Nonstandard Methods in Combinatorics of Numbers: a few examples

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Introduction

In combinatorics of numbers one can find deep and fruitful interactions among diverse *non-elementary* methods, namely:

- Ergodic theory
- Fourier analysis
- Topological dynamics
- Algebra in the space of ultrafilters $\beta\mathbb{N}$

Also nonstandard analysis has recently started to give contributions in this area, starting from the following result:

Theorem (R.Jin 2000)

If A and B are sets of integers with positive upper Banach density, then $A + B$ is piecewise syndetic.

(A set is *piecewise syndetic* if it has bounded gaps on arbitrarily large intervals. The *Banach density* is a refinement of the upper asymptotic density.)

The goal of this talk is to present a few examples to illustrate the use of *nonstandard analysis* in this area of research.

1. Quick introduction to the *hyper-integers* of nonstandard analysis.
2. Hyper-integers as *ultrafilters* and an ultrafilter proof of *Rado's theorem* on monocromatic injective solutions of diophantine equations.
3. Nonstandard characterization of *Banach density* and applications in *additive number theory*.

Nonstandard Analysis, hyper-quickly

Nonstandard analysis is essentially grounded on the following two properties:

1. Every object X can be extended to an object ${}^*X$.
2. ${}^*X$ is a sort of "weakly isomorphic" copy of X, in the sense that it satisfies exactly the same "elementary properties" as X.

E.g., ${}^*\mathbb{R}$ is an *ordered field* that properly extends the real line $\mathbb{R}$. The two structures $\mathbb{R}$ and ${}^*\mathbb{R}$ cannot be distinguished by any "elementary property".

Star-map

To every mathematical object X is associated its *hyper-extension* (or *nonstandard extension*) ${}^*X$.

$$X \longmapsto {}^*X$$

If $r \in \mathbb{R}$ is a number, we assume that ${}^*r = r$. We also assume the non-triviality condition $A \subsetneq {}^*A$ for all infinite $A \subseteq \mathbb{R}$.

- ${}^*\mathbb{N}$ is the set of hyper-natural numbers,
- ${}^*\mathbb{Z}$ is the set of hyper-integer numbers,
- ${}^*\mathbb{Q}$ is the set of hyper-rational numbers,
- ${}^*\mathbb{R}$ is the set of hyper-real numbers, and so forth.

Transfer principle

If $P(x_1, \ldots, x_n)$ is any property expressed in "elementary terms", then

$$P(A_1, \ldots, A_n) \Longleftrightarrow P(^*A_1, \ldots, ^*A_n)$$

P is expressed in "elementary terms" if it is written in the *first-order language of set theory*, *i.e.* everything is expressed by only using the *equality* and the *membership* relations.
Not a limitation: (virtually) *all* mathematical objects can be "coded" as sets.

Moreover, quantifiers must be used in the *bounded forms*:

$$\text{"}\forall x \in A \;\; P(x, \ldots)\text{"} \quad \text{and} \quad \text{"}\exists x \in A \;\; P(x, \ldots)\text{"}.$$

By *transfer*, the following are easily proved.

1. $A \subseteq B \Leftrightarrow {}^*A \subseteq {}^*B$.
2. ${}^*(A \cup B) = {}^*A \cup {}^*B$
3. ${}^*(A \cap B) = {}^*A \cap {}^*B$
4. ${}^*(A \setminus B) = {}^*A \setminus {}^*B$
5. ${}^*(A \times B) = {}^*A \times {}^*B$
6. $f : A \to B \Leftrightarrow {}^*f : {}^*A \to {}^*B$
7. The function f is 1-1 $\Leftrightarrow$ the function ${}^*f$ is 1-1
8. *etc.*

By *transfer*, ${}^*\mathbb{R}$ is an *ordered field* where the sum and product operation are the hyper-extensions of the binary functions $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; and the order relation is the hyper-extension ${}^*\{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a < b\}$.

