Proof mining in ergodic theory - a survey

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Proof Mining

The idea of proof mining is to use proof theoretic techniques to extract additional information from sufficiently formal proofs, in particular from existence proofs in mathematics.

Additional information can be:

- Quantitative - algorithms, bounds.
- Qualitative - uniformities, weakening of premises.

Proof Mining

The main technique in proof mining are proof interpretations:

Given a formal system – a language, constants, axioms and rules – we want to give computational interpretations of:

- constants by some computational constants or terms,
- axioms by suitable terms realizing existential quantifiers,
- derivation rules by rules for combining realizers.

Then we can give computational interpretations of formal proofs and the theorems they prove.

Proof Mining - Metatheorems

Based on Gödel's ('Dialectica') functional interpretation, one may develop general logical metatheorems that describe classes of theorems and proofs from which additional information may be extracted.

These metatheorems both give a-priori criteria for when and what kind of information – bounds, uniformities, etc. – may be extracted, as well as describing an algorithm for the extraction.

These metatheorems cover theories for intutionistic and classical arithmetic, but extend to full classical analysis and also abstract metric spaces, normed linear spaces, Hilbert spaces, etc.

Proof Mining - Example

The principle of convergence for bounded monotone sequences – short: PCM – says the following:

Every bounded monotone sequence of real numbers converges.

More formally, this can be expressed as:

$$\forall (a_n)_{n\in \mathbb{N}} \forall b \in \mathbb{N} \forall \varepsilon > 0 \exists n \in \mathbb{N} \forall m_1, m_2 > n \\ \big(\forall k (a_k \leq a_{k+1} \leq b) \to |a_{m_1} - a_{m_2}| \leq \varepsilon\big).$$

Can we compute a rate of convergence?

Proof Mining - Example

It is well known that there exist computable bounded monotone sequences of rational numbers that do not converge to a computable limit, i.e. there is no computable rate of convergence.

However, PCM is classically equivalent to:

$$\forall (a_n)_{n\in \mathbb{N}} \forall b \in \mathbb{N} \forall \varepsilon > 0 \forall M : \mathbb{N} \to \mathbb{N} \exists n \in \mathbb{N} \forall m_1, m_2 \in [n, M(n)] \\ \big(\forall k (a_k \leq a_{k+1} \leq b) \to |a_{m_1} - a_{m_2}| \leq \varepsilon\big),$$

which is the Dialectica transform of PCM. This version is also known as the no-counterexample version of PCM and this weakened form of convergence has been called local stability.

Here, a bound on n in the parameters ε, b, M is computed easily.

Ergodic Theory

Ergodic theory studies the long-time and limit behaviour of dynamical systems.

Let $(X, \mathcal{B}, \mu)$ be a (finite) measure space, let $T : X \to X$ be a measure-preserving transformation – together: a measure preserving system – and let $f \in L^1(X, \mathcal{B}, \mu)$.

Then ergodic theory studies long-time behaviour of e.g. elements $T^i f$, where $(T^i f)(x) = f(T^i x)$, and the properties of sums, products, averages, etc. of such elements.

Two Ergodic Theorems

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and define the average $A_n f := \frac{1}{n} \sum_{i=0}^{n-1} T^i f$.

Mean Ergodic Theorem *For any $f \in L^2(X, \mathcal{B}, \mu)$ the averages $A_n f$ converge in the L^2-norm.*

Pointwise Ergodic Theorem *For any $f \in L^1(X, \mathcal{B}, \mu)$ the averages $A_n f$ converge pointwise almost everywhere.*

Moreover, if the space is ergodic – i.e. X and $\emptyset$ are the only T-invariant sets – the averages converge to the integral of f.

