From Ramsey to Ehrenfeucht: a reduction between games

Oleg Verbitsky

Humboldt Universität Berlin, Germany and IAPMM Lviv, Ukraine

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Joint work with Frank Harary and Wolfgang Slany.

Basics of Graph Ramsey theory

Definition. $G \to F$ if, for any coloring of $E(G)$ in red and blue, G contains a monochromatic copy of F.

Ramsey theorem. There is a function $N = N(n)$ such that $K_N \to K_n$ (and hence $K_N \to F$ for any F on n vertices).

Burr (Garey and Johnson GT6):
Deciding if $G \to K_3$ is coNP-complete.

Ramsey games on (G, F)

$\mathcal{A}$ and $\mathcal{B}$ color $E(G)$

> alternately, one edge per move
> $\mathcal{A}$ in red, $\mathcal{B}$ in blue
> $\mathcal{A}$ moves first

Player's objective in

$\mathrm{ACHIEVE}(G, F)$: create a monochromatic F

$\mathrm{AVOID}(G, F)$: avoid such an F

Strong version: $\mathcal{A}$ and $\mathcal{B}$ have the same objective.

Observation: If $G \to F$, then the game never ends in a draw!

Weak version: $\mathcal{A}$ has the objective, $\mathcal{B}$ plays against (most studied but out the scope of this talk).

Example.

$$\mathrm{AVOID}(K_6, K_3) = \text{SIM}$$

Mead, Rosa, Huang 74: SIM is won by $\mathcal{B}$

Open question (József Beck 08). Who wins $\mathrm{AVOID}(K_{18}, K_4)$?

Symmetry breaking-preserving game

Rules of $\mathrm{SYM}(G)$:

A round: $\mathcal{A}$ ' move + $\mathcal{B}$'s move
Objective of $\mathcal{B}$: to keep the red and the blue subgraphs of G isomorphic after each round

Observation: If $\mathcal{B}$ wins $\mathrm{SYM}(G)$, then he does not lose $\mathrm{AVOID}(G, F)$ for any F.

Mirror strategy in $\mathrm{SYM}(G)$

$\mathcal{B}$ wins $\mathrm{SYM}(G)$ whenever G has a good automorphism.
An automorphism is *good* if it

is involutory and
leaves no edge fixed.

$\mathcal{C}_{\mathrm{auto}}$ denotes the class of graphs with a good automorphism.

$\mathcal{C}_{\mathrm{auto}}$ includes

- Paths and cycles of even length.
- Platonic graphs except the tetrahedron.
- Cubes.
- $K_{s,t}$ if st is even.

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$\mathcal{C}_{\mathrm{auto}}$ denotes the class of graphs with a good automorphism.

$\mathcal{C}_{\mathrm{auto}}$ is closed with respect to the

- sum
- Cartesian, lexicographic, categorical products

$\mathcal{C}_{\mathrm{auto}}$ is NP-complete.

Length of the game

$$L_{\text{sym}}(G) = \text{max } k \text{ s.t. } \mathcal{B} \text{ wins the } k\text{-round } \mathrm{SYM}(G).$$

Known:

- $L_{\text{sym}}(K_n) \le 6$
- $L_{\text{sym}}(G) = |E(G)|/2$ if $G \in \mathcal{C}_{\text{auto}}$. In particular,
  - $L_{\text{sym}}(P_n) = L_{\text{sym}}(C_n) = n/2$ if n is even, where P_n (resp. C_n) denotes the path (resp. cycle) of length n.
  - $L_{\text{sym}}(K_{n,n}) = n^2/2$ if n is even
- $\frac{n-1}{2} \le L_{\text{sym}}(K_{n,n}) \le 2n + 38$ if n is odd (Pikhurko 03)

Length of the game

$$L_{\rm sym}(G) = \text{max } k \text{ s.t. } \mathcal{B} \text{ wins the } k\text{-round } \mathrm{SYM}(G).$$

Theorem. If n is odd, then

1. $L_{\rm sym}(P_n) = \Omega(\log n)$ and $L_{\rm sym}(C_n) = \Omega(\log n)$,
2. $L_{\rm sym}(P_n) = O(\log^2 n)$ and $L_{\rm sym}(C_n) = O(\log^2 n)$.