Moreover:

- The *hyper-rational numbers* ${}^*\mathbb{Q}$ are dense in ${}^*\mathbb{R}$.
- Every $\xi \in {}^*\mathbb{R}$ has an *integer part*, *i.e.* there exists a unique hyper-integer $\nu \in {}^*\mathbb{Z}$ such that $\nu \leq \xi < \nu + 1$.

and so forth.

As a proper extension of the reals, the hyper-real field ${}^*\mathbb{R}$ contains infinitesimal numbers $\varepsilon \neq 0$ such that:

$$-\frac{1}{n} < \varepsilon < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}$$

as well as infinite numbers

$$|\Omega| > n \quad \text{for all } n \in \mathbb{N}.$$

So, ${}^*\mathbb{R}$ is *not* Archimedean, and hence it is *not* complete (the bounded set of infinitesimals does not have a least upper bound).

- How about the *transfer principle*?

Here is the correct formalization of the Archimedean property in elementary terms:

$$\forall x, y \in \mathbb{R} \;\; 0 < x < y \Rightarrow \exists n \in \mathbb{N} \text{ s.t. } n \cdot x > y$$

By *transfer*:

$$\forall \xi, \eta \in {}^*\mathbb{R} \;\; 0 < \xi < \eta \Rightarrow \exists \nu \in {}^*\mathbb{N} \text{ s.t. } \nu \cdot \xi > \eta$$

Remark that the above property does not express the *Archimedean property* of ${}^*\mathbb{R}$. In fact, ${}^*\mathbb{N}$ also contains *infinite* numbers.

The completeness property of the real numbers:

$$\forall X \in \mathcal{P}(\mathbb{R}) \;\; X \text{ nonempty bounded } \Rightarrow \exists r \in \mathbb{R} \;\; r = \sup X$$

transfers to:

$$\forall X \in {}^*\mathcal{P}(\mathbb{R}) \;\; X \text{ nonempty bounded } \Rightarrow \exists \xi \in {}^*\mathbb{R} \;\; \xi = \sup X$$

The point is that ${}^*\mathcal{P}(\mathbb{R})$ is a proper subfamily of $\mathcal{P}({}^*\mathbb{R})$.

Sets in ${}^*\mathcal{P}(\mathbb{R})$ are the "well-behaved" ones. They are called internal sets.

Hyper-finite sets

Definition

A hyper-finite set $A \subset {}^*\mathbb{R}$ is an element of ${}^*\{F \subset \mathbb{R} \mid F \text{ is finite}\} \subset {}^*\mathcal{P}(\mathbb{R})$.

Hyper-finite are a fundamental tool in nonstandard analysis, because they "behave" as *finite sets*. For instance:

- A is *hyper-finite* $\Leftrightarrow$ there exists an *internal* bijection $f : \{1, \ldots, \nu\} \to A$ for some $\nu \in {}^*\mathbb{N}$.
- Every *hyperfinite set* $A \subset {}^*\mathbb{R}$ has a least and a greatest element.

(An *internal function* is an element of ${}^*\{f \subset \mathbb{R} \times \mathbb{R} \mid f \text{ is a function}\}$.)

Models of the hyper-reals

Models of hyper-real numbers ${}^*\mathbb{R}$ are easily obtained by algebraic means.

- Take the *ring of real sequences* $\mathrm{Fun}(\mathbb{N}, \mathbb{R})$.
  (One may replace $\mathbb{N}$ with any infinite set of indexes).
- Take a *maximal ideal*

$$\mathfrak{m} \supset \{\sigma : \mathbb{N} \to \mathbb{R} \mid \sigma(n) = 0 \text{ for all but finitely many } n\}$$

- Let ${}^*\mathbb{R}$ be the *quotient field*

$${}^*\mathbb{R} = \mathrm{Fun}(\mathbb{N}, \mathbb{R})/\mathfrak{m}$$

- The *hyper-extensions* of sets $A \subseteq \mathbb{R}$ are defined by

$$^*A = \text{Fun}(\mathbb{N}, A)/\mathfrak{m} \subseteq {}^*\mathbb{R}$$

- The *hyper-extensions* of functions $f : A \to B$ are defined by

$$^*f : [\sigma]_\mathfrak{m} \longmapsto [f \circ \sigma]_\mathfrak{m}$$

Equivalently, the same construction can be presented as an *ultrapower* $\mathbb{R}^\mathbb{N}/\mathcal{U}$ of the real numbers modulo a *non-principal ultrafilter* $\mathcal{U}$ on $\mathbb{N}$.

