

Ramsey Theory and Reverse Mathematics

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<http://www.nd.edu/~cholak/papers/>

<http://www.nd.edu/~cholak/papers/italy.pdf>

Ramsey's theorem

- $[X]^n = \{Y \subseteq X : |Y| = n\}$.
- A k -coloring C of $[X]^n$ is a function from $[X]^n$ into a set of size k .
- H is *homogeneous* for C if C is constant on $[H]^n$, i.e. all n -element subsets of H are assigned the same color by C .
- RT_k^n is the statement that every k -coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

Reverse mathematics

What are the consequences of Ramsey's theorem (and its natural special cases) as a formal statement in second order arithmetic?

Questions

Question (Relation between 2nd order statements)

Does RT_2^2 imply WKL? Does $RT_{<\infty}^2$ imply WKL?

Question (First order consequences)

Is $RT_2^2 \Pi_1^1$ -conservative over $RCA_0 + B\Sigma_2$? Is $RT_{<\infty}^2 \Pi_1^1$ -conservative over $RCA_0 + B\Sigma_3$?

Question (Π_2^0 consequences)

Is $RT_2^2 \Pi_2^0$ -conservative over RCA_0 ? In particular, does RT_2^2 prove the consistency of $P^- + I\Sigma_1$? Does RT_2^2 prove that Ackerman's function is total?

Computability theory

Study the complexity (in terms of the arithmetical hierarchy or degrees) of infinite homogeneous sets for a coloring C relative to that of C . (For simplicity, assume that C is computable (recursive) and relativize.)

Language

Use the sorted language with the symbols: $=, \in, +, \times, 0, 1, <$;
Number variables: $n, m, x, y, z \dots$; Set Variables: $X, Y, Z \dots$

p is prime.

This just uses bounded quantification.

$$\forall \delta \exists \epsilon [|x - c| < \epsilon \Rightarrow |x^2 - c^2| < \delta]$$

This is an example of a Π_2^0 formula. The negation is Σ_2^0 . A formula which is logically equivalent (over our base theory) to both a Π_n^0 formula and a Σ_n^0 formula is Δ_n^0 .

Induction

$I\Sigma_n$ is the following statement: For every $\varphi(x)$, a Σ_n^0 formula, if $\varphi(0)$ and $\forall x[\varphi(x) \Rightarrow \varphi(x + 1)]$ then $\forall x[\varphi(x)]$.

Over our base theory, $I\Sigma_n^0$ and $II\Pi_n^0$ are equivalent. $I\Sigma_n$ is also equivalent to every Π_n^0 -definable set (Σ_n^0 set) has a least element.

Comprehension

Δ_1^0 comprehension is the statement: For every $\varphi(x)$, a Δ_1^0 formula, there is an X such that $X = \{x : \varphi(x)\}$.

For example, Δ_1^0 comprehension implies the set of all primes exists.

Arithmetic comprehension is the statement: For every $\varphi(x)$, a Δ_n^0 formula, there is an X such that $X = \{x : \varphi(x)\}$.

Bounding

Statement ($B\Sigma_n$)

For every $\varphi(x, y)$, a Σ_n^0 formula, if $\forall x \exists y [\varphi(x, y)]$ then for all a there is a b such that $\forall x \leq a \exists y \leq b [\varphi(x, y)]$.

Every initial segment of a Σ_n^0 function is bounded.

$B\Sigma_{n+1}$ is stronger than $I\Sigma_n$ but not as strong as $I\Sigma_{n+1}$.

2nd-order arithmetic

We work over models of 2nd-order arithmetic.

The intended model: $(\mathbb{N}, \mathcal{P}(\mathbb{N}), +, \times, 0, 1, <)$.

P^- is the theory of finite sets. The base theory, RCA_0 , is the logical closure of $P^- + I\Sigma_1^0$ and Δ_1^0 comprehension. PA is P^- plus arithmetic induction.

$$(\mathbb{N}, \{\text{all computable sets}\}, +, \times, 0, 1, <) \models RCA_0.$$

ACA_0 is RCA_0 plus arithmetic comprehension.

$$(\mathbb{N}, \{\text{all arithmetic sets}\}, +, \times, 0, 1, <) \models ACA_0.$$

Statements in 2nd-order arithmetic

Statement (WKL)

Every infinite tree of binary strings has an infinite branch.

$\forall T \exists P$ [if T is an infinite binary branching tree then P is an infinite path through T].

Statement (RT_k^n)

For every infinite set X and for every k -coloring of $[X]^n$ there is an infinite homogeneous set H .

Statement ($RT_{<\infty}^n$)

For every k , RT_k^n .

