Homogeneous Structures, Ramsey Classes, and Constraint Satisfaction

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May 2011

Outline

A method to apply structural Ramsey theory to analyze definability in homogeneous structures + two applications

1. Computational Complexity of Constraint Satisfaction Problems
2. Classification of First-Order Reducts in Model Theory

Constraint Satisfaction Problems

Informal description

Constraint Satisfaction Problem (CSP)

A computational problem:

Input: a set of variables and a set of constraints imposed on these variables

Question: is there an assignment of values to the variables such that all the constraints are satisfied?

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Examples and Applications of CSPs in:
Artificial Intelligence, Type Systems for Programming Languages, Computational Linguistics, Database Theory, Computational Biology, Graph Theory, Finite Model Theory, Computational Real Geometry, Computer Algebra, Operations Research, Boolean Satisfiability, Complexity Theory, . . .

Constraint Satisfaction Problems

Let Γ be a structure with a finite relational signature τ.
Γ also called the template.

Definition 1 (CSP).

$\mathrm{CSP}(\Gamma)$ is the computational problem to decide whether a given finite τ-structure homomorphically maps to Γ.

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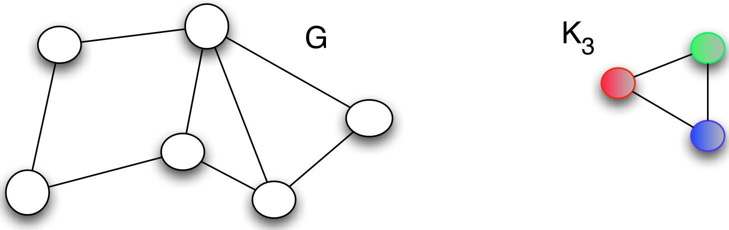
Example: 3-colorability is $\text{CSP}(K_3)$



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More Examples of CSPs

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Strongest evidence comes from the so-called universal algebraic approach.

Primitive Positive Definability

A first-order formula is called primitive positive if it is of the form

$$\exists x_1, \ldots, x_n.\, \psi_1 \wedge \cdots \wedge \psi_m \;,$$

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Lemma 1 (Jeavons et al.).

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Lemma 1 (Jeavons et al.).

Let $\Gamma = (D; R_1, \dots, R_k)$ be a relational structure, and
let R be a relation that has a primitive positive definition in Γ.
Then $\mathrm{CSP}(\Gamma)$ and $\mathrm{CSP}(D; R, R_1, \dots, R_k)$ are polynomial-time equivalent.

How do we recognize whether a relation R is primitive positive definable in Γ?

Polymorphisms

A function $f : D^k \to D$ preserves $R \subseteq D^m$ if $\big(f(a^1_1, \dots, a^k_1), \dots, f(a^1_m, \dots, a^k_m)\big) \in R$ whenever $(a^i_1, \dots, a^i_m) \in R$ for all $i \leq m$.

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Polymorphisms $\leftrightarrow$ Algorithms

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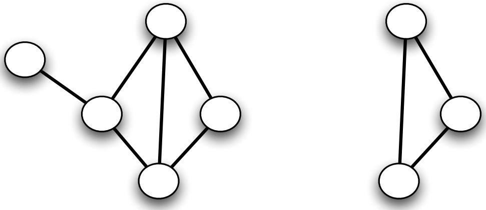

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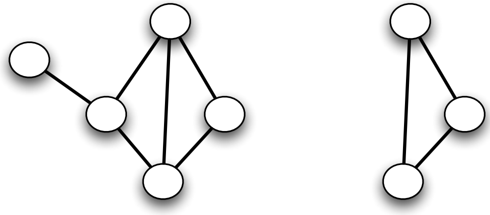

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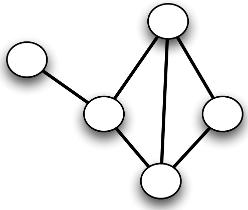
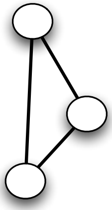

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Consequence: When studying $\text{CSP}(\Gamma)$ we can assume wlog that Γ contains the unary relation $\{c\}$ for every element c of Γ.