The hyper-integers

By *transfer*, one can easily show that the hyper-integers ${}^*\mathbb{Z}$ are a *discretely ordered ring* whose positive part are the hyper-natural numbers ${}^*\mathbb{N}$.

$$ {}^*\mathbb{N} = \{ \underbrace{1, 2, \ldots, n, \ldots}_{\text{finite numbers}} \quad \underbrace{\ldots, N-2, N-1, N, N+1, N+2, \ldots}_{\text{infinite numbers}} \} $$

Hyper-integers can be used as a convenient setting for the study of certain *density properties* and certain aspects of *additive number theory*.

By *transfer*, one directly obtains that

An internal set $A \subset {}^*\mathbb{Z}$ is hyper-finite $\Leftrightarrow$ A is bounded.

In particular, all *intervals* of hyper-integers are hyper-finite:

$$[\mu, \nu] = \{\xi \in {}^*\mathbb{Z} \mid \mu \leq \xi \leq \nu\}$$

Remark that the *hyper-finite* set $[\mu, \nu]$ contains *infinitely many* numbers when the hyper-natural number $\nu - \mu$ is infinite.

As a first example of possible uses of *hyper-integers* in combinatorics, let us consider the infinite version of *Ramsey Theorem* (for pairs).

Theorem (Ramsey 1928 – *Infinite version*)

Let X be infinite and let $[X]^2 = C_1 \cup \ldots \cup C_r$ be a finite coloring. Then exists an infinite homogeneous $H \subseteq X$, i.e. the pairs $[H]^2 \subseteq C_i$ for some i.

The following proof uses the *hyper-hyper-natural numbers* ${}^{**}\mathbb{N}$.

A nonstandard proof of Ramsey theorem

Proof.

Pick an infinite $\nu \in {}^*\mathbb{N}$. Then $\{\nu, {}^*\nu\} \in {}^{**}C_i$ for some i.
$\nu \in \{\xi \in {}^*\mathbb{N} \mid \{\xi, {}^*\nu\} \in {}^{**}C_i\} = {}^*\{n \in \mathbb{N} \mid \{n, \nu\} \in {}^*C_i\} = {}^*A$.
Pick $a_1 \in A$, so $\{a_1, \nu\} \in {}^*C_i$. Then
$\nu \in \{\xi \in {}^*\mathbb{N} \mid \{a_1, \xi\} \in {}^*C_i\} = {}^*\{n \in \mathbb{N} \mid \{a_1, n\} \in C_i\} = {}^*B_1$.
$\nu \in {}^*A \cap {}^*B_1 \Rightarrow A \cap B_1$ is infinite: pick $a_2 \in A \cap B_1$ with $a_2 > a_1$.
$a_2 \in B_1 \Rightarrow \{a_1, a_2\} \in C_i$.
$a_2 \in A \Rightarrow \{a_2, \nu\} \in {}^*C_i \Rightarrow \nu \in {}^*\{n \in \mathbb{N} \mid \{a_2, n\} \in {}^*C_1\} = {}^*B_2$.
$\nu \in {}^*A \cap {}^*B_1 \cap {}^*B_2 \Rightarrow$ we can pick $a_3 \in A \cap B_1 \cap B_2$ with $a_3 > a_2$.
$a_3 \in B_1 \cap B_2 \Rightarrow \{a_1, a_3\}, \{a_2, a_3\} \in C_i$, and so forth.
$H = \{a_n \mid n \in \mathbb{N}\}$ is homogeneous: $[H]^2 \subset C_i$.

