Density Ramsey Theory for trees

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Bertinoro, May 2011

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Conventions/Definitions

- All trees in this talk will be **uniquely rooted** and **finitely branching**.
- A tree *T* will be called homogeneous if there exists an integer b_T ≥ 2, called the branching number of *T*, such that every t ∈ T has exactly b_T immediate successors;
 e.g., every dyadic or triadic tree is homogeneous.
- A vector tree is a finite sequence of (possibly finite) trees having common height. The level product of a vector tree T = (T₁,..., T_d), denoted by ⊗T, is defined to be the set

$$\bigcup_{n < h(\mathbf{T})} \otimes \mathbf{T}(n)$$

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where $\otimes \mathbf{T}(n) = T_1(n) \times ... \times T_d(n)$.

The concept of a strong subtree

A **strong subtree** of a tree T is a subset S of T with the following properties:

- (1) *S* is uniquely rooted and balanced (that is, all maximal chains of *S* have the same cardinality);
- (2) there exists a subset L_T(S) = {I_n : n < h(S)} of N, called the level set of S in T, such that for every n < h(S) we have S(n) ⊆ T(I_n);
- (3) for every non-maximal s ∈ S and every immediate successor t of s in T, there exists a *unique* immediate successor s' of s in S such that t ≤ s'.

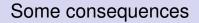
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The Halpern-Läuchli Theorem (strong subtree version)

Theorem (Halpern & Läuchli – 1966)

For every integer $d \ge 1$ we have that HL(d) holds: for every d-tuple $(T_1, ..., T_d)$ of uniquely rooted and finitely branching trees without maximal nodes and every finite coloring of the level product of $(T_1, ..., T_d)$ there exist strong subtrees $(S_1, ..., S_d)$ of $(T_1, ..., T_d)$ of infinite height and with common level set such that the level product of $(S_1, ..., S_d)$ is monochromatic.

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The following result is one of the earliest applications of the Halpern-Läuchli Theorem.

Theorem (Milliken – 1979 and 1981)

The class of strong subtrees (both finite and infinite) of a tree T is partition regular.

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The reason why this result is powerful lies in the rich "geometric" properties of strong subtrees.

The problem

- (i) The natural problem whether there exists a density version of the Halpern-Läuchli Theorem was first asked by Laver in the late 1960s who actually conjectured that there is such a version.
- (ii) Bicker & Voigt (1983) observed that one has to restrict attention to the category of homogeneous trees. They also showed that for a single homogeneous there is a density version.

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The infinite version

Theorem (D, Kanellopoulos & Karagiannis – 2010)

For every integer $d \ge 1$ we have that DHL(d) holds: for every d-tuple ($T_1, ..., T_d$) of homogeneous trees and every subset D of the level product of ($T_1, ..., T_d$) satisfying

$$\limsup_{n\to\infty}\frac{|D\cap (T_1(n)\times\ldots\times T_d(n))|}{|T_1(n)\times\ldots\times T_d(n)|}>0$$

there exist strong subtrees $(S_1, ..., S_d)$ of $(T_1, ..., T_d)$ of infinite height and with common level set such that the level product of $(S_1, ..., S_d)$ is a subset of *D*.

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The finite version

Theorem (D, Kanellopoulos & Tyros – 2011)

For every $d \ge 1$, every $b_1, ..., b_d \ge 2$, every $k \ge 1$ and every $0 < \varepsilon \le 1$ there exists an integer N with the following property. If $\mathbf{T} = (T_1, ..., T_d)$ is a vector homogeneous tree with $b_{T_i} = b_i$ for all $i \in \{1, ..., d\}$, L is a subset of \mathbb{N} of cardinality at least N and D is a subset of the level product of \mathbf{T} such that

$|D \cap (T_1(n) \times ... \times T_d(n))| \ge \varepsilon |T_1(n) \times ... \times T_d(n)|$

for every $n \in L$, then there exist strong subtrees $(S_1, ..., S_d)$ of $(T_1, ..., T_d)$ of height k and with common level set such that the level product of $(S_1, ..., S_d)$ is a subset of D. The least integer N with this property will be denoted by $\text{UDHL}(b_1, ..., b_d | k, \varepsilon)$.

Comments

- The proof of the finite version is effective and gives explicit upper bounds for the numbers UDHL(b₁, ..., b_d|k, ε). These upper bounds, however, have an Ackermann-type dependence with respect to the "dimension" d.
- The one-dimensional case (that is, when "d = 1") is due to Pach, Solymosi and Tardos (2010):

$$\text{UDHL}(b|k,\varepsilon) = O_{b,\varepsilon}(k).$$

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This bound is clearly optimal.

On the proofs

- The proof of the infinite version is based on stabilization arguments.
- The proof of the finite version is based on a density increment strategy and uses probabilistic (i.e. averaging) arguments. Following Furstenberg and Weiss (2003), for every finite vector homogeneous tree T define a probability measure on ⊗T by the rule

$$\mu_{\mathsf{T}}(\mathsf{A}) = \mathbb{E}_{n < h(\mathsf{T})} \frac{|\mathsf{A} \cap \otimes \mathsf{T}(n)|}{| \otimes \mathsf{T}(n)|}$$

The crucial observation is that "lack of density increment" implies a strong concentration hypothesis for the probability measure μ_{T} .