Ergodic Theory - Applications to Combinatorics

Furthermore, there is a (no longer) surprising connection between ergodic theory and finite combinatorics:

Furstenberg Recurrence Theorem *If* $(X, \mathcal{B}, \mu)$ *is a measure space and* $T_1, \ldots, T_l$ *are commuting measure preserving transformations, then for any set* $A \in \mathcal{B}$ *with* $\mu(A) > 0$ *there exists an integer* $n \geq 1$ *with*

$$\mu(A \cap T_1^{-n} A \cap T_2^{-n} A \cap \ldots \cap T_l^{-n} A) > 0.$$

Ergodic Theory - Applications to Combinatorics

This recurrence theorem allows one to proof Szemeredi's theorem:

Szemeredi's Theorem *For any* $\delta > 0$ *and any* $k \in \mathbb{N}$ *there exists an* $N = N(\delta, k)$ *such that for any interval* $[a, b] \subset \mathbb{Z}$ *of length* $\geq N$ *and* $A \subseteq [a, b]$ *of density* $\geq \delta$*, i.e.* $\frac{|A|}{b-a} \geq \delta$*, contains an arithmetic progression of length* k*.*

The challenge is: How to translate the abstract concepts and techniques of ergodic theory – limits, projections, etc. – into concrete combinatorial results?

Mean Ergodic Theorem

Let us state the Mean Ergodic theorem in a Hilbert space setting:

Mean Ergodic Theorem *Let* $(X, \langle\cdot,\cdot\rangle)$ *be a Hilbert space, let* $T : X \to X$ *be an isometry and let* $f \in X$ *be given. Then*

$$\forall\varepsilon > 0 \exists n \in \mathbb{N} \forall m > n (\|A_m f - A_n f\| \leq \varepsilon).$$

This also holds if T is nonexpansive, i.e. $\|Tf\| \leq \|f\|$ for all $f \in X$.

Can we compute (a bound on) n in the parameters, i.e. f, T, ε and possibly depending on the Hilbert space?

Mean Ergodic Theorem - Noncomputability Results

Just as with PCM, we may construct a Hilbert space and an isometry $T : X \to X$ such that there can be no computable rate of convergence for the averages.

The same can be done for measure spaces – so this is not a feature of the more general setting of Hilbert spaces.

The general idea is to code the Halting problem into a measure space and a measure preserving transformation, such that a computable rate of convergence would solve the halting problem.

Mean Ergodic Theorem - Computability Results

Instead we consider, as with PCM, the Dialectica-transform:

Mean Ergodic Theorem *Let* $(X, \langle \cdot, \cdot \rangle)$ *be a Hilbert space, let* $T : X \to X$ *be an isometry and let* $f \in X$ *be given. Then*

$$\forall \varepsilon > 0 \forall M : \mathbb{N} \to \mathbb{N} \exists n \in \mathbb{N} (\|A_{M(n)} f - A_n f\| \leq \varepsilon),$$

where we assume that M is monotone increasing; again, we could add notation to express local stability.

This now has the suitable logical form for logical metatheorems to *guarantee* that a bound on n can be extracted. The bound will *not* depend on the particular space, *nor* on the transformation T, but *only* on ε, M and a bound $\|f\| \leq b$.

Mean Ergodic Theorem - Computability Results

The sketch of the proof for the Mean Ergodic Theorem is as follows:

- We can decompose the space X into components $U = \overline{\{u - Tu | u \in X\}}$ and $V = \{v \in X | v = Tv\}$.
- For elements $u - Tu$, we have $\|A_n(u - Tu)\| \leq 2\|u\|/n$.
- For elements $v \in V$, we have $A_n v = v$.

So for $f = u + v$, the averages $A_n f$ converge to v, where the rate can be given in terms of $\|u\|$, i.e. the projection of f onto U.

It is this projection onto U that makes the proof non-constructive!

Mean Ergodic Theorem - Computability Results

Since the averages $A_n f$ exist in the subspace of X spanned by $\{T^i f\}$, it suffices to consider the projection of f onto the subspace $U_f := \overline{\{T^i f - T^{i+1} f | i = 0, 1, \dots\}}$.