Ultrafilters as hyper-integers

In a nonstandard setting, every *hyper-natural number* $\nu \in {}^*\mathbb{N}$ generates an *ultrafilter*:

$$\mathfrak{U}_\nu = \{A \subseteq \mathbb{N} \mid \nu \in {}^*A\}$$

Definition

An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ such that:

1. $\mathbb{N} \in \mathcal{U}, \emptyset \notin \mathcal{U}$;
2. $A \in \mathcal{U}, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$;
3. $A \in \mathcal{U}, A \subseteq B \Rightarrow B \in \mathcal{U}$.
4. $A_1 \cup \ldots \cup A_n \in \mathcal{U} \;\; \Rightarrow \;\; A_i \in \mathcal{U}$ for some i.

By suitably choosing the nonstandard model:

Every ultrafilter is generated by some number $\nu \in {}^*\mathbb{N}$.

(It takes the so-called $\mathfrak{c}^+$-*enlarging property*, a form of *saturation*).

So, in a nonstandard setting, every ultrafilter is a "*principal*" ultrafilter!

A celebrated theorem in combinatorics of numbers is the following:

Theorem (Hindman 1974)

For every finite coloring of $\mathbb{N}$ there exists an infinite X such that all sums of distinct elements of X are monocromatic.

The original proof consisted in really intricate combinatorial arguments.

"Anyone with a very masochistic bent is invited to wade through the original combinatorial proof." (Neil Hindman)

The very next year, Galvin and Glazer found an elegant (and much "simpler") proof by using a strange algebra on the *space of ultrafilters* $\beta\mathbb{N}$ over the natural numbers.

That proof is grounded on ultrafilters $\mathcal{U} = \mathcal{U} \oplus \mathcal{U}$ that are idempotent with respect to a "pseudo-sum" operation:

$$A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \mid A - n \in \mathcal{V}\} \in \mathcal{U}$$

where $A - n = \{m \mid m + n \in A\}$

The existence of *idempotent ultrafilters* follows from iterated applications of *Zorn's lemma* in the *compact topological right semi-group* $(\beta\mathbb{N}, \oplus)$. *Idempotent ultrafilters* yield important applications in combinatorics. *E.g.*

Let $\mathcal{U}$ be an *idempotent* ultrafilter. Then

1. Every $A \in \mathcal{U}$ includes the set of all sums of distinct elements of some infinite set.
2. If $\mathcal{U}$ is "minimal" then every $A \in \mathcal{U}$ contains *arbitrarily long arithmetic progressions*.

(1) $\Rightarrow$ Hindman theorem

(2) $\Rightarrow$ van der Waerden theorem

Nonstandard characterization

An ultrafilter $\mathcal{U}$ is *idempotent* if and only if there exists $\nu \in {}^*\mathbb{N}$ such that

$$\mathcal{U} \;=\; \mathfrak{U}_\nu \;=\; \mathfrak{U}_{\nu + {}^*\nu}$$

Note that $\nu + {}^*\nu \in {}^{**}\mathbb{N}$.
Also numbers $\theta \in {}^{**}\mathbb{N}$ or in ${}^{***}\mathbb{N}$ and so forth generate ultrafilters:

$$\mathfrak{U}_\theta \;=\; \{A \subseteq \mathbb{N} \mid \theta \in {}^{**}A\}$$

The above characterization makes it possible to handle *linear combinations of idempotent ultrafilters* in a manageable manner.

Theorem (Bergelson-Hindman 1990)

Let $\mathcal{U}$ be an idempotent ultrafilter. Then every $A \in 2\mathcal{U} \oplus \mathcal{U}$ contains an arithmetic progression of length 3.

The nonstandard proof reduces to the following simple observation.