Lower bound: a connection to the Ehrenfeucht game

Rules of $\mathrm{EF}(G_0, G_1)$, the Ehrenfeucht-Fraïssé game on graphs G_0 and G_1

Players: Spoiler
Duplicator

i-th round: Spoiler selects $u_i \in V(G_a)$
Duplicator selects $v_i \in V(G_{1-a})$

Duplicator's objective: to keep the correspondence '$u_i \leftrightarrow v_i$' being a partial isomorphism between G_0 and G_1.

$L_{\mathrm{EF}}(G_0, G_1) =$ max k s.t. $\mathcal{B}$ wins the k-round $\mathrm{EF}(G_0, G_1)$.

Lower bound: a connection to the Ehrenfeucht game

Ehrenfeucht's theorem. No first order sentence of quantifier depth $L_{\mathrm{EF}}(G_0, G_1)$ distinguishes between non-isomorphic G_0 and G_1. On the other hand, depth $L_{\mathrm{EF}}(G_0, G_1) + 1$ suffices.

Theorem (textbooks in Finite Model Theory).
For every n,

1. $\log n - 2 < L_{\mathrm{EF}}(P_n, P_{n+1}) < \log n + 2$.

2. $\log n - 1 < L_{\mathrm{EF}}(C_n, C_{n+1}) < \log n + 1$.

Proof of the lower bound

$$L_{\mathrm{sym}}(C_n) \geq \frac{1}{4}\log n - \frac{1}{4} \text{ for odd } n.$$

"$L_{\mathrm{sym}}(G) \geq k$" is expressible by a first order sentence Φ_k with $4k$ quantifiers.

Let $k = \frac{[\log n - 1]}{4}$.

Since $C_{n+1} \in \mathcal{C}_{\mathrm{auto}}$, we have $C_{n+1} \models \Phi_k$.

Since $L_{\mathrm{EF}}(C_n, C_{n+1}) > \log n - 1$, we have $C_n \models \Phi_k$ too.

Constructivization?

$$
\begin{array}{ccc}
 & \mathrm{EF}(C_n, C_{n+1}) & \\
\nearrow & & \searrow \\
\mathrm{SYM}(C_n) & & \mathrm{SYM}(C_{n+1})
\end{array}
$$

Question: We know a strategy for $\mathcal{B}$ in $\mathrm{SYM}(C_{n+1})$.
Can we know it in $\mathrm{SYM}(C_n)$?

Answer: Yes, because we know Duplicator's strategy in $\mathrm{EF}(C_n, C_{n+1})$!

Preliminaries: the line graph

$\mathcal{L}(H)$ denotes the line graph of a graph H:

$V(\mathcal{L}(H)) = E(H)$,
e_1 and e_2 are adjacent in $\mathcal{L}(H)$ if they have a common vertex in H.

Example: $\mathcal{L}(C_n) = C_n, \quad \mathcal{L}(P_n) = P_{n-1}$

Clearly, $H_1 \cong H_2 \Rightarrow \mathcal{L}(H_1) \cong \mathcal{L}(H_2)$.

The Whitney theorem. $\mathcal{L}(H_1) \cong \mathcal{L}(H_2) \Rightarrow H_1 \cong H_2$
for all connected H_1 and H_2 unless $\{H_1, H_2\} = \{K_3, K_{1,3}\}$.

Constructivization!

Our former approach generalizes to

$$L_{\mathrm{sym}}(G_1) \geq \min\left\{L_{\mathrm{sym}}(G_0), \frac{1}{4}L_{\mathrm{EF}}(G_0, G_1)\right\}$$

Now we prove: If G_1 is triangle-free, then

$$L_{\mathrm{sym}}(G_1) \geq \min\left\{L_{\mathrm{sym}}(G_0), \frac{1}{2}L_{\mathrm{EF}}(\mathcal{L}(G_0), \mathcal{L}(G_1))\right\}$$