We can explicitly describe a sequence $g_i = u_i - Tu_i$ converging to the projection of f onto U_f:

$$g_0 = \frac{\langle f, f - Tf \rangle}{\|f - Tf\|^2}(f - Tf),$$

$$g_{i+1} = g_i + \frac{\langle f - g_i, T^i f - T^{i+1} f \rangle}{\|T^i f - T^{i+1} f\|^2}(T^i f - T^{i+1} f)$$

Mean Ergodic Theorem - Computability Results

The computation then roughly goes as follows:

- $\|A_{M(n)}f - A_n f\| \leq$
  $\|A_{M(n)}(f - g_i) - A_n(f - g_i)\| + \|A_M(n)g_i\| + \|A_n g_i\|$.
- Let $a_i = \|g_i\|$. If $|a_i - a_j|$ is small, then $\|g_i - g_j\|$ is small.
- If $\|g_i - g_j\|$ is suitably small for a suitable j, then $\|A_{M(n)}(f - g_i) - A_n(f - g_i)\|$ is small.
- If $n, M(n)$ are large relative to $\|u_i\|$, $\|A_M(n)g_i\|, \|A_n g_i\|$ are small.

Thus sufficient local stability for the bounded monotone sequence a_i together with upper bounds on $\|u_i\|$ allows us to derive local stability for $A_n f$.

Mean Ergodic Theorem - Computability Results

But: to compute upper bounds on $\|u_i\|$ we need lower bounds on $\|T_i f - T^{i+1} f\|$ - which in general could be $= 0$.

The solution is the following observation:

- If e.g. $\|f - Tf\|$ is very small, we may get local stability for $\|A_{M(0)} f - f\|$ via the triangle inequality.
- Otherwise, we have a lower bound on $\|f - Tf\|$.

In other words, the final step is giving an appropriate computational interpretation to $r = 0 \vee r \neq 0$ for a particular real number r.

Mean Ergodic Theorem - Computability Results

With this analysis, the following bounds were obtained by Avigad-Towsner-G.:

Define:

$$i_0 = 0, \qquad n_k = \lceil \frac{b}{\varepsilon^2} \sum_{l=0}^{i_k} M(\frac{2lb}{\varepsilon}) \rceil$$

$$i_k + 1 = i_k + \lceil \frac{2^{15} M(n_k)^4 b^4}{\varepsilon^4} \rceil$$

Let $d = \frac{512 b^2}{\varepsilon^2}$, then for some $n \leq N(b, \varepsilon, M) = \frac{2 n_d b}{\varepsilon}$, we have that $\|A_{M(n)} f - A_n f\| < \varepsilon$.

Mean Ergodic Theorem - Observations

- Instead of full convergence of the averages, one can obtain only local stability.
- To obtain local stability of the averages, it suffices to obtain local stability (not convergence) of the projections onto U_f.
- Observations like, "$\|f - Tf\| = 0$ implies convergence of the averages" are weakened to "$\|f - Tf\| < \delta$ implies local stability of the averages", allowing the crucial upper bounds on the norms $\|u_i\|$.

Mean Ergodic Theorem - Further Comments

- If the measure preserving system is ergodic, we can compute a full rate of convergence for $A_n f$ - this is because here $A_n f$ converges to the integral of f, and one can compute a full rate of convergence from $\|f^*\|$, where $f^* = \lim_{n\to\infty} A_n f$.
- Kohlenbach and Leustean analysed a proof by Birkhoff of the generalization of Mean Ergodic Proof to uniformly convex Banach spaces. This analysis yields much improved bounds.

Szemeredi's Theorem - Sketch of Proof

We call a measure preserving system weak mixing if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i<n} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)| = 0,$$

for any measurable sets A, B.

We call a measure preserving system compact, if the orbit

$$\{A, T^{-1}A, T^{-2}A, T^{-3}A, \ldots\}$$

is compact for any measurable set A.

Informally, weak mixing corresponds to randomness and compactness to regularity.