If ν is such that $\mathcal{U} = \mathfrak{U}_\nu = \mathfrak{U}_{\nu + {}^*\nu}$ then

- $\xi = 2\nu + \ 0 \ + {}^{**}\nu$
- $\zeta = 2\nu + {}^*\nu + {}^{**}\nu$
- $\vartheta = 2\nu + 2{}^*\nu + {}^{**}\nu$

form an *arithmetic progression* of length 3 in ${}^{***}\mathbb{N}$, and

$$\mathfrak{U}_\xi = \mathfrak{U}_\zeta = \mathfrak{U}_\vartheta = 2\mathcal{U} \oplus \mathcal{U}$$

Then for every $A \in 2\mathcal{U} \oplus \mathcal{U}$, the numbers $\xi, \zeta, \vartheta \in {}^{***}A$ and so, by *transfer*, there exist 3 elements in A in arithmetic progression.

More generally, one obtains the following:

Theorem

Let $\mathcal{U}$ be any given idempotent ultrafilter. Then for every diophantine equation

$$c_1X_1 + \ldots + c_nX_n = 0$$

where $\sum_{i=1}^{n} c_i = 0$ there exists a linear combination

$$\mathcal{W} = \mathcal{U} \oplus a_1\mathcal{U} \oplus \ldots \oplus a_{n-2}\mathcal{U}$$

such that an injective solution $\xi_1, \ldots, \xi_n$ is found in every $A \in \mathcal{W}$.

As a straight consequence, one gets an *ultrafilter proof* of a restricted version of *Rado theorem*.

Corollary

Let $c_1X_1 + \ldots + c_nX_n = 0$ be a diophantine equation where $\sum_{i=1}^{n} c_i = 0$. Then for every finite coloring of $\mathbb{N}$ there exists an injective monocromatic solution.

Proof. By working in a hyper-hyper-…-hyper-extension of $\mathbb{N}$, one find suitable coefficients a_i and distinct generators $\xi_1, \ldots, \xi_n$ of the same ultrafilter $\mathcal{U} \oplus a_1\mathcal{U} \oplus \ldots \oplus a_{n-2}\mathcal{U}$ that form a solution.

This technique can also be applied to certain *non-linear* equations. *(Work in progress with L. Luperi)*

Largeness notions for sets of integers

We now recall a few basic notions of "largeness" for sets of integers, along with their *nonstandard* characterizations:

Definition

A is thick if for every k there exists x such that $[x, x+k] \subseteq A$.

Definition (Nonstandard)

A is thick if there exists an *infinite interval* $[\nu, \mu] \subseteq {}^*A$.

Definition

A is syndetic if there exists $k \in \mathbb{N}$ such that every interval $[x, x+k] \cap A \neq \emptyset$. (That is, if A^c is *not* thick.)
Equivalently, there exists a finite F such that $F + A = \mathbb{Z}$.

Definition (Nonstandard)

A is syndetic if ${}^*A$ has only *finite gaps*,
i.e. ${}^*A \cap I \neq \emptyset$ for every infinite interval I.

Definition

A is piecewise syndetic if $A = B \cap C$ where B is *thick* and C is *syndetic*.
Equivalently, there exists a finite F such that $F + A$ is thick.

Definition (Nonstandard)

A is piecewise syndetic if ${}^*A$ has only *finite gaps* on some *infinite interval*.

In every finite partition $A = C_1 \cup \ldots \cup C_r$ of a *piecewise syndetic* set, one of the pieces C_i is *piecewise syndetic*.

Nonstandard proof.

By induction, it is enough to check the property for 2-partitions $A = \text{BLUE} \cup \text{RED}$.

By *transfer*, ${}^*A = {}^*\text{BLUE} \cup {}^*\text{RED}$ is a 2-partition. Pick an infinite interval I where ${}^*A$ has only finite gaps.

If the *blue elements of ${}^*A$ has only finite gaps in I, then BLUE is piecewise syndetic.

Otherwise, there exists an infinite interval $J \subseteq I$ that only contains *red elements of ${}^*A$. But then ${}^*\text{RED}$ has only finite gaps in J, and hence RED is piecewise syndetic.

An important "measure" of largeness is given by the asymptotic density.

