Density Ramsey Theory for trees

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Conventions/Definitions

- All trees in this talk will be **uniquely rooted** and **finitely branching**.
- A tree T will be called **homogeneous** if there exists an integer $b_T \geqslant 2$, called the **branching number** of T, such that every $t \in T$ has exactly b_T immediate successors; e.g., every dyadic or triadic tree is homogeneous.
- A **vector tree** is a finite sequence of (possibly finite) trees having common height. The **level product** of a vector tree $\mathbf{T} = (T_1, ..., T_d)$, denoted by $\otimes\mathbf{T}$, is defined to be the set

$$\bigcup_{n<h(\mathbf{T})} \otimes\mathbf{T}(n)$$

where $\otimes\mathbf{T}(n) = T_1(n) \times ... \times T_d(n)$.

The concept of a strong subtree

A **strong subtree** of a tree T is a subset S of T with the following properties:

(1) S is uniquely rooted and balanced (that is, all maximal chains of S have the same cardinality);

(2) there exists a subset $L_T(S) = \{l_n : n < h(S)\}$ of $\mathbb{N}$, called the **level set** of S in T, such that for every $n < h(S)$ we have $S(n) \subseteq T(l_n)$;

(3) for every non-maximal $s \in S$ and every immediate successor t of s in T, there exists a *unique* immediate successor s' of s in S such that $t \leqslant s'$.

The Halpern-Läuchli Theorem (strong subtree version)

Theorem (Halpern & Läuchli – 1966)

For every integer $d \geqslant 1$ we have that $\mathrm{HL}(d)$ holds: for every d-tuple $(T_1, \ldots, T_d)$ of uniquely rooted and finitely branching trees without maximal nodes and every finite coloring of the level product of $(T_1, \ldots, T_d)$ there exist strong subtrees $(S_1, \ldots, S_d)$ of $(T_1, \ldots, T_d)$ of infinite height and with common level set such that the level product of $(S_1, \ldots, S_d)$ is monochromatic.

Some consequences

The following result is one of the earliest applications of the Halpern-Läuchli Theorem.

Theorem (Milliken – 1979 and 1981)

The class of strong subtrees (both finite and infinite) of a tree T is partition regular.

The reason why this result is powerful lies in the rich "geometric" properties of strong subtrees.

The problem

(i) The natural problem whether there exists a density version of the Halpern-Läuchli Theorem was first asked by Laver in the late 1960s who actually conjectured that there is such a version.

(ii) Bicker & Voigt (1983) observed that one has to restrict attention to the category of homogeneous trees. They also showed that for a single homogeneous there is a density version.

The infinite version

Theorem (D, Kanellopoulos & Karagiannis – 2010)

For every integer $d \geqslant 1$ we have that $\mathrm{DHL}(d)$ holds: for every d-tuple $(T_1, ..., T_d)$ of homogeneous trees and every subset D of the level product of $(T_1, ..., T_d)$ satisfying

$$\limsup_{n\to\infty} \frac{|D \cap \big(T_1(n) \times ... \times T_d(n)\big)|}{|T_1(n) \times ... \times T_d(n)|} > 0$$

there exist strong subtrees $(S_1, ..., S_d)$ of $(T_1, ..., T_d)$ of infinite height and with common level set such that the level product of $(S_1, ..., S_d)$ is a subset of D.

The finite version

Theorem (D, Kanellopoulos & Tyros – 2011)

For every $d \geqslant 1$, every $b_1, ..., b_d \geqslant 2$, every $k \geqslant 1$ and every $0 < \varepsilon \leqslant 1$ there exists an integer N with the following property. If $\mathbf{T} = (T_1, ..., T_d)$ is a vector homogeneous tree with $b_{T_i} = b_i$ for all $i \in \{1, ..., d\}$, L is a subset of $\mathbb{N}$ of cardinality at least N and D is a subset of the level product of $\mathbf{T}$ such that

$$|D \cap (T_1(n) \times ... \times T_d(n))| \geqslant \varepsilon |T_1(n) \times ... \times T_d(n)|$$

for every $n \in L$, then there exist strong subtrees $(S_1, ..., S_d)$ of $(T_1, ..., T_d)$ of height k and with common level set such that the level product of $(S_1, ..., S_d)$ is a subset of D. The least integer N with this property will be denoted by $\mathrm{UDHL}(b_1, ..., b_d | k, \varepsilon)$.

Comments

- The proof of the finite version is effective and gives explicit upper bounds for the numbers $\mathrm{UDHL}(b_1, ..., b_d | k, \varepsilon)$. These upper bounds, however, have an Ackermann-type dependence with respect to the "dimension" d.
- The one-dimensional case (that is, when "$d = 1$") is due to Pach, Solymosi and Tardos (2010):

$$\mathrm{UDHL}(b | k, \varepsilon) = O_{b,\varepsilon}(k).$$

  This bound is clearly optimal.

On the proofs

- The proof of the infinite version is based on stabilization arguments.
- The proof of the finite version is based on a density increment strategy and uses probabilistic (i.e. averaging) arguments. Following Furstenberg and Weiss (2003), for every finite vector homogeneous tree $\mathbf{T}$ define a probability measure on $\otimes\mathbf{T}$ by the rule

$$\mu_{\mathbf{T}}(A) = \mathbb{E}_{n<h(\mathbf{T})} \frac{|A \cap \otimes\mathbf{T}(n)|}{|\otimes\mathbf{T}(n)|}.$$

The crucial observation is that "lack of density increment" implies a strong concentration hypothesis for the probability measure $\mu_{\mathbf{T}}$.