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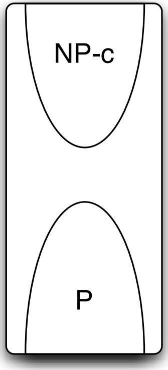
Tractability is already known when there is a polymorphism that is

- majority, that is, satisfies $\forall x, y.\ f(x,x,y) = f(x,y,x) = f(y,x,x) = x$.
- Maltsev, that is, satisfies $\forall x, y.\ f(x,y,y) = f(y,y,x) = x$.
- semi-lattice, that is, is binary commutative, associative, idempotent.
- ...

The Original Motivation of Feder and Vardi

NP



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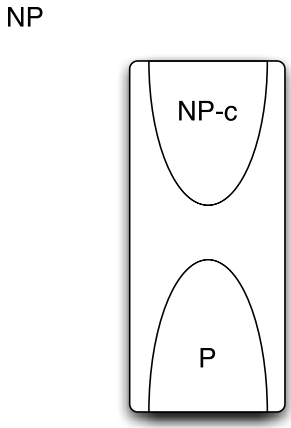

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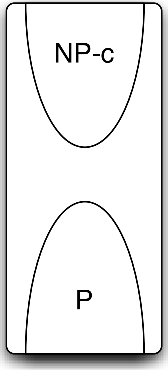

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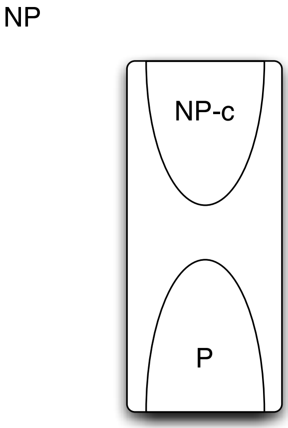

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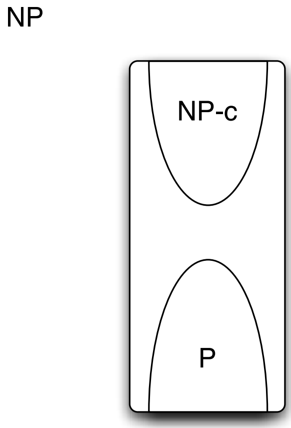
Related to Gurevich's (open) question: Is there an (effective) logic for P?



Syntactic Fragments of NP

Theorem 3 (Fagin's theorem).

An isomorphism-closed class $\mathcal{C}$ of finite τ-structures is in NP if and only if there is an existential second-order sentence Φ whose class of finite models is $\mathcal{C}$.

Inspection of the proof: we can choose Φ of the form

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Definition (Feder+Vardi)

An SNP sentence is monotone if all free relation symbols occur negatively.
An SNP sentence is monadic if all existentially quantified relations are unary.

The class of all problems that can be described by a (monotone, monadic) SNP sentence is also called (monotone, monadic) SNP.

Dichotomy versus Non-Dichotomy

Non-dichotomy:

Theorem 4 (Feder+Vardi'93).

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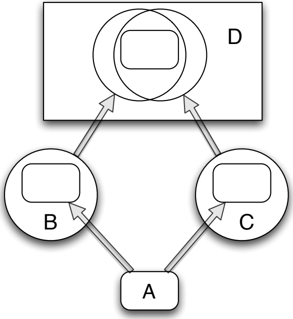
Dichotomy (probably...):

Theorem 5 (Feder+Vardi'93,Kun'07).

The class of problems in monotone monadic SNP has a complexity dichotomy if and only if the Feder-Vardi conjecture about finite domain CSPs is true.

Amalgamation SNP

An amalgamation SNP sentence is a monotone SNP sentence $\Phi = \exists R_1, \ldots, R_n.\ \varphi$ such that the class $\mathcal{C}$ of all finite models of φ has the amalgamation property.



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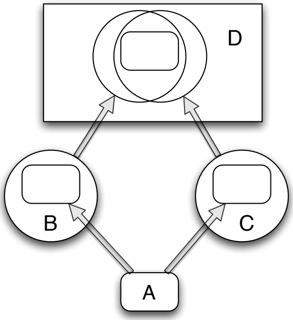

Example.

$$\exists T. \forall x, y, z. \big(x < y \rightarrow T(x, y)\big) \wedge \big((T(x, y) \wedge T(y, z)) \Rightarrow T(y, z)\big) \wedge \neg T(x, x)$$

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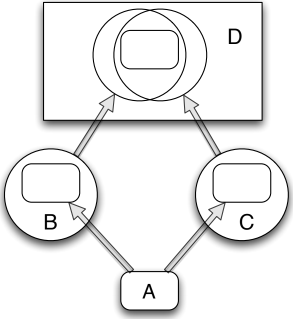

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Describes $\text{CSP}(\mathbb{Q}; <)$.