In particular,

$$L_{\mathrm{sym}}(C_n) \geq \frac{1}{2}\log n - \frac{1}{2}.$$

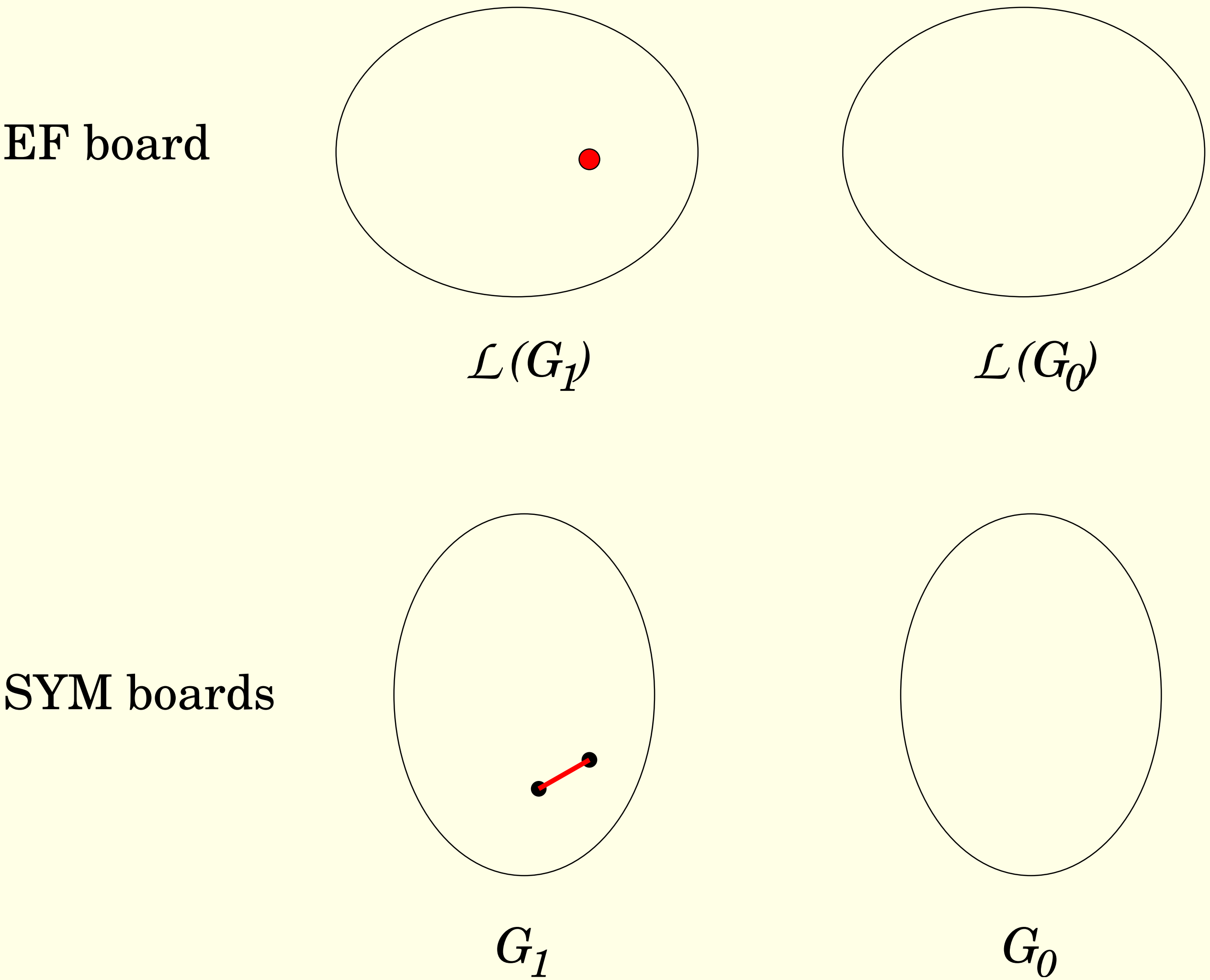
Reduction

Let S_0 denote a strategy of $\mathcal{B}$ in $\mathrm{SYM}(G_0)$.
Let D denote a strategy of Duplicator in $\mathrm{EF}(\mathcal{L}(G_0), \mathcal{L}(G_1))$.

We describe $S_1 = S_1(S_0, D)$, a strategy for $\mathcal{B}$ in $\mathrm{SYM}(G_1)$, such that