Definition

The upper asymptotic density of a set $A \subseteq \mathbb{N}$ is defined by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}$$

Let α be a non-negative real number. The following are equivalent:

1. $\overline{d}(A) \geq \alpha$
2. $\frac{|{}^*A \cap [1,\nu]|}{\nu} \approx \alpha$ for some infinite $\nu \in {}^*\mathbb{N}$.

A useful generalization of the *upper asymptotic density* is the following.

Definition

The Banach density of a set $A \subseteq \mathbb{Z}$ is defined by

$$\mathrm{BD}(A) = \lim_{n \to \infty} \left(\max_{k \in \mathbb{Z}} \frac{|A \cap [k+1, k+n]|}{n} \right)$$

- $\mathrm{BD}(A) = 1 \Leftrightarrow A$ is *thick*.

Clearly $\mathrm{BD}(A) \geq \overline{d}(A)$
(In fact, there are sets thick sets A with $\overline{d}(A) = 0$.)

Definition

B is a basis of order h if

$$\underbrace{B + \ldots + B}_{h \text{ times}} = h \cdot B = \mathbb{N}$$

A classic result in additive number theory is the following.

Theorem (Plünnecke 1970)

If B is a basis of order h (i.e. if $\sigma(h \cdot B) = 1$) then $\sigma(A + B) \geq \sigma(A)^{1-\frac{1}{h}}$ for all A.

By applying Plünnecke's theorem jointly with methods of *nonstandard analysis*, recently Jin proved the following

Theorem (Jin 2009)

1. *If B is a Banach basis of order h (i.e. if $BD(h \cdot A) = 1$) then $BD(A+B) \geq BD(A)^{1-\frac{1}{h}}$ for all A.*
2. *If B is a lower asymptotic basis of order h (i.e. if $\underline{d}(h \cdot A) = 1$) then $\underline{d}(A+B) \geq \underline{d}(A)^{1-\frac{1}{h}}$ for all A.*

Remark. The same result does not hold for the *upper asymptotic density* $\overline{d}$.

The proof is derived from the following *nonstandard characterizations*, which provide a bridge connecting *Banach density*, *Schnirel'man density*, *lower asymptotic density* and the *upper asymptotic density*.

Theorem

Let A be a set of natural numbers and let $0 \leq \alpha \leq 1$. The following are equivalent:

1. $BD(A) \geq \alpha$
2. $\sigma_\nu({}^*A) = \inf_{n\in\mathbb{N}} \frac{|{}^*A\cap[\nu+1,\nu+n]|}{n} \geq \alpha$ *for some* $\nu \in {}^*\mathbb{N}$.
3. $\underline{d}_\nu({}^*A) = \liminf_{n\to\infty} \frac{|{}^*A\cap[\nu+1,\nu+n]|}{n} \geq \alpha$ *for some* $\nu \in {}^*\mathbb{N}$.
4. $\overline{d}_\nu({}^*A) = \limsup_{n\to\infty} \frac{|{}^*A\cap[\nu+1,\nu+n]|}{n} \geq \alpha$ *for some* $\nu \in {}^*\mathbb{N}$.

The generalization of Plünnecke's theorem for lower asymptotic density needs following stronger nonstandard property, obtained by applying *Birkoff Ergodic Theorem*.

If $\mathrm{BD}(A) = \alpha$ and I is an infinite interval such that $\frac{|{}^*A \cap I|}{|I|} \approx \alpha$, then for "almost all" $\nu \in I$:

$$\underline{d}_\nu({}^*A) = \liminf_{n\to\infty} \frac{|{}^*A \cap [\nu+1, \nu+n]|}{n} = \alpha$$

Jin's theorem

Let us go back...
About ten years ago, by using the methods of nonstandard analysis, Renling Jin proved an interesting results about sum sets.

Theorem (R.Jin 2000)

If A and B are sets of integers with positive upper Banach density, then $A+B$ is piecewise syndetic.