Amalgamation SNP

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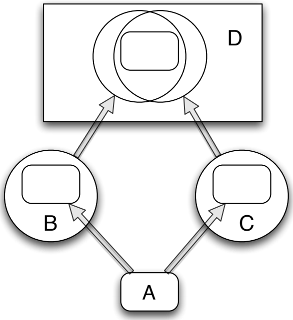

Example.

$$\exists T. \forall x, y, z. \big(x < y \rightarrow T(x,y)\big) \wedge \big((T(x,y) \wedge T(y,z)) \Rightarrow T(y,z)\big) \wedge \neg T(x,x)$$

Describes $\mathrm{CSP}(\mathbb{Q}; <)$.
Is in amalgamation SNP.

Dichotomy for Amalgamation CSP?

Conjecture. Amalgamation SNP has a complexity dichotomy into P and NP-complete.

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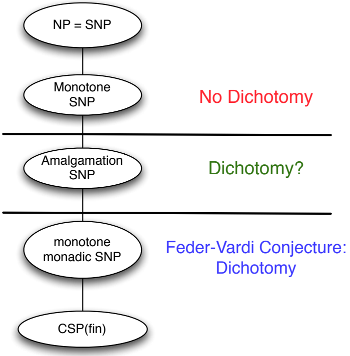
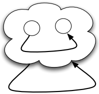
Remark. Every problem in monotone monadic SNP is a finite union of problems in amalgamation SNP (follows from Cherlin+Shelah+Shi'99).



Fraïssé Limits

A structure is called homogeneous if isomorphisms between finite substructures can be extended to automorphisms.



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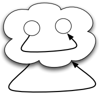

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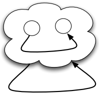

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Let $\mathcal{C}$ be a non-empty amalgamation class. Then there is a countable homogeneous τ-structure Γ whose age is $\mathcal{C}$. The structure Γ is unique up to isomorphism, and called the Fraïssé-limit of $\mathcal{C}$.

Examples: $(\mathbb{Q};<)$ is the Fraïssé-limit of the class of all finite linear orders. The Random graph is the Fraïssé-limit of the class of all finite graphs.

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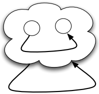

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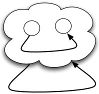

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Observation: every problem in amalgamation SNP can be formulated as CSP(Γ) for an ω-categorical structure Γ.

Generalizing the Universal-Algebraic Approach

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- Γ has a 4-ary canonical polymorphism f and automorphisms $\alpha_1, \alpha_2, \beta_1, \ldots, \beta_8$ such that for all $x, y \in V$

$$f(y,y,x,x) = \alpha_1 f(\beta_1 x, \beta_2 x, \beta_3 x, \beta_4 y) = \alpha_2 f(\beta_5 y, \beta_6 x, \beta_7 y, \beta_8 x) \; ,$$

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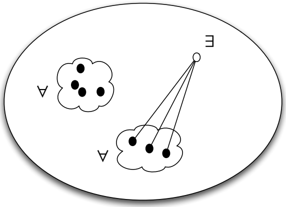

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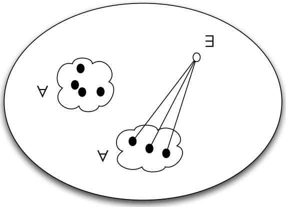

Two structures Γ and Δ are $\left\{ \begin{array}{l} \text{primitive positive} \\ \text{existential positive} \\ \text{first-order} \end{array} \right\}$ interdefinable

iff for every relation of Γ there is a $\{\dots\}$ definition in Δ, and vice versa.

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Thomas' Conjecture: every homogeneous structure with finite signature has only finitely many reducts.

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All reducts of $(V; E)$ (and all its reducts!) has less than $3^{\binom{n}{2}}$ many orbits of n-tuples, and hence is ω-categorical.

Locally Closed Permutation Groups

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Basic fact: $\mathcal{G} = \text{Aut}(D; R_1, R_2, \dots)$ is locally closed: whenever a permutation f of D is such that for every finite $A \subseteq D$ there exists $g \in \mathcal{G}$ such that $f(x) = g(x)$ for all $x \in A$, then $f \in \text{Aut}(\Gamma)$.