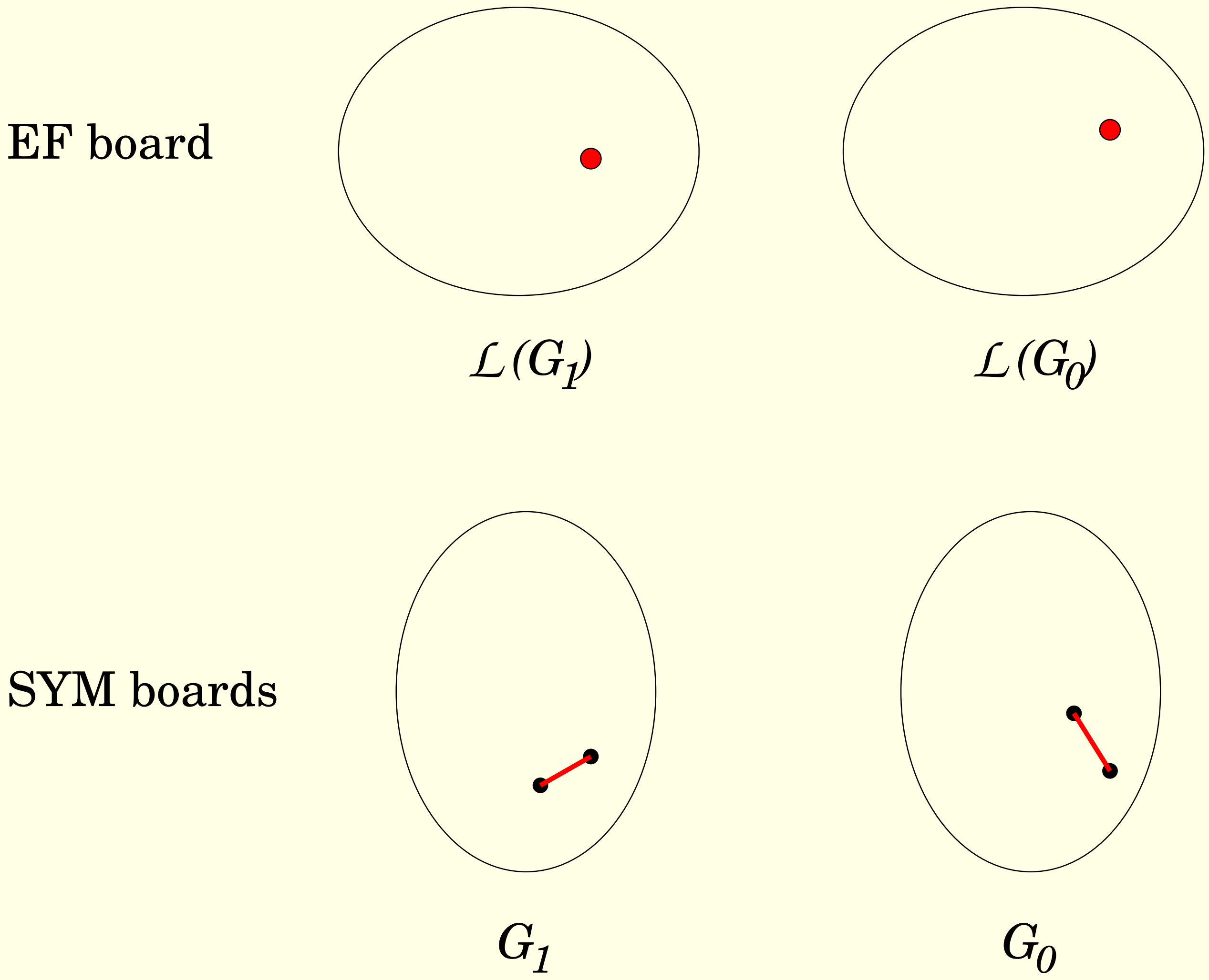
> if S_0 succeeds in k rounds of $\mathrm{SYM}(G_0)$ and
> D in $2k$ rounds of $\mathrm{EF}(\mathcal{L}(G_0), \mathcal{L}(G_1))$, then
> S_1 succeeds in k rounds of $\mathrm{SYM}(G_1)$.

A round of $\mathrm{SYM}(G_1)$



1. $\mathcal{A}$'s move in $\mathrm{SYM}(G_1)$
2. Spoiler's move in $\mathrm{EF}(G_1, G_0)$ (simulation)