Statement (RT)

For every n , $RT_{<\infty}^n$.

These are Π_2^1 sentences: look at the set quantifiers ignore the (inside) number quantifiers.

Conservation

Definition

If T_1 and T_2 are theories and Γ is a set of sentences then T_2 is Γ -conservative over T_1 if $\forall \varphi [(\varphi \in \Gamma \wedge T_2 \vdash \varphi) \Rightarrow T_1 \vdash \varphi]$.

Theorem (H. Friedman)

ACA_0 is arithmetically conservative over PA .

Computability Theory

$A \leq_T B$ iff there is a computer which using an oracle for B can compute A . For all A , $\emptyset \leq_T A$.

A' (read A -jump) is all those programs e which using A as an oracle halt on input e . $A^{(n)}$ is the n^{th} jump of A . The jump operation is order preserving.

So for all A , $\emptyset^{(n)} \leq_T A^{(n)}$. A is low_n iff $A^{(n)} \leq_T \emptyset^{(n)}$.

Key idea: if A is low_n then sets which are Δ_{n+1}^0 in A are Δ_{n+1}^0 in \emptyset .

WKL

Theorem (Jockusch and Soare)

[The Low Basis Theorem] Every infinite computable tree of binary strings has a low path (working in the standard model).

Theorem (Harrington)

$RCA_0 + WKL$ is Π_1^1 -conservative over RCA_0 .

Adding a path

Lemma (Harrington)

If $\mathcal{M} = (\mathbb{X}, \mathcal{F}, +, \times, 0, 1, <)$ is a model of RCA_0 , $T \in \mathcal{F}$ and T codes an infinite tree of binary strings then there is a $G \subset \mathbb{X}$ such that $\mathcal{M}' = (\mathbb{X}, \mathcal{F} \cup G, +, \times, 0, 1, <)$ is a model of $I\Sigma_1$ and P^- and G is an infinite path through T .

Lemma (H. Friedman)

Any model of P^- and $I\Sigma_n$ can be expanded to a model of $RCA_0 + I\Sigma_n$ by only adding reals.

Iterating the addition of a path

Corollary (Harrington)

Every countable model \mathcal{M} of RCA_0 is a ω -submodel (the integers do not change) of some countable model \mathcal{M}' of $RCA_0 + WKL$.

By Gödel completeness, this implies Theorem 12. All Σ_1^1 sentences true in \mathcal{M} are true in \mathcal{M}' .

Lemma

Every countable model of $RCA_0 + I\Sigma_n$ is a ω -submodel of some countable model of $RCA_0 + I\Sigma_n + WKL$.

Ramsey's Theorem – Known Results

Theorem (Specker)

There is a computable 2-coloring of $[\mathbb{N}]^2$ with no infinite computable homogeneous set.

Corollary (Specker)

RT_2^2 is not provable in RCA_0 .

Results of Jockusch

Theorem (Jockusch)

1. *For any n and k , any computable k -coloring of $[\mathbb{N}]^n$ has an infinite Π_n^0 homogeneous set.*
2. *For any $n \geq 2$, there is a computable n -coloring of $[\mathbb{N}]^n$ which has no infinite Σ_n^0 homogeneous set.*
3. *For any n and k and any computable k -coloring of $[\mathbb{N}]^n$, there is an infinite homogeneous set A with $A' \leq_T 0^{(n)}$.*
4. *For each $n \geq 2$, there is a computable 2-coloring of $[\mathbb{N}]^n$ such that $0^{(n-2)} \leq_T A$ for each infinite homogeneous set A .*

Jockusch's results into Reverse Mathematics

Theorem (Simpson)

1. For each $n \geq 3$ and $k \geq 2$ (both n and k standard), the statements RT_k^n are equivalent to ACA_0 over RCA_0 .
2. The statement RT is not provable in ACA_0 .
3. RT is equivalent to ACA_0 plus for all n , for all X , the n^{th} -jump of X exists.
4. RT does not prove ATR .
5. ATR proves RT .

Cone Avoidance

Theorem (Seetapun)

For any computable 2-coloring C of $[\mathbb{N}]^2$ and any noncomputable sets C_0, C_1, \dots , there is an infinite homogeneous set X such that $(\forall i)[C_i \not\leq_T X]$.

Corollary (Seetapun)

RT_2^2 does not imply ACA_0 . Hence, over RCA_0 , RT_2^2 is strictly weaker than RT_2^3 .

First order consequences

Theorem (Hirst)

RT_2^2 proves $B\Sigma_2$.

Corollary (Hirst)

- RT_2^2 is stronger than RCA_0 .
- RT_2^2 is not Σ_3^0 -conservative over RCA_0 .

