The singular world of singular cardinals

Mirna Džamonja

University of East Anglia

May, 2011

Infinite Ramsey theorem

For every finite n and k,

Infinite Ramsey theorem

For every finite n and k,

$$\omega \to (\omega)^n_k \quad .$$

Infinite Ramsey theorem

For every finite n and k,

$$\omega \to (\omega)^n_k \quad .$$

For which infinite cardinals is there any sort of analogue?

Singular and regular cardinals

Definition

(1) A *cardinal* is an ordinal which is not bijective with any smaller ordinal (infinite if not specified otherwise).

Singular and regular cardinals

Definition

(1) A *cardinal* is an ordinal which is not bijective with any smaller ordinal (infinite if not specified otherwise).
(2) A function $f : \alpha \to \beta$ is *cofinal* if $\sup(\mathrm{rng}(f)) = \beta$.

Singular and regular cardinals

Definition

(1) A *cardinal* is an ordinal which is not bijective with any smaller ordinal (infinite if not specified otherwise).
(2) A function $f : \alpha \to \beta$ is *cofinal* if $\sup(\mathrm{rng}(f)) = \beta$.
(3) If κ is a cardinal ,
$\mathrm{cf}(\kappa) = \min\{\alpha : \text{ there is a cofinal } f : \alpha \to \kappa\}$ (*cofinality*).

Singular and regular cardinals

Definition

(1) A *cardinal* is an ordinal which is not bijective with any smaller ordinal (infinite if not specified otherwise).
(2) A function $f : \alpha \to \beta$ is *cofinal* if $\sup(\mathrm{rng}(f)) = \beta$.
(3) If κ is a cardinal ,
$\mathrm{cf}(\kappa) = \min\{\alpha : \text{ there is a cofinal } f : \alpha \to \kappa\}$ (*cofinality*).
(4) A cardinal κ is *regular* iff $\mathrm{cf}(\kappa) = \kappa$. Otherwise, $\mathrm{cf}(\kappa) < \kappa$ and the cardinal is *singular*.

Singular and regular cardinals

Definition

(1) A *cardinal* is an ordinal which is not bijective with any smaller ordinal (infinite if not specified otherwise).
(2) A function $f : \alpha \to \beta$ is *cofinal* if $\sup(\mathrm{rng}(f)) = \beta$.
(3) If κ is a cardinal ,
$\mathrm{cf}(\kappa) = \min\{\alpha : \text{ there is a cofinal } f : \alpha \to \kappa\}$ (*cofinality*).
(4) A cardinal κ is *regular* iff $\mathrm{cf}(\kappa) = \kappa$. Otherwise, $\mathrm{cf}(\kappa) < \kappa$ and the cardinal is *singular*.
(5) For a cardinal λ, λ^+ is the next larger cardinal (always exists).

Singular and regular cardinals

Definition

(1) A *cardinal* is an ordinal which is not bijective with any smaller ordinal (infinite if not specified otherwise).
(2) A function $f : \alpha \to \beta$ is *cofinal* if $\sup(\mathrm{rng}(f)) = \beta$.
(3) If κ is a cardinal ,
$\mathrm{cf}(\kappa) = \min\{\alpha :$ there is a cofinal $f : \alpha \to \kappa\}$ (*cofinality*).
(4) A cardinal κ is *regular* iff $\mathrm{cf}(\kappa) = \kappa$. Otherwise, $\mathrm{cf}(\kappa) < \kappa$ and the cardinal is *singular*.
(5) For a cardinal λ, λ^+ is the next larger cardinal (always exists).
(6) κ is a *successor cardinal* if $\kappa = \lambda^+$ for some λ.

Singular and regular cardinals

Definition

(1) A *cardinal* is an ordinal which is not bijective with any smaller ordinal (infinite if not specified otherwise).
(2) A function $f : \alpha \to \beta$ is *cofinal* if $\sup(\mathrm{rng}(f)) = \beta$.
(3) If κ is a cardinal ,
$\mathrm{cf}(\kappa) = \min\{\alpha :$ there is a cofinal $f : \alpha \to \kappa\}$ (*cofinality*).
(4) A cardinal κ is *regular* iff $\mathrm{cf}(\kappa) = \kappa$. Otherwise, $\mathrm{cf}(\kappa) < \kappa$ and the cardinal is *singular*.
(5) For a cardinal λ, λ^+ is the next larger cardinal (always exists).
(6) κ is a *successor cardinal* if $\kappa = \lambda^+$ for some λ. Otherwise, it is a *limit* cardinal.

Examples and facts

- $\omega = \aleph_0$ is a regular limit cardinal.

Examples and facts

- $\omega = \aleph_0$ is a regular limit cardinal.
- every successor cardinal is regular.

