then

$$n \leq \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$

2. If $q = 1$ then

$$r \leq \frac{m(m-1)}{2}.$$

Ideas Used in Proof

$A \subseteq G_{n,m}$, rectangle free.

x_i is number of points in i^{th} column.

	1	\dots	m
1		\dots	
\vdots		\vdots	
n		\dots	
	x_1 points $\binom{x_1}{2}$ pairs of points	\dots \dots	x_m points $\binom{x_m}{2}$ pairs of points

Ideas Used in Proof

$A \subseteq G_{n,m}$, rectangle free.

x_i is number of points in i^{th} column.

	1	...	m
1		...	
\vdots		\vdots	
n		...	
	x_1 points $\binom{x_1}{2}$ pairs of points	...	x_m points $\binom{x_m}{2}$ pairs of points

$$\sum_{i=1}^m \binom{x_i}{2} \leq \binom{n}{2}.$$

We define and use Strong c -Colorings to get c -Colorings

Strong c -Colorings

Definition

Let $c, n, m \in \mathbb{N}$. $\chi : G_{n,m} \rightarrow [c]$. χ is a **strong c -coloring** if the following holds: CANNOT have a rectangle with the two right most corners are same color and the two left most corners the same color.

Example: A strong 3-coloring of $G_{4,6}$.

R	R	G	R	G	G
B	G	R	G	R	G
G	B	B	G	G	R
G	G	G	B	B	B

Strong Coloring Lemma

Let $c, n, m \in \mathbb{N}$. If $G_{n,m}$ is strongly c -colorable then $G_{n,cm}$ is c -colorable.

Example:

R	R	G	R	G	G	B	B	R	B	R	R	G	G	B	G	B	B
B	G	R	G	R	G	G	R	B	R	B	R	R	B	G	B	G	B
G	B	B	G	G	R	R	G	G	R	R	B	B	R	R	B	B	G
G	G	G	B	B	B	R	R	R	G	G	G	B	B	B	R	R	R

Combinatorial Coloring Theorem

Let $c \geq 2$.

1. There is a strong c -coloring of $G_{c+1, \binom{c+1}{2}}$.
2. There is a c -coloring of $G_{c+1, m}$ where $m = c \binom{c+1}{2}$.

Example: Strong 5-coloring of $G_{6,15}$.

O	O	O	O	O	R	R	R	R	R	R	R	R	R	R
O	R	R	R	R	O	O	O	O	B	B	B	B	B	B
R	O	B	B	B	O	B	B	B	O	O	O	G	G	G
B	B	O	G	G	B	O	G	G	O	G	G	O	O	P
G	G	G	O	P	G	G	O	P	G	O	P	O	P	O
P	P	P	P	O	P	P	P	O	P	P	O	P	O	O

Coloring Using Primes!

Theorem

Let p be a prime.

1. There is a strong p -coloring of $G_{p^2, p+1}$.
2. There is a p -coloring of G_{p^2, p^2+p} .

Proof.

Uses geometry over finite fields. □

Note: Have more general theorem.

Using a Generalization of Strong Coloring

Theorem

Let $c \geq 2$.

1. There is a c -coloring of $G_{c+2, m'}$ where $m' = \lfloor c/2 \rfloor \binom{c+2}{2}$.
2. There is a c -coloring of $G_{2c, 2c^2 - c}$.

We will **EXACTLY** Characterize which $G_{n,m}$ are **3-colorable!**

Theorem

1. *The following grids are not 3-colorable.*

$G_{4,19}$, $G_{19,4}$, $G_{5,16}$, $G_{16,5}$, $G_{7,13}$, $G_{13,7}$, $G_{10,12}$, $G_{12,10}$, $G_{11,11}$.

2. *The following grids are 3-colorable.*

$G_{3,19}$, $G_{19,3}$, $G_{4,18}$, $G_{18,4}$, $G_{6,15}$, $G_{15,6}$, $G_{9,12}$, $G_{12,9}$.

Proof.

Follows from tools.



$G_{10,10}$ is 3-colorable

Theorem

$G_{10,10}$ is 3-colorable.

Proof.

UGLY! TOOLS DID NOT HELP AT ALL!!

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

$G_{10,11}$ is not 3-colorable

Theorem

$G_{10,11}$ is not 3-colorable.

Proof.