$B\Sigma_2$ is strictly between $I\Sigma_1$ and $I\Sigma_2$.

Our Work – Computability Theory

Theorem

For any computable 2-coloring of $[\mathbb{N}]^2$, there is an infinite homogeneous set X which is low_2 , i.e. $X'' \leq_T 0''$.

Definition

An infinite set X is r -cohesive if for each computable set R , $X \subseteq^* R$ or $X \subseteq^* \overline{R}$.

Theorem (Jockusch and Stephan)

There exists a low_2 r -cohesive set.

Theorem

For each Δ_2^0 set A there is an infinite low_2 set X which is contained in A or \overline{A} .

Definition (S)

A k -coloring of $[X]^2$ is called *stable* if for each a , the color assigned to the pair $\{a, b\}$ has a fixed color c_a for all sufficiently large b (i.e., there is a d_a such that for all b greatly than d_a , the color of $\{a, b\}$ is c_a).

Now any computable coloring of pairs (from \mathbb{N}) becomes stable when it is restricted to an r -cohesive set.

Lemma

For any computable stable 2-coloring C of $[\mathbb{N}]^2$, there are 2 disjoint Δ_2^0 sets A_i such that $\bigsqcup_{i < 2} A_i = \mathbb{N}$ and any infinite subset of some A_i computes an infinite homogeneous set for C .

Our work – Reverse mathematics

Theorem

$RCA_0 + RT_2^2$ is Π_1^1 -conservative over $RCA_0 + I\Sigma_2$.

Corollary

RT_2^2 does not imply PA over RCA_0 .

This improves Seetapun's result that RT_2^2 does not imply ACA_0 over RCA_0 .

$$RT_2^2 = COH + SRT_2^2$$

Statement (COH)

Let R_i be a sequence of sets. Then there is a set G such that for all i , either $G \subseteq^* R_i$ or $G \subseteq^* \overline{R_i}$. (There “modulo finite” is coded by finite sets in our models of arithmetic.)

Over RCA_0 , RT_2^2 follows from $COH + SRT_2^2$.

Theorem (Mileti, Lempp and Jockusch)

Over RCA_0 , RT_2^2 and $COH + SRT_2^2$ are equivalent.

COH

Theorem

$RCA_0 + COH$ is Π_1^1 -conservative over RCA_0 .

Theorem

Every countable model of $RCA_0 (+I\Sigma_2)$ is a ω -submodel of some countable model of $RCA_0 + COH (+I\Sigma_2)$.

Start with any model of RCA_0 . Given a sequence of sets, R_i , add a \vec{R} -cohesive set G while preserving $I\Sigma_1$. Close to get a model of RCA_0 . Iterate over all such sequences to get a model of $RCA_0 + COH$.

Adding a cohesive set G

Force using the conditions $\langle D, L \rangle$, where D is finite, L is an infinite set in the ground model and $\max D < \min L$. We say $\langle D', L' \rangle$ extends $\langle D, L \rangle$ iff $D \subseteq D' \subset D \cup L$ and $L' \subseteq L$.

If G is any generic set then either $G \subseteq^* R_i$ or $G \subseteq^* \bar{R}_i$.

To preserve $I\Sigma_1$, for all $\psi(x, G)$, a Σ_1^0 -formula and all numbers a , we want to ensure either $\forall x \leq a[\psi(x, G)]$ or for some b , $\neg\psi(b, G) \wedge \forall x < b\psi(x, G)$.

The key to preserving $I\Sigma_1$

Let $\langle D, L \rangle$ be a given condition. By $I\Sigma_1$ in the ground model, there is a least $c \leq a$ (if any) such that $\langle D, L \rangle$ cannot be extended to a condition which forces $\forall y \leq c [\psi(y, G)]$ (whether such an extension of $\langle D, L \rangle$ exists is Σ_1^L). If there is no such c , then $\langle D, L \rangle$ has an extension which forces $\forall x \leq a [\psi(x, G)]$. If there is such a c , then by minimality, there is a condition $\langle D', L' \rangle$ extending $\langle D, L \rangle$ which forces $\forall x < c [\psi(x, G)]$ and $\neg\psi(c, G)$ (since it has no extension which forces $\psi(c, G)$).

COH and WKL

Theorem

COH and WKL are independent over RCA_0 .

SRT_2^2

Theorem

$RCA_0 + SRT_2^2 + WKL$ is Π_1^1 -conservative over $RCA_0 + I\Sigma_2$.

Theorem

Every countable model of $RCA_0 + I\Sigma_2$ is a ω -submodel of some countable model of $RCA_0 + I\Sigma_2 + WKL + SRT_2^2$.