Examples and facts

- $\omega = \aleph_0$ is a regular limit cardinal.
- every successor cardinal is regular.
- $\aleph_\omega$ is a singular limit cardinal.

Examples and facts

- $\omega = \aleph_0$ is a regular limit cardinal.
- every successor cardinal is regular.
- $\aleph_\omega$ is a singular limit cardinal.
- the existence of regular limit cardinals $> \aleph_0$ cannot be proved in ZFC.

Examples and facts

- $\omega = \aleph_0$ is a regular limit cardinal.
- every successor cardinal is regular.
- $\aleph_\omega$ is a singular limit cardinal.
- the existence of regular limit cardinals $> \aleph_0$ cannot be proved in ZFC. (These are “large cardinals”).

Examples and facts

- $\omega = \aleph_0$ is a regular limit cardinal.
- every successor cardinal is regular.
- $\aleph_\omega$ is a singular limit cardinal.
- the existence of regular limit cardinals $> \aleph_0$ cannot be proved in ZFC. (These are “large cardinals”).
- if $\kappa \rightarrow (\kappa)^2_2$ then it is a regular limit (in fact weakly compact) and this characterises weakly compact cardinals.

ω_1

- $\omega_1 \not\to (\omega_1)^2_2$ (Sierpiński).

ω_1

- $\omega_1 \not\to (\omega_1)^2_2$ (Sierpiński). Well order a subset A of $\mathbb{R}$ in type ω_1 by $\prec$. Let $c(x, y) = 1$ iff $\prec$ and $<$ agree, 0 otherwise.
- $\omega_1 \not\to [\omega_1]^2_{\omega_1}$ (Todorčević).

ω_1

- $\omega_1 \not\to (\omega_1)^2_2$ (Sierpiński). Well order a subset A of $\mathbb{R}$ in type ω_1 by $\prec$. Let $c(x, y) = 1$ iff $\prec$ and $<$ agree, 0 otherwise.
- $\omega_1 \not\to [\omega_1]^2_{\omega_1}$ (Todorčević). Uses minimal walks, a technique discovereded by Todorčević.

ω_1

- $\omega_1 \not\to (\omega_1)^2_2$ (Sierpiński). Well order a subset A of $\mathbb{R}$ in type ω_1 by $\prec$. Let $c(x, y) = 1$ iff $\prec$ and $<$ agree, 0 otherwise.
- $\omega_1 \not\to [\omega_1]^2_{\omega_1}$ (Todorčević). Uses minimal walks, a technique discovereded by Todorčević.
- $\omega_1 \to (\omega_1, \omega)$.
- The analogue of Todorčević's result holds at other successors of regulars (Shelah) [a proof that does not cover ω_1].

ω_1

- $\omega_1 \not\rightarrow (\omega_1)^2_2$ (Sierpiński). Well order a subset A of $\mathbb{R}$ in type ω_1 by $\prec$. Let $c(x, y) = 1$ iff $\prec$ and $<$ agree, 0 otherwise.
- $\omega_1 \not\rightarrow [\omega_1]^2_{\omega_1}$ (Todorčević). Uses minimal walks, a technique discovereded by Todorčević.
- $\omega_1 \rightarrow (\omega_1, \omega)$.
- The analogue of Todorčević's result holds at other successors of regulars (Shelah) [a proof that does not cover ω_1].

Singular cardinals

Observation If $\lambda = \mathrm{cf}(\kappa) < \kappa$, then $\kappa \nrightarrow (\kappa, \lambda^+)$.

Singular cardinals

Observation If $\lambda = \mathrm{cf}(\kappa) < \kappa$, then $\kappa \not\rightarrow (\kappa, \lambda^+)$.

[Let $\langle \kappa_i : i < \lambda \rangle$ increase to κ with each $\kappa_i < \kappa$. Let $c(\alpha, \beta) = 1$ iff they first appear in the same κ_i, and 0 otherwise.]

Singular cardinals

Observation If $\lambda = \mathrm{cf}(\kappa) < \kappa$, then $\kappa \not\rightarrow (\kappa, \lambda^+)$.

[Let $\langle \kappa_i : i < \lambda \rangle$ increase to κ with each $\kappa_i < \kappa$. Let $c(\alpha, \beta) = 1$ iff they first appear in the same κ_i, and 0 otherwise.] For example, $\aleph_\omega \not\rightarrow (\aleph_\omega, \omega_1)$.

A well known difference between singular and regular cardinals is to do with cardinal arithmetic:

A well known difference between singular and regular cardinals is to do with cardinal arithmetic:

- modulo the König Lemma and monotonicity, it is consistent to have 2^κ for κ regular to be whatever we want (Easton)

A well known difference between singular and regular cardinals is to do with cardinal arithmetic:

- modulo the König Lemma and monotonicity, it is consistent to have 2^κ for κ regular to be whatever we want (Easton)
- but if $\aleph_\omega$ is a strong limit then $2^{\aleph_\omega} < \aleph_{\omega_4}$ (Shelah).