A round of $\mathrm{SYM}(G_1)$



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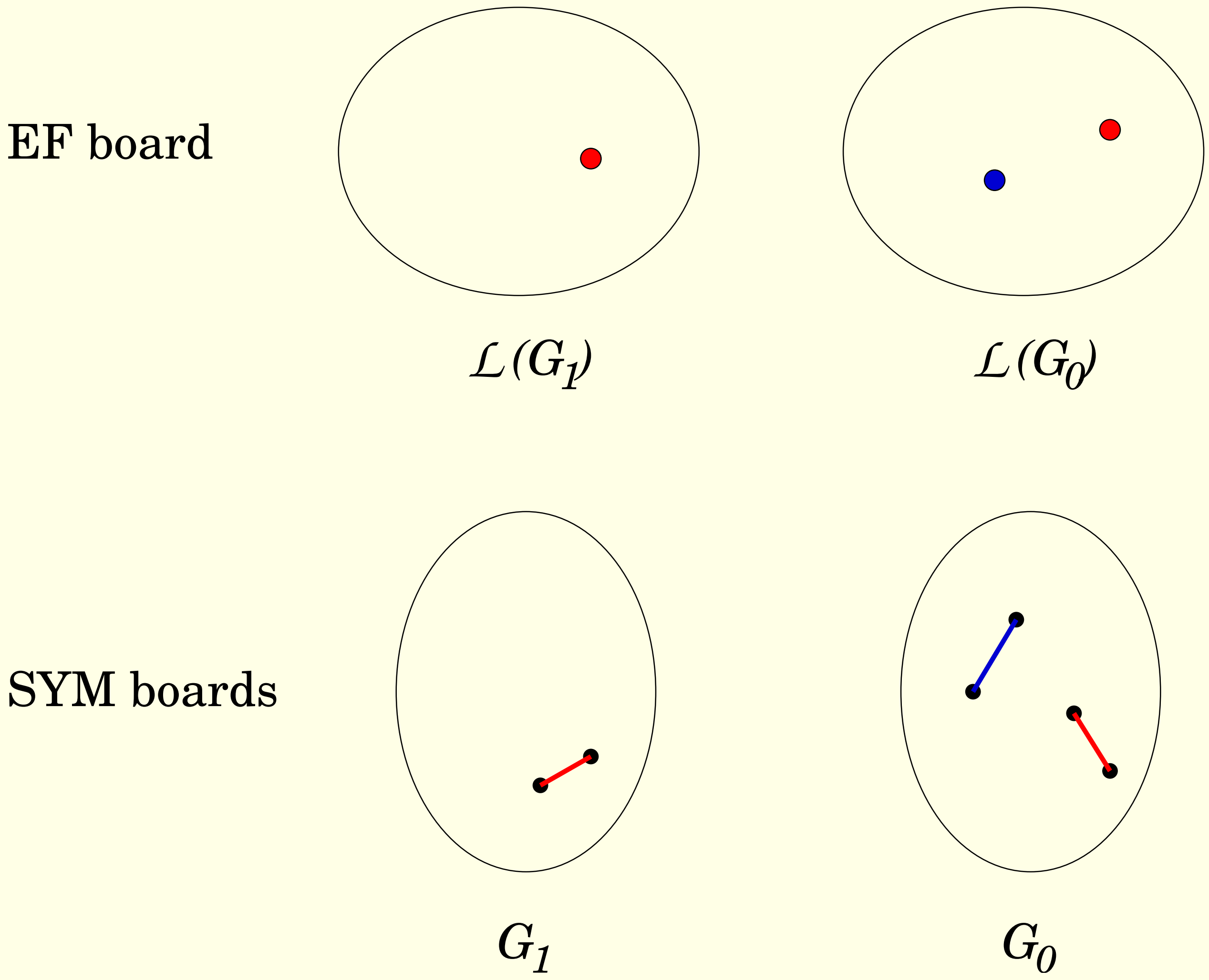
2. Spoiler's move in $\mathrm{EF}(G_1, G_0)$ (simulation)
3. Duplicator's move in $\mathrm{EF}(G_1, G_0)$ (according to D)
4. $\mathcal{A}$'s move in $\mathrm{SYM}(G_0)$ (simulation)