You don't want to see this. UGLY case hacking. □

Complete Char of 3-colorability

Theorem

$\text{OBS}_3 =$

$$\{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\}.$$

Proof.

Follows from above results on grids being or not being 3-colorable. □

We will **MAKE PROGRESS ON** Characterizing which $G_{n,m}$ are 4-colorable.

Theorem

1. The following grids *are NOT* 4-colorable:

$G_{5,41}$, $G_{41,5}$, $G_{6,31}$, $G_{31,6}$, $G_{7,29}$, $G_{29,7}$, $G_{9,25}$, $G_{25,9}$, $G_{10,23}$,
 $G_{23,10}$, $G_{11,22}$, $G_{22,11}$, $G_{13,21}$, $G_{21,13}$, $G_{17,20}$, $G_{20,17}$, $G_{18,19}$,
 $G_{19,18}$.

2. The following grids *are* 4-colorable:

$G_{4,41}$, $G_{41,4}$, $G_{5,40}$, $G_{40,5}$, $G_{6,30}$, $G_{30,6}$, $G_{8,28}$, $G_{28,8}$, $G_{16,20}$,
 $G_{20,16}$.

Proof.

Follows from tools.



Theorems with UGLY Proofs

Theorem

1. $G_{17,19}$ *is NOT 4-colorable: Used some tools.*
2. $G_{24,9}$ *is 4-colorable: Used strong coloring of $G_{9,6}$.*

Theorems with UGLY Proofs

Theorem

1. $G_{17,19}$ *is NOT* 4-colorable: Used some tools.
2. $G_{24,9}$ *is* 4-colorable: Used strong coloring of $G_{9,6}$.

P	R	R	P	R	R
P	B	B	R	P	B
P	G	G	B	B	P
R	P	G	P	G	R
B	P	R	B	P	G
G	P	B	G	R	P
G	B	P	P	B	G
R	G	P	G	P	R
B	R	P	R	G	P

Rectangle Free Conjecture

Recall the following lemma:

Lemma

Let $n, m, c \in \mathbb{N}$. If $\chi(G_{n,m}) \leq c$ then $\text{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$.

Rectangle Free Conjecture

Recall the following lemma:

Lemma

Let $n, m, c \in \mathbb{N}$. If $\chi(G_{n,m}) \leq c$ then $\text{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$.

Rectangle-Free Conjecture (RFC) is the converse:

Let $n, m, c \geq 2$. If $\text{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$ then $G_{n,m}$ is c -colorable.

Theorem

If RFC then

$$\text{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\} \cup \\ \{G_{13,21}, G_{11,22}, G_{10,23}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}\}.$$

Proof.

Follows from known 4-colorability and non-4-colorability results, and from some Rect Free Sets we found by computer search. \square

Theorem

(Bipartite Ramsey Theorem) For all a, c there exists $n = BR(a, c)$ such that for all c -colorings of the edges of $K_{n,n}$ there will be a monochromatic $K_{a,a}$. (See Graham-Rothchild-Spencer [1] for history and refs.)

PART VI: BIPARTITE RAMSEY THEORY

Theorem

(Bipartite Ramsey Theorem) For all a, c there exists $n = BR(a, c)$ such that for all c -colorings of the edges of $K_{n,n}$ there will be a monochromatic $K_{a,a}$. (See Graham-Rothchild-Spencer [1] for history and refs.)

Equivalent to:

Theorem

For all a, c there exists $n = BR(a, c)$ so that for all c -colorings of $G_{n,n}$ there will be a monochromatic $a \times a$ submatrix.

Theorem

1. $BR(2, 2) = 5$. (Already known.)
2. $BR(2, 3) = 11$.
3. $17 \leq BR(2, 4) \leq 19$.
4. $BR(2, c) \leq c^2 + c$.
5. If p is a prime and $s \in \mathbb{N}$ then $BR(2, p^s) \geq p^{2s}$.
6. For almost all c , $BR(2, c) \geq c^2 - c^{1.525}$.

PART VII: OPEN QUESTIONS

1. Is $G_{17,17}$ 4-colorable? (Other 4-col also open.)
2. What is OBS_4 ? OBS_5 ?
3. Prove or disprove **Rectangle Free Conjecture**.
4. Have $\Omega(\sqrt{c}) \leq |OBS_c| \leq O(c^2)$. Get better bounds!
5. Refine tools so can prove **ugly** results **cleanly**.

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