SANE BOUNDS ON SOME VDW-TYPE NUMBERS

A collaboration spanning many papers and authors.
http://www.cs.umd.edu/~gasarch/sane/sane.html.

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BROAD RESEARCH PLAN

- VDW-type theorems Often show f exists by showing it's bounded by another function F (E.g., VDW numbers [Go,GRS,Sh1,VDW]).

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- Often F is INSANE!!!!!!!!!
- Big Open Question: What is the true growth rate of f?
- Our Angle: *variants* and *special cases* of VDW-type theorems.

Part I: Squares and Rectangles

Fenner, Gasarch, Glover, Purewal [FGGP]
Molina, Oza, Puttagunta (Mentor: Gasarch) [MOP]
Apon and Purtilo (Question Asker: Gasarch) (unpublished)

SQUARES HARD!. RECTANGLES?

Theorem

(Gallai-Witt Thm, [R1,R2,Wi,GRS]) For all c, there exists $G = G(c)$ such that for every c-coloring of $[G] \times [G]$ there exists a monochromatic square.

$$\begin{array}{ccccc} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & R & \cdots & R & \cdots \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ \cdots & R & \cdots & R & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

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\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & R & \cdots & R & \cdots \\
\cdots & \vdots & \cdots & \vdots & \cdots \\
\cdots & R & \cdots & R & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

1. Known Bounds on G HUGE!
2. What if we look at Rectangles instead?

UPPER AND LOWER BOUNDS

$G_{n,m}$ is the grid $[n] \times [m]$.

1. If $G_{n,m}$ is c-colorable then the color that appears the most often is a rectangle free set of size at least $\geq \lceil nm/c \rceil$.

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3. Find colorings: Comb, Proj Geom, Finite Fields, Tournaments
4. Express results: $(\forall c)(\exists OBS_c)$ such that

   $G_{n,m}$ c-col iff $G_{n,m}$ does not contain any element of OBS_c.

OBS_2 AND OBS_3

1. $\mathrm{OBS}_2 = \{G_{3,7}, G_{5,5}, G_{7,3}\}$.
2. $\mathrm{OBS}_3 = \{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\}$.
3. OBS_4 contains
   $G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}$.

MANY MONO RECTANGLES-CAN BE DIFF COLORS

Definition

OBS_c^s is obstruction set for c-coloring grids and avoiding getting s monochromatic rectangles (can be diff colors).

1. $OBS_2^2 = \{G_{3,8}, G_{4,7}, G_{5,5}, G_{7,4}, G_{8,3}\}$
2. $OBS_2^3 = \{G_{3,9}, G_{4,8}, G_{5,6}, G_{6,5}, G_{8,4}, G_{9,3}\}$
3. $OBS_2^4 = \{G_{3,10}, G_{4,8}, G_{5,6}, G_{6,5}, G_{8,4}, G_{10,3}\}$
4. $OBS_2^5 = \{G_{3,11}, G_{4,9}, G_{5,7}, G_{6,6}, G_{7,5}, G_{9,4}, G_{11,3}\}$
5. $OBS_2^6 = \{G_{3,12}, G_{4,9}, G_{5,7}, G_{6,6}, G_{7,5}, G_{9,4}, G_{12,3}\}$
6. $OBS_3^2 = \{G_{4,20}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{20,4}\}$

OPEN QUESTIONS

1. Is $G_{17,17}$ 4-colorable? (Other 4-col also open.)
2. What is OBS_4? OBS_5?
3. Better Tools.

CASH PRIZE!

The first person to email me both (1) plaintext, and (2) LaTeX, of a 4-coloring of the 17×17 grid that has no monochromatic rectangles will receive \$289.00.

SQUARES- WHAT IS KNOWN?

B	B	B	B	B	B	R	B	B	R	R	R	R
B	R	B	R	R	B	B	R	B	R	B	R	B
B	B	R	R	B	B	R	R	R	R	B	B	R
R	R	R	B	R	B	B	B	R	B	B	R	R
R	B	R	R	R	R	R	B	B	R	B	B	B
B	R	R	B	B	R	B	B	R	R	R	R	B
R	R	B	R	B	B	R	B	R	B	R	B	B
R	B	B	B	B	R	R	R	R	B	B	R	B
R	B	R	R	B	B	B	R	B	B	R	R	R
B	B	B	R	R	R	B	B	R	B	R	B	R
B	R	B	R	B	R	R	B	B	B	B	R	R
R	R	B	B	B	R	B	R	B	R	B	B	R
B	R	R	B	R	R	R	R	B	B	R	B	B

2-coloring of $G_{13,13}$ without mono squares. Is better known? I ask non-rhetorically.