A round of $\mathrm{SYM}(G_1)$



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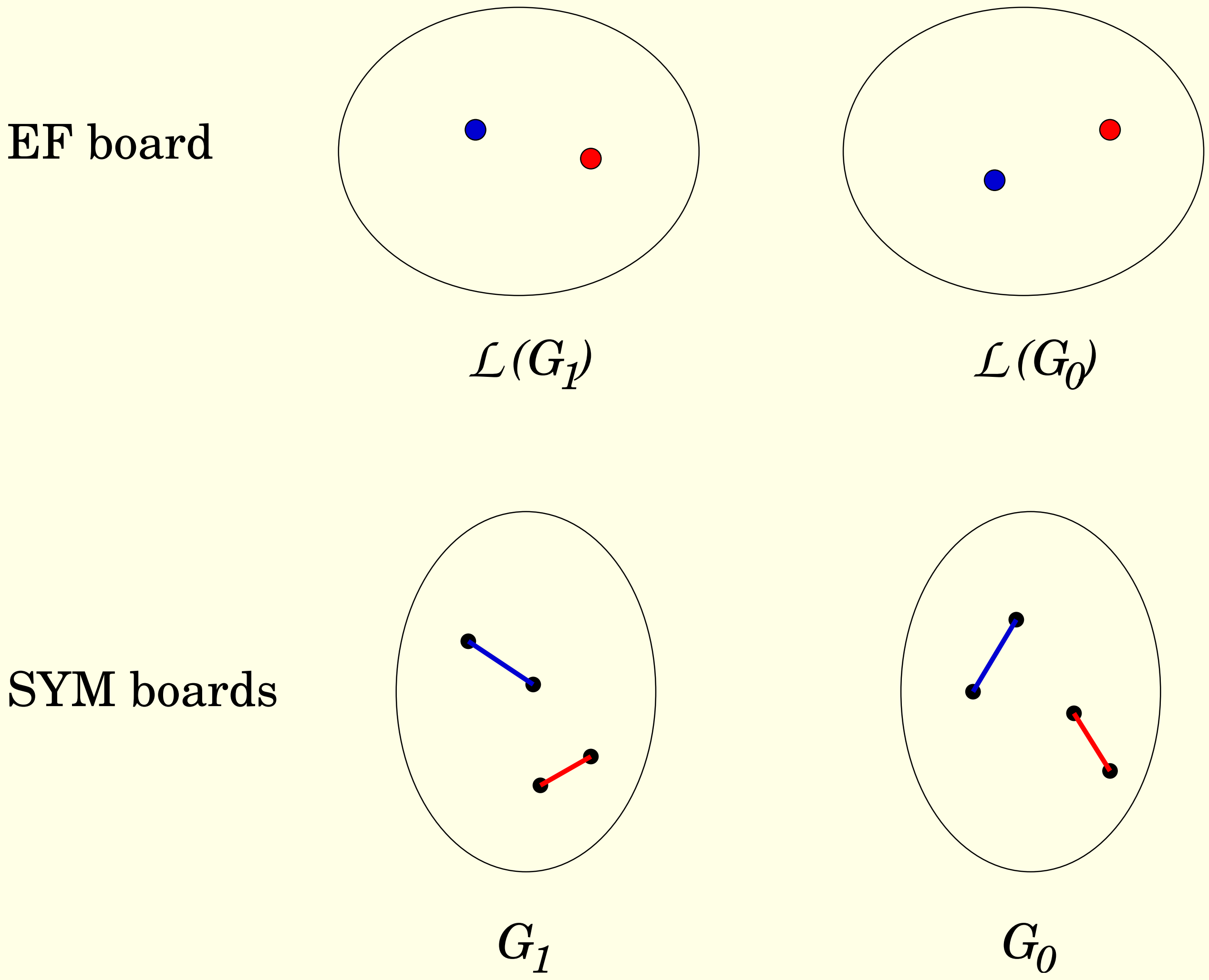
5. $\mathcal{B}$'s move in $\mathrm{SYM}(G_0)$ (according to S_0)
6. Spoiler's move in $\mathrm{EF}(G_1, G_0)$ (simulation)

A round of $\mathrm{SYM}(G_1)$



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6. Spoiler's move in $\mathrm{EF}(G_1, G_0)$ (simulation)
7. Duplicator's move in $\mathrm{EF}(G_1, G_0)$ (according to D)
8. $\mathcal{B}$'s move in $\mathrm{SYM}(G_1)$ (this defines S_0)

Analysis of the strategy

Fix a strategy of $\mathcal{A}$ in $\mathrm{SYM}(G_1)$. Denote

A_i – red edges of G_i colored up to the k-th round,

B_i – blue edges of G_i colored up to the k-th round.

Note that A_0 is constructed from A_1 and B_1 from B_0.

$$
\begin{array}{ccccl}
A_0 & \cong & B_0 & & \text{because } S_0 \text{ succeeds} \\
 & \Downarrow & & & \\
\mathcal{L}(A_0) & \cong & \mathcal{L}(B_0) & & \\
\| & & \| & & \\
\mathcal{L}(G_0)[A_0] & \cong & \mathcal{L}(G_0)[B_0] & & \\
\|\wr & & \|\wr & & \text{because } D \text{ succeeds} \\
\mathcal{L}(G_1)[A_1] & \cong & \mathcal{L}(G_1)[B_1] & & \\
\| & & \| & & \\
\mathcal{L}(A_1) & \cong & \mathcal{L}(B_1) & & \\
 & \Downarrow & & & \text{by Whitney's theorem} \\
A_1 & \cong & B_1 & &
\end{array}
$$

Thank you!