- Jin's result raised the attention of people in additive number theory. But they did not understand his nonstandard proof.
- Jin's himself then published a "standard" proof, which was a direct translation of the original *nonstandard* arguments. But it was awkward and complicated.
- In 2006, by using ergodic theory, Bergelson, Furstenberg and Weiss re-proved Jin's theorem in strengthened form, by showing that $A + B$ must be *piecewise Bohr*.
- In 2009, again by ergodic theory, Griesmer generalized BFW's result to cases where one of the two sets has null Banach density.
- Last year, Beiglböck found a nice proof of Jin's theorem by using ultrafilters and some measure theory.

Inspired by Beiglböck's ultrafilter argument, I went back to the nonstandard setting of hyper-integers, and tried to simplify Jin's original proof. The following is a crucial step:

If $BD(A) = \alpha > 0$ and $BD(B) = \beta > 0$, then there exists $\nu \in {}^*\mathbb{N}$ and an infinite interval I such that

$$\frac{|(\nu + {}^*A) \cap {}^*B \cap I|}{|I|} \approx \alpha \cdot \beta.$$

The above property is obtained by applying a simple combinatorial property of finite sets to a suitable *hyper-finite* interval I.

- PROBLEM:
  There is no "standard" meaning for the *infinite* translation $\nu + {}^*A$.

However, a related notion turns out to be appropriate.

Definition

$X \lhd Y$ (read X is finitely embeddable in Y) if Y contains a copy of every finite configuration of X. That is, every finite $F \subseteq X$ has a translated copy $k + F \subseteq Y$.

If $X \lhd Y$, then

- X piecewise syndetic $\Rightarrow$ Y piecewise syndetic.
- X contains a k-term arithmetic progression $\Rightarrow$ Y contains a k-term arithmetic progression.
- X thick $\Rightarrow$ Y thick.
- $\mathrm{BD}(X) \leq \mathrm{BD}(Y)$.

The following *nonstandard characterization* holds:

- $X \lhd Y$ if and only if $\nu + X \subseteq {}^*Y$ for some $\nu \in {}^*\mathbb{N}$.

In the early versions of his paper, Renling Jin asked whether one can estimate the number k needed for $A + B + [0, k]$ to be thick, in terms of $\mathrm{BD}(A)$ and $\mathrm{BD}(B)$.
He later proved that such a k *does not* directly depend on $BD(A)$ and $BD(B)$. In fact:

For any $\alpha + \beta < 1$ and for every $k \in \mathbb{N}$, there exist sets A, B s.t.

1. $BD(A) > \alpha$
2. $BD(B) > \beta$
3. $A + B + [0, k]$ is <u>not</u> thick.

However, if one takes *arbitrary* finite sets F in place of initial segments $[0, k]$, a bound can be given.

Theorem

Assume that $BD(A) = \alpha > 0$ and $BD(B) = \beta > 0$. Then for every infinite X there exist a finite subset $F \subset X$ such that

1. $|F| \leq \lfloor 1/\alpha\beta \rfloor$
2. $X \lhd A + B + F$.

By taking $X = \mathbb{N}$ (or any other thick set) we obtain the following

Corollary (Jin – with bound)

Let $BD(A) = \alpha > 0$, $BD(B) = \beta > 0$ and $k = \lfloor 1/\alpha\beta \rfloor$. Then $A + B$ is piecewise k-syndetic, i.e. $A + B + F$ is thick for some finite set $|F| \leq k$.

Conclusions

- Certain *ultrafilter techniques* can be conveniently accommodated in a nonstandard setting.
  In fact, there is a natural way of identifying ultrafilters with the *hyper-integers* of nonstandard analysis.
- *Ultralimits* and *ultraproducts techniques* are also naturally accommodated in the setting of nonstandard analysis.
  (I did not have time to discuss about this...)
- In the setting of hyper-integers, one can directly use *finite* combinatorial arguments (with some caution!) and prove results about *infinite* sets which depends on their *density*.

THANK YOU

for your attention