A well known difference between singular and regular cardinals is to do with cardinal arithmetic:

- modulo the König Lemma and monotonicity, it is consistent to have 2^κ for κ regular to be whatever we want (Easton)
- but if $\aleph_\omega$ is a strong limit then $2^{\aleph_\omega} < \aleph_{\omega_4}$ (Shelah).

The singular behaviour of singulars also influences their successors:

Successors of singulars

Open question: Suppose that $\lambda = \kappa^+$ and κ is singular. Does it necessarily hold that $[\lambda] \not\rightarrow [\lambda]^2_\lambda$?

Successors of singulars

Open question: Suppose that $\lambda = \kappa^+$ and κ is singular. Does it necessarily hold that $[\lambda] \nrightarrow [\lambda]^2_\lambda$?

Eisworth and Eisworth-Shelah obtain the conclusion from a set-theoretic statement whose negation is not known to be consistent (saturation of a certain ideal related to club guessing).

Structured Ramsey theory

We'll only look at graphs.

Structured Ramsey theory

We'll only look at graphs.

- for every κ, μ there is θ such that any colouring of K_θ into μ colours has a monochromatic of *size* κ (Erdös-Rado).

Structured Ramsey theory

We'll only look at graphs.

- for every κ, μ there is θ such that any colouring of K_θ into μ colours has a monochromatic of *size* κ (Erdös-Rado).No structure preserved!

Structured Ramsey theory

We'll only look at graphs.

- for every κ, μ there is θ such that any colouring of K_θ into μ colours has a monochromatic of *size* κ (Erdös-Rado).No structure preserved!
- Consistently, there is a graph X such that every graph has an edge colouring into two colours and no monochromatic copy of X. (Hajnal and Komjáth).

Structured Ramsey theory

We'll only look at graphs.

- for every κ, μ there is θ such that any colouring of K_θ into μ colours has a monochromatic of *size* κ (Erdös-Rado).No structure preserved!
- Consistently, there is a graph X such that every graph has an edge colouring into two colours and no monochromatic copy of X. (Hajnal and Komjáth).
- For any X and μ, consistently there is a graph X^* such that for every edge colouring of X^* into μ colours, there is a monochromatic copy of X.

Rainbow colourings

- Consistently, there is a graph on ω_1 such that for every graph Y there is an edge colouring of Y into $\aleph_1$ colours such that evey copy of X gets all the colours (Komjáth).

Rainbow colourings

- Consistently, there is a graph on ω_1 such that for every graph Y there is an edge colouring of Y into $\aleph_1$ colours such that evey copy of X gets all the colours (Komjáth).

Definition

Let $Q(\mu^+)$ denote the statement that for every graph X with chromatic number μ^+ there is an edge colouring c into μ^+ such that for every vertex colouring g of X into μ colours, there is a g^{-1} class on which c takes every colour.

Rainbow colourings

- Consistently, there is a graph on ω_1 such that for every graph Y there is an edge colouring of Y into $\aleph_1$ colours such that evey copy of X gets all the colours (Komjáth).

Definition

Let $Q(\mu^+)$ denote the statement that for every graph X with chromatic number μ^+ there is an edge colouring c into μ^+ such that for every vertex colouring g of X into μ colours, there is a g^{-1} class on which c takes every colour.

Open question Is $Q(\aleph_1)$ true? (in ZFC).

Rainbow colourings

- Consistently, there is a graph on ω_1 such that for every graph Y there is an edge colouring of Y into $\aleph_1$ colours such that evey copy of X gets all the colours (Komjáth).

Definition

Let $Q(\mu^+)$ denote the statement that for every graph X with chromatic number μ^+ there is an edge colouring c into μ^+ such that for every vertex colouring g of X into μ colours, there is a g^{-1} class on which c takes every colour.

Open question Is $Q(\aleph_1)$ true? (in ZFC). (Todorčević's colouring is a special case for $X = K_{\aleph_1}$).
Introduced and studied by Hajnal and Komjáth.

Q was introduced and studied by Hajnal and Komjáth. They proved:

Q was introduced and studied by Hajnal and Komjáth. They proved:

- Consistently $Q(\aleph_1)$ holds.

Q was introduced and studied by Hajnal and Komjáth. They proved:

- Consistently $Q(\aleph_1)$ holds.
- Consistency results for μ regular.

Q was introduced and studied by Hajnal and Komjáth. They proved:

- Consistently $Q(\aleph_1)$ holds.
- Consistency results for μ regular.