Theorem

SRT_2^2 proves $B\Sigma_2$. Hence SRT_2^2 is not Σ_3^0 -conservative over RCA_0 .

Why Σ_2 and WKL ?

- Which color? Red or Blue?
- Now whether we can extend a condition to another which forces a Σ_1^0 (Π_1^0) statement is no longer Σ_1^0 (Π_1^0).
- But using WKL , whether we can extend a condition to another which forces a Σ_2^0 (Π_2^0) statement is Σ_2^0 (Π_2^0).

RT_2^2

Theorem

$RCA_0 + RT_2^2 + WKL$ is Π_1^1 -conservative over $RCA_0 + I\Sigma_2$.

Theorem

Every countable model of $RCA_0 + I\Sigma_2$ is a ω -submodel of some countable model of $RCA_0 + I\Sigma_2 + WKL + SRT_2^2 + COH$.

Finitely many colors

Theorem

$RCA_0 + RT_{<\infty}^2$ is Π_1^1 -conservative over $RCA_0 + I\Sigma_3$.

Theorem

Every countable model of $RCA_0 + I\Sigma_3$ with a real of greatest Turing degree is a ω -submodel of some countable model of $RCA_0 + I\Sigma_3 + WKL + SRT_{<\infty}^2 + COH$.

Theorem

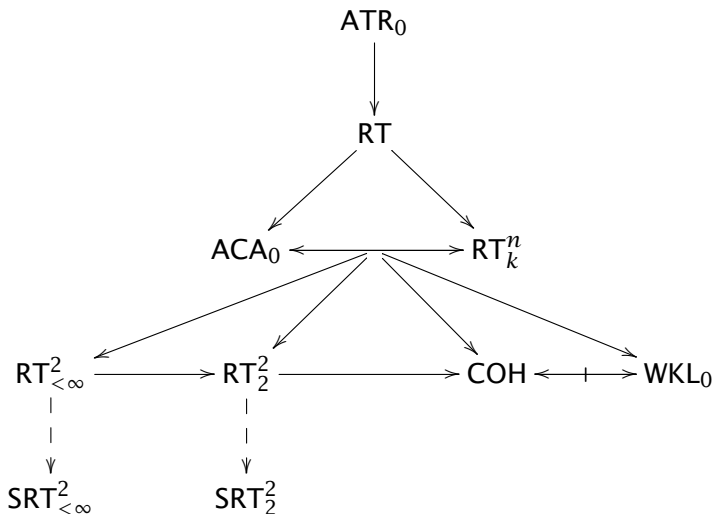
$RCA_0 + SRT_{<\infty}^2 \vdash B\Sigma_3$.

Since $B\Sigma_3$ is stronger than $I\Sigma_2$, we have that RT_2^2 does not imply $RT_{<\infty}^2$.

Why $\mathcal{I}\Sigma_3$ and a real of greatest Turing degree?

- To determine which color we need $\mathcal{I}\Sigma_3$ to hold in the ground model.
- Again using *WKL*, whether we can extend a condition to another which forces a Σ_2^0 (Π_2^0) statement is Σ_2^0 (Π_2^0).
- But we need to preserve $\mathcal{I}\Sigma_3$. This requires us to quantify over the conditions.

Some second order consequences



Some first order consequences

Theorem

Let $(\varphi)^1$ be the first order consequences of $\varphi + RCA_0$.

1. $(RCA_0)^1 = (WKL_0)^1 = (COH)^1$.
2. $(RCA_0)^1 \subsetneq (B\Sigma_2)^1 \subseteq (SRT_2^2)^1 \subseteq (RT_2^2)^1 \subseteq (I\Sigma_2)^1$.
3. $(I\Sigma_2)^1 \subsetneq (B\Sigma_3)^1 \subseteq (SRT_{<\infty}^2)^1 \subseteq (RT_{<\infty}^2)^1 \subseteq (I\Sigma_3)^1$.
4. $(I\Sigma_3)^1 \subsetneq PA = (RT_2^3)^1 = (RT_n^k)^1$ (for any standard $k \geq 3$ and $n \geq 2$).

Questions

Question (Relation between 2nd order statements)

Does RT_2^2 imply WKL? Does $RT_{<\infty}^2$ imply WKL?

Question (First order consequences)

Is $RT_2^2 \Pi_1^1$ -conservative over $RCA_0 + B\Sigma_2$? Is $RT_{<\infty}^2 \Pi_1^1$ -conservative over $RCA_0 + B\Sigma_3$?

Question (Π_2^0 consequences)

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