Part 2: Rado's Theorem [R1,R2,GRS]

Gasarch and Moriarty [GM]

EXT VDW THM

(See [GRS]) Extended VDW theorem:

Lemma

For all $c, k, s \in \mathsf{N}$, *there exists* $E = EW(k, s, c)$ *for any* c*-coloring of* $[E]$, *there exists* a, d *such that*

$$a, a+d, \ldots, a+(k-1)d, \ \textit{AND } sd$$

are the same color.

RADO'S THM (Traditional)

Theorem

Let $b_1, \ldots, b_n \in Z$. If $(\exists J)[\sum_{i \in J} b_i = 0]$ then, for all c there exists $R = R(\vec{b}, c)$ such that for all c-colorings of $[R]$ there exists MONO SOLUTION.

Example: 4-coloring to get mono solution of
$2x_1 + 3x_2 - 5x_3 + 8x_4 + x_5$.

$$x_1 = a + e_1 d, \quad x_2 = a + e_2 d, \quad x_3 = a + e_3 d, \quad x_4 = x_5 = sd.$$

$$2x_1 + 3x_2 - 5x_3 + 8x_4 + x_5 = (2+3-5)a + (2e_1 + 3e_2 - 5e_3)d + 9sd = 0$$

Can choose e_1, e_2, e_3 to make $2e_1 + 3e_2 - 5e_3 + 9s = 0$.
Note: Bound used Extended VDW number- LARGE!!!

Note: RADO''s theorem is actually iff.

BETTER BOUNDS

KEY: Don't really need FULL Extended VDW. Will just use WEAK EXT VDW:

Lemma

For all $c, L, s \in \mathsf{N}$ there exists $WEW = WEW(m, s, c)$ such that for all c-colorings of $[WEW]$ there exists a, d such that

$$a, a + Ld, \ \ AND \ sd$$

are the same color.

WEAK EXT VDW IMPLIES RADO

Theorem

Let $b_1, \ldots, b_n \in Z$. If $(\exists J)[\sum_{i \in J} b_i = 0]$ then, for all c there exists $R = R(\vec{b}, c)$ such that for all c-colorings of $[R]$ there exists MONO SOLUTION. Can take $R = WEW(\max(b_1), -\sum_{i \notin J} b_i, c)$

Example: 4-coloring to get mono solution of $2x_1 + 3x_2 - 5x_3 + 8x_4 + x_5$.

$$x_1 = a, \quad x_2 = x_3 = a + Ld, \quad x_4 = x_5 = sd.$$

$$2x_1+3x_2-5x_3+8x_4+x_5 = (2+3-5)a+(3-5)Ld+9sd = -2Ld+9sd.$$

Can choose L, s to make $-2L + 9s = 0$.

Note: Much BETTER Upper Bounds! Can do better with LCM's.

IS THIS AN IMPROVEMENT?

Need better bounds on $WEW(L, s, c)$.

Theorem

$WEW(L, 1, 2) \leq 1 + 3L + L^2$. *(a, a + d, d same color)*

Proof Idea: Cases and forced colorings.

Example: if 1 is RED then $1 + L$ is BLUE else $a = 1$, $d = L$ works.

SANE BOUNDS ON RADO NUMBERS

Num Colors	equation	VDW-bounds	new bound
2	$x - y + z$	$W(3,2) = 9$	5
2	$x - y + 2z$	$W(7,2) \geq 3703$	11
2	$x - y + 3z$	$W(13,2) \geq 2^{14}$	19
2	$x - y + 4z$	$W(21,2) \geq 2^{23}$	49
2	$x - y + 5z$	$W(31,2) \geq 2^{32}$	101
3	$x - y + z$	$W(W(3,3)+1,3) = W(28,3)$	14
3	$x - y + 2z$	$W(2W(7,3)+1,3)$	75
3	$x - y + 3z$	$W(3W(13,3)+1,3)$	253
4	$x - y + z$	$W(W(W(3,4)+1,4)+1,4)$	61

OPEN PROBLEMS

1. Get better upper bounds on $WEW(L, s, c)$ and hence on Rado Numbers. Especially $c \geq 3$.
2. Get a handle on DISTINCT-Rado: all of the x_i distinct.

Part III: Better Bounds on the Poly VDW numbers [BL,Sh2,Wa].

