Proof mining in topological dynamics

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Topological Dynamics - Definitions

In topological dynamics, we model the behaviour of a dynamical system by:

- a compact metric space (X, d),
- a self-map $T : X \to X$ (potentially a homeomorphism).

We write (X, T) for such a dynamical system.

If we have a group G of self-maps (resp. homeomorphisms) of X, we write (X, G).

Topological Dynamics - Definitions

We call a system (X, G) *minimal*, if there is no non-trivial subset $A \subset X$ that is invariant under all actions of G.

We call a subset $A \subseteq X$ *homogeneous*, if there is a group G' commuting with G, such that (A, G') is minimal.

We call a point $x \in X$ *recurrent* in (X, T), if

$$\forall \varepsilon > 0 \exists n \in \mathbb{N} (d(T^n x, x) < \varepsilon).$$

We call a point $x \in X$ *uniformly recurrent* in (X, T), if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m \in \mathbb{N} \exists n \leq N (d(T^{m+n} x, x) < \varepsilon).$$

Topological Dynamics - Properties of Minimal Systems

Let (X, G) be minimal, then

- Every orbit $\{Gx\}, x \in X$ is dense in X.
- For every $\varepsilon > 0$ there exists a finite set $g_1, g_2, \ldots, g_m$ such that $\min_{1 \le i \le m} d(x, g_i y) \le \varepsilon$ for all $x, y \in X$.

For a minimal dynamical system (X, T), we furthermore get

- Every $x \in X$ is uniformly recurrent.

Lemma. *Every dynamical system has a minimal subsystem.*

Topological Dynamics - Multiple Birkhoff Recurrence

Multiple Birkhoff Recurrence Theorem. *Let (X, d) be a compact metric space and $T_1, \ldots, T_l$ commuting homeomorphisms of X. Then there exists a point $x \in X$ and a sequence $n_k \to \infty$ with $T_i^{n_k} x \to x$ simultaneously for $i = 1, \ldots, l$.*

An easy corollary is:

Weak Multiple Birkhoff Recurrence Theorem. *Let (X, d) be a compact metric space and $T_1, \ldots, T_l$ comm. homeomorphisms of X. Then for every $\varepsilon > 0$ there exist $x \in X, n \in N$ such that $d(T_i^n x, x) < \varepsilon$ simultaneously for $i = 1, \ldots, l$.*

The reverse direction can be shown using compactness.

Topological Dynamics - van der Waerden's Theorem

van der Waerden's Theorem. *For any $q, k \in \mathbb{N}$ there is an $N = N(q, k)$ such that for any q-colouring of $[1, N]$ some colour contains an arithmetic progression of length k.*

van der Waerden follows from WMBR in the following way:

- Let f be a q-colouring of $\mathbb{N}$ and let T be the 1-shift, then $\{T^i f\}$ is a compact metric space with the usual metric.
- Two colourings f, g with distance < 1 satisfy $f(1) = g(1)$.
- A multiply recurrent point (in the weak sense) and the $n \in \mathbb{N}$ yields an arithmetic progression.

How do we compute a multiply recurrent $x \in X$ and an $n \in \mathbb{N}$?

Furstenberg-Weiss' proof

Lemma. *Let (X, d) be a compact metric space and let $T : X \to X$ be a self-map of X. Then for every $\varepsilon > 0$ there is an $x \in X$ and an $n \in \mathbb{N}$ such that $d(T^n x, x) < \varepsilon$.*

Proof: Take any $x_0 \in X$ and consider the sequence $x_0, Tx_0, T^2x_0, \ldots$. By compactness two elements $T^i x_0, T^j x_0$ with $i < j$ are close. Let $x = T^i x_0$ and $n = j - i$. □

- The point $x \in X$ and $n \in \mathbb{N}$ are constructed explicitly – given a modulus of total boundedness.
- The values $i, n \in \mathbb{N}$ are bounded uniformly in x_0.
- Completeness of the space is not needed.