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Definition

For sets of permutations $\mathcal{F}, \mathcal{G}$, we say that $\mathcal{F}$ generates $\mathcal{G}$ iff $\mathcal{G}$ is the smallest locally closed permutation group that contains $\mathcal{F}$ and $\text{Aut}(E; V)$.

Theorem of Thomas, reformulated

Let – be any isomorphism between $(V; E)$ and $(V; N)$.

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Theorem 11 (Thomas'91; permutation group reformulation).

Let $\mathcal{G}$ be a locally closed permutation group containing $\text{Aut}(V;E)$. Then exactly one out of (1)-(5) is true:

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3. $\mathcal{G}$ is the group generated by $\{sw\}$.
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5. $\mathcal{G}$ is the group of all permutations on V.

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Let Δ_1, Δ_2 be structures. An operation $f : \Delta_1 \rightarrow \Delta_2$ is canonical if for all k-types t_1 in Δ_1 there exists a k-type t_2 in Δ_2 such that f maps every tuple of type t_1 to a tuple of type t_2.

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Can be generalized to higher-ary operations

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Theorem 12 (Thomas'96).

Let f be a permutation, $f \notin \mathrm{Aut}(G)$. Then f generates $-$ or sw.

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Theorem 13 (Abramson+Harrington,Nešetřil+Rödl).

For any relational signature τ, the class of all finite ordered τ-structures is a Ramsey class.

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Problem: If $\text{Age}(\Gamma)$ is Ramsey, is $\text{Age}(\Gamma, c_1, \ldots, c_n)$ still Ramsey?

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Observation:

Open subgroups of extremely amenable groups are extremely amenable.

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An ordered homogeneous structure Γ is Ramsey if and only if $G = \mathrm{Aut}(\Gamma)$ is extremely amenable, i.e., if every action of G on a compact Hausdorff space has a fixed point.

Observation:

Open subgroups of extremely amenable groups are extremely amenable.

Proposition 15 (B+Pinsker+Tsankov'11).

If Γ is ordered homogeneous Ramsey, then so is $(\Gamma, c_1, \ldots, c_n)$.

Application 1: Decidability of Definability

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'Decidability of Definability' in reducts of ordered Ramsey structures

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Theorem 16 (B+Pinsker+Tsankov'11).

When Δ is the reduct of an ordered homogeneous Ramsey structure whose age is described by finitely many finite forbidden induced substructures, then Expr(Δ) is decidable.

Application 2: Schaefer's theorem for graphs

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- Γ has a 4-ary (canonical) polymorphism f and $\alpha_1, \alpha_2 \in \mathrm{Aut}(G)$ such that for all $x, y \in V$

$$f(y,y,x,x) = \alpha_1 f(x,x,x,y) = \alpha_2 f(y,x,y,x) \; ,$$

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Application 3: Temporal CSPs

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- there is a primitive positive interpretation of $(\{0,1\}; \{(0,0,1),(0,1,0),(1,0,0)\})$ in an expansion of Γ by finitely many constants, and CSP(Γ) is NP-hard, or
- Γ has a binary polymorphism f and $\alpha, \beta_1, \beta_2 \in \text{Aut}(\mathbb{Q}; <)$ such that for all $x, y \in V$

$$f(x,y) = \alpha f(\beta_1 x, \beta_2 y) \ ,$$

  and CSP(Γ) is in P.

Open Problem

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Find a homogeneous structure in a finite relational signature that can not be expanded by finitely many relations so that the expansion

- is again homogeneous, and
- has an age with the Ramsey property

What is next?

Reducts of	First-order Reducts	Ramsey Class	CSP Dichotomy	Application, Motivation
$(X; =)$	Trivial	Ramsey's theorem	Yes	Equality Constraints
$(\mathbb{Q}; <)$	Cameron	Ramsey's theorem	Yes	Temporal Reasoning
$(V; E)$	Thomas	Nešetřil + Rödl	Yes	Schaefer for graphs
Homogeneous universal poset	?	Nešetřil+Rödl, Paoli+Trotter+Walter	?	Temporal Reasoning
Homogeneous C-relation	Adeleke et al	Deuber, Miliken	?	Phylogeny Reconstruction
Allen's Interval Algebra	?	Yes	?	Temporal Reasoning
Ctbl. atomless Bool. algebra	?	Graham, Leeb, Rothschild	?	Set Constraints