Structured successors of singulars

Theorem. (Dž, Komjáth and Morgan) Modulo a supercompact cardinal,

Structured successors of singulars

Theorem. (Dž, Komjáth and Morgan) Modulo a supercompact cardinal, it is consistent to have a μ singular of cofinality ω, $\mu < \tau \leq \kappa$ as large as desired and

Structured successors of singulars

Theorem. (Dž, Komjáth and Morgan) Modulo a supercompact cardinal, it is consistent to have a μ singular of cofinality ω, $\mu < \tau \leq \kappa$ as large as desired and every graph of size $\leq \kappa$ and chromatic number τ has an edge colouring *c* such that for every vertex colouring *g* with into $< \tau$ colours there is a $g^{-1}(i)$ on which *c* assumes all colours.

Structured successors of singulars

Theorem. (Dž, Komjáth and Morgan) Modulo a supercompact cardinal, it is consistent to have a μ singular of cofinality ω, $\mu < \tau \leq \kappa$ as large as desired and every graph of size $\leq \kappa$ and chromatic number τ has an edge colouring c such that for every vertex colouring g with into $< \tau$ colours there is a $g^{-1}(i)$ on which c assumes all colours.

Countable cofinality necessary in the proof.

SuSiFA

Theorem (Džamonja, Magidor and Shelah) Modulo a supercompact cardinal we prove it consistent that for all reasonable values of the relevant parameters,

SuSiFA

Theorem (Džamonja, Magidor and Shelah) Modulo a supercompact cardinal we prove it consistent that for all reasonable values of the relevant parameters,

- (i) $2^\kappa = 2^{\kappa^+} = \lambda$ and $\kappa^+ < \partial = \mathrm{cf}(\lambda)$ while $\mathcal{P}(\kappa^+) = \bigcup_{i<\partial} B_i$ where sets B_i are increasing with i and each has size $< 2^\kappa$.

SuSiFA

Theorem (Džamonja, Magidor and Shelah) Modulo a supercompact cardinal we prove it consistent that for all reasonable values of the relevant parameters,

- (i) $2^\kappa = 2^{\kappa^+} = \lambda$ and $\kappa^+ < \partial = \mathrm{cf}(\lambda)$ while $\mathcal{P}(\kappa^+) = \bigcup_{i<\partial} B_i$ where sets B_i are increasing with i and each has size $< 2^\kappa$.
- (ii) There exists a Radin utrafilter u of length $\theta < \kappa$ with the top point κ and a collection $\{name(v_\alpha) : \alpha < \kappa\}$ of $\mathbb{R}(u)$-names for a truth value such that:

SuSiFA

Theorem (Džamonja, Magidor and Shelah) Modulo a supercompact cardinal we prove it consistent that for all reasonable values of the relevant parameters,

- (i) $2^\kappa = 2^{\kappa^+} = \lambda$ and $\kappa^+ < \partial = \mathrm{cf}(\lambda)$ while $\mathcal{P}(\kappa^+) = \bigcup_{i<\partial} B_i$ where sets B_i are increasing with i and each has size $< 2^\kappa$.
- (ii) There exists a Radin utrafilter u of length $\theta < \kappa$ with the top point κ and a collection $\{name(v_\alpha) : \alpha < \kappa\}$ of $\mathbb{R}(u)$-names for a truth value such that:
  for every family $\{name(\tau_j) : j < j^* < \partial\}$ of $\mathbb{R}(u)$-names for a truth value there is $A \in \mathfrak{F}(u)$ and an $\mathbb{R}(u)$-name for a function $name(h) : j^* \to \kappa$ such that $\langle (u, A) \rangle$ decides all $name(\tau_j)$ and for every j

$$\langle (u, A) \rangle \text{forces}_{\mathbb{R}(u)} \text{"} name(\tau_j) = name(v)_{name(h(j))} \text{"}.$$

- (iii) for every forcing notion $\mathbb{P}$ of size $< \lambda$ which is stationary κ^+-cc $(< \kappa)$-directed closed with $(2, \omega)$-lubs

- (iii) for every forcing notion $\mathbb{P}$ of size $< \lambda$ which is stationary κ^+-cc $(< \kappa)$-directed closed with $(2, \omega)$-lubs and for every family $\mathcal{D} = \{\mathcal{D}_k : k < k^* \leq \lambda\}$ of dense open sets in $\mathbb{P}$, there is a decomposition $\lambda = \bigcup_{i<\partial} X_i$ into increasing sets X_i of size $< \lambda$ such that for every i there a filter $H_i \subseteq \mathbb{P}$ such that $\mathcal{D}_k \cap H_i \neq \emptyset$ for each $k \in X_i$.