Furstenberg-Weiss' proof

Lemma. *Assume that for any $\delta < 0$ and $T_1, \ldots T_l$ there exists a $z \in X$ and an $n > 0$ such that simultaneously $d(T_i^n z, z) < \delta$. Then for any $\varepsilon > 0$ and $S_1, \ldots, S_{l+1}$ there exist $x, y \in X$ and an $m > 0$ such that simultaneously $d(S_i^m x, y) < \varepsilon$.*

Proof. Define $T_i = S_i S_{l+1}^{-1}$ and let $x = S_{l+1}^{-n} z$, $y = z$ and $m = n$. □

This is the start of the induction step to prove multiple recurrence for any l commuting homeomorphisms.

Again, we have explicit constructions for $x, y \in X$ which – assuming the construction of $z \in X$ is uniform – are uniform in a similar way. Again, completeness is not used.

Furstenberg-Weiss' proof

Lemma. *Let $A \subseteq X$ be homogeneous (for a group G). If for any $\delta > 0$ there exist $u, v \in A$ and $n > 0$ such that $d(T^n u, v) < \delta$. Then for every $x \in A$ and $\varepsilon > 0$ there is a $y \in A$ and an $m \in \mathbb{N}$ such that $d(x, T^m y) < \varepsilon$.*

Proof. Using minimality, we obtain $g_1, \dots, g_l$ such that $\min_{1 \leq i \leq l} d(g_i z, z') < \varepsilon/2$. Using continuity, we find u, v such that $d(T^n g_i u, g_i v) < \varepsilon/2$. Combine with $d(g_i v, x) < \varepsilon/2$. □

This is to be applied to a suitable $(l+1)$-fold product of (X, T_i), yielding the result simultaneously for $T_1, \dots, T_{l+1}$.

Nothing of the previous (uniform) constructions is used here!

Furstenberg-Weiss' proof

Proof of WMBR. Induction start by lemma.

For induction step, pick $z_0 \in X$. Find $z_1 \in X$ such that $d(T_i^{n_1} z_1, z_0) < \varepsilon_1$.

Pick small enough $\varepsilon_2 > 0$ (using continuity and n_1), find $z_2 \in X$ such that $d(T_i^{n_2} z_2, z_1) < \varepsilon_2$.

Construct sequence z_i s.t. $i < j \rightarrow d(T_i^{n_j + \ldots + n_{i+1}} z_j, z_i) < \varepsilon/2$. By compactness, some $z_i, z_j \in X$ are $\varepsilon/2$-close. □

Furstenberg-Weiss' proof

Observations:

- Most of the constructions in the proof are explicit.
- Details of the constructions are forgotten.
- Minimality is used to recover/replace "lost" information.

Girard modified the Furstenberg-Weiss proof, using the "forgotten" constructions – in particular their uniformity – to obtain a proof that does not use minimality.

Girard's proof

Girard proved the following variant of the Multiple Birkhoff Recurrence Theorem:

WMBR, Girard's variant. *Let (X, d) be a compact metric space, let $T_1, \ldots, T_l$ commuting homeomorphisms of X and let G be the commutative group generated by $T_1, \ldots, T_l$. Then*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \exists S_1, \ldots S_M \in G \forall z_0 \in X$$
$$\exists n \leq N \exists i \leq M (d(T_1^n S_i z_0, S_i z_o) < \varepsilon \wedge \ldots d(T_l^n S_i z_0, S_i z_0) < \varepsilon).$$

The key here is to explicitly construct the elements S_i which will be in the group generated by $T_1, \ldots, T_l$.

Observations

- Except for the constructions involving minimality, one easily make the constructions explicit, i.e. describing n and the finite set of group elements.
- Furstenberg and Weiss use minimality for a forward construction of a potentially infinite sequence z_i.
- Girard uses a backwards construction of sequences of arbitrary *finite* length, using the constructed group elements and their uniformities. This does not need minimality.

Final observation: At no point the completeness is used, the result already holds for totally bounded spaces.

Comparison: Girard - van der Waerden

Combinatorics	Topological Dynamics
Finite set	Totally bounded space
Number of colours	Number of ε-neighbourhoods
Colouring of $\mathbb{N}$	Point in space
Colouring of finite block	ε-neighbourhood
Length of progression	Number of homeomorphisms
Blocks within blocks	Continuity
q nested appeals to IH	$\gamma(\varepsilon)$ nested appeals to IH

Girard's proof is a topological reformulation of van der Waerden's proof and yields the same Ackermanian bounds.

Furstenberg-Weiss' proof is a less explicit, more complicated version of Girard's proof (and yields worse bounds).