Szemeredi's Theorem - Sketch of proof

Both conditions easily imply Szemeredi's Theorem - but not every measure preserving system is either weak mixing or compact.

However, if a measure preserving system $(X, \mathcal{B}, \mu, T)$ is *not* weak mixing, it has a non-trivial T-invariant compact factor.

A factor can be thought of as a coarsening of the σ-algebra $\mathcal{B}$ to a T-invariant sub-σ-algebra $\mathcal{B}'$.

Szemeredi's Theorem - Sketch of proof

Furthermore, if a measure preserving system $(X, \mathcal{B}, \mu, T)$ is *not* weak mixing relative to a factor $\mathcal{B}'$, it has an intermediate non-trivial T-invariant compact factor $\mathcal{B}''$ such that $(X, \mathcal{B}'', \mu, T)$ is compact relative to $\mathcal{B}'$.

Iterating this construction and taking unions at limit stages, this process will – assuming the system is separable – terminate at some countable ordinal.

Szemeredi's Theorem - Sketch of proof

A factor $\mathcal{B}$ is Szemeredi (short: SZ), if it satisfies

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n}(\mu(A \cap T_1^{-i}A \cap T_2^{-i}A \cap \ldots \cap T_l^{-i}A)) > 0,$$

for every set $A \in \mathcal{B}$.

- The trivial factor $\mathcal{B}_0$ is SZ.
- SZ is preserved under compact extensions.
- SZ is preserved under limits.
- SZ is preserved under weak mixing extensions.

Szemeredi's Proof - Comments on the Proof

- The construction of the measure space from the "dense" subset of the natural numbers corresponds to an application of Weak (binary) Koenig's Lemma.
- Once weak mixing (relative to an SZ factor) is established, obtaining recurrence is straightforward.
- The transfinite iteration is constructively "acceptable".
- The source of non-constructivity is the construction of the compact extension using projections and limits akin to the Mean Ergodic Theorem.

Szemeredi's Proof - Comments on the Proof

- The transfinite sequence of compact extensions exhausts the countable ordinals, i.e. for every countable ordinal there is a measure space where the induction stops only at that ordinal.
- Avigad and Towsner have shown that the proof can be formalized in ID_1 and have given a functional interpretation of ID_1 thus yielding a computational interpretation of this proof.
- A more refined analysis by Avigad and Towsner yields improved ordinal bounds for the application of Furstenberg's structure theorem to Szemeredi's Theorem.

Szemeredi's Theorem - Improved Ordinal Bounds

In Furstenberg's proof of Szmeredi's Theorem one establishes weak mixing relative to a factor. Weak mixing, in general, was:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i<n} |\mu(T^{-i} A \cap B) - \mu(A)\mu(B)| = 0.$$

To eventually apply this property to obtain recurrence it suffices to have approximate weak mixing

$$\forall m \geq n \frac{1}{m} \sum_{i<m} |\mu(T^{-i} A \cap B) - \mu(A)\mu(B)| < \varepsilon,$$

for a suitable ε and suitably many factors in the sequence of compact extensions. Avigad and Towsner show approximate weak mixing can always be obtained below ω^{ω^ω}.

Final Remarks

- Non-constructivity arises from use of limits and projections, not "choice" or "compactness arguments" or "transfinite constructions".
- In general, use of limits and projections to obtain combinatoric results is weakened to use of approximate limits and projections.
- Plenty of avenues for future work - extracting "exact" bounds from Furstenberg's proof, analysing methods by Gowers, Tao-Green, etc.

Final Remarks

References:

- Kohlenbach, "Applied Proof Theory: Proof Interpretations and their Use in Mathematics", Springer Monographs in Mathematics, 2008.
- Avigad, G., Towsner, "Local Stability of Ergodic Averages", TAMS 362, pp. 261-280 (2010).
- Avigad, "The Metamathematics of Ergodic Theory", APAL 157, pp. 64-79 (2009).
- Avigad, Towsner, "Metastability in the Furstenberg-Zimmer Tower", draft.