Shelah's proof of Hales-Jewett

Shelah gave a new combinatorial proof of the following:

Hales-Jewett Theorem. *Let $q \in \mathbb{N}$ be the number of colours and let A be a finite alphabet of size $r \in \mathbb{N}$. Then there exists an $N = N(q, r)$ such that for every q-colouring of the finite words over A, there is a 1-parameter word of length $\leq N$ that is monochromatic.*

- The result easily extends to k-parameter words.
- HJ implies vdW/WMBR – consider the alphabet $T_1, \ldots, T_l$.
- HJ can be thought of as a non-commutative version of the Weak Multiple Birkhoff Recurrence Theorem.

Comments on Shelah's proof

All proofs proceed by induction on number of homeomorphisms / length of progressions / size of alphabet.

van der Waerden's/Girard's proof can be summed up as:

- Base case: appeal to compactness.
- Induction step: Compactness $\Rightarrow k$, k nested appeals to the induction hypothesis.

Shelah's proof can be summed up as:

- Base case: appeal to compactness.
- Induction step: Induction hypothesis $\Rightarrow k$, k nested appeals to compactness.

Comments on Shelah's proof

- Shelah's approach yields primitive recursive bounds (all previous approaches yielded Ackermanian bounds).
- Is there a natural counterpart of Shelah's proof in the setting of topological dynamics?
- Is there a simple, combinatorial counterpart to Gower's proof (of Szemeredi's Theorem) yielding elementary upper bounds (tower of exponentials of height four).

Furstenberg-Katznelson's proof

For (X, T), consider $G = \mathbb{N}$ as a group acting on X (by applying T n times). Then G is a semi-group, the closure $E(G)$ of G in the space of functions $X \to X$ is called the enveloping semi-group.

Furstenberg and Katznelson show how to obtain van der Waerden's Theorem and Hales-Jewett Theorem using results semi-group theory.

To obtain some constructive content from this proof, one would need to make explicit:

- The compactification of $E(G)$.
- The complexity of approximating elements in the boundary of $E(G)$ by elements in G.

Extensions of van der Waerden's Theorem

Multidimensional van der Waerden's Theorem. Colourings of $\mathbb{N}^k$, shifts of dilations of finite configurations.

Folkman's Theorem. *For any $q, k \in \mathbb{N}$ there is an $N = N(q, k)$ such that for any q-colouring of $[1, N]$, some colour contains a set A of size k and all finite sums of elements of A.*

Hindman's Theorem. *For any q-colouring of $\mathbb{N}$ some colour contains an infinite sequence $n_1 < n_2 < n_3 < \dots$ and all finite sums of elements of that sequence.*

The first two follow from vdW. Hindman's Theorem has a topological dynamics proof heavily using minimality.

Folkman - ? - Hindman

Consider a "relative largeness" version of Folkman/Hindman:

finitary Hindman's Theorem. *For any* $q, k \in \boldsymbol{N}$ *there is an* $N = N(q, k)$ *such that any* q*-colouring of* $[1, N]$ *some colour contains an relatively long, finite sequence* $n_1 < n_2 < n_3 < \ldots < n_l$ *(of length* $\max(k, n_1)$*) and all finite sums of elements of that sequence.*

- fHT follows from HT + weak König's Lemma - but in what formal system is fHT provable?
- Can we relativize minimality (and consequences thereof) to get a direct proof of fHT?

Proof of Hindman's Theorem

Two points $x, y \in X$ are called *proximal*, if

$$\forall \varepsilon > 0 \forall m \in \mathbb{N} \exists n > m (d(T^n x, T^n y) < \varepsilon).$$

Lemma. *Let (X, T) be given and let (Y, T) be a minimal subsystem. Then for every $x \in X$ there is a $y \in Y$ that is proximal to x and uniformly recurrent.*

Can we weaken this lemma (and the minimality appealed to) to give a proof of finitary Hindman's Theorem?

Conclusions and future work

- One can extract bounds (for combinatorial statements) from proofs in topological dynamics . . .
- . . . but so far this does not yield new or better bounds.
- It would be interesting to develop a nice computational interpretation of minimality (and weakenings thereof) to obtain bounds and proof-theoretic strength of the "relatively long" version of Hindman's Theorem.