Gasarch, C. Kruskal, J. Kruskal [GKK]
Molina, Oza, Puttagunta (Mentor: Gasarch) [MOP]
Beigel and Gasarch [BG]

POLY VDW THM (One Equation Case)

Theorem
For all $p(x) \in \mathbb{Z}[x]$ *such that* $p(0) = 0]$, *for all* c, *there exists* $W_{\rm poly} = W_{\rm poly}(p, c)$ *such that for all* c*-colorings of* $[W_{\rm poly}]$ *there exists* a, d *such that*

$$a, a + p(d)$$

are all the same color.

How to prove?

1. Bergelson and Leibman [BL]. No bounds! (Debatable)
2. Walters [Wa]. Elem. ω^{ω} induction. Bounds INSANE.
3. Shelah [Sh]. Primitive Recursive.

POLY VDW- TWO COLORS

Theorem

If p is a poly of deg n and $p(0) = 0$ then

1. $W_{\rm poly}(p(x), 2) \le 2\max\{|p(1)|, \ldots, |p(n+1)|\}$.
2. $W_{\rm poly}(p(x), 2) \le \min\{|p(2|p(i)|)| : i \in N\}$.

POLY VDW- 3-COL: UMCP MATH COMP

If $\{1, 2, \ldots, 2006\}$ is 3-colored then there exists two numbers, a square apart, same color. Let $COL : [2006] \rightarrow \{R, B, G\}$. Fix $x \geq 10$.

1. $COL(x)$, $COL(x+16)$, $COL(x-9)$ ALL DIFFERENT!
2. Assume $COL(x) = R$, $COL(x+16) = B$, $COL(x-9) = G$.

$COL(x+7) = COL((x+16)-9) \neq COL(x+16) = B$.
$COL(x+7) = COL((x-9)+16) \neq COL(x-9) = G$.

$$(\forall x \geq 10)[COL(x) = COL(x+7)]!!$$

$COL(10) = COL(10+7) = COL(10+2\times7) = \cdots = COL(10+7\times7)$.

Can replace 2006 with $10 + 7 \times 7 = 59$. With more work: 29 is optimal.

KEY TO THAT PROOF

Show that if $\{1, 2, \ldots, 2006\}$ is 3-colored then there exists two nums a square apart that are the same color.
We used

$$16 + 9 = 25.$$

For general $p(x)$ we will need to find x, y, z such that

$$p(x) + p(y) = p(z).$$

1. Approach can't work for $p(x) = x^3$.
2. We will only work with quadratics.

(ALMOST) GENERAL QUADRATIC CASE

Theorem

Let $a, b \in \mathrm{N}$ such that a divides b. Then

$$W_{\text{poly}}(ax^2 + bx, 3) \leq 72b^2/a + 1.$$

Proof.

Let

$$x = 5b/a, y = 6b/a, z = 8b/a.$$

$$\begin{aligned} p(x) = & \quad a(5b/a)^2 + b(5b/a) = 25b^2/a + 5b^2/a = 30b^2/a \\ p(y) = & \quad a(6b/a)^2 + b(6b/a) = 36b^2/a + 6b^2/a = 42b^2/a \\ p(z) = & \quad a(8b/a)^2 + b(8b/a) = 64b^2/a + 8b^2/a = 72b^2/a \end{aligned}$$

$$p(x) + p(y) = p(z).$$

Still more to do, but this is key ingredient. □

GENERAL QUADRATIC CASE

Theorem

Let $a, b \in \mathsf{N}$. *Let* $m = \max\{a, b\}$. *Then*

$$W_{\text{poly}}(ax^2 + bx, 3) \leq O(m^7).$$

Proof.

You don't want to see the proof. □

WHAT IF CONSTANT TERM NONZERO? (I)

POLY VDW had hypothesis that polys had zero-constant. Why?

$p(x) = 1$. 2-coloring is $RBRB\ldots$.

But what about OTHER polys with non-zero constant?
Does Poly VDW hold? What are bounds?

WHAT IF CONSTANT TERM NONZERO (II)?

1. $W_{\text{poly}}(x^2 + a, 2) = O(a^2)$.
2. If $a \equiv 1 \pmod 3$ then $W_{\text{poly}}(x^2 + a, 3) = \infty$.
3. If $a \not\equiv 1 \pmod 3$ then $W_{\text{poly}}(x^2 + a, 3) = O(a^3)$.
4. If $a \equiv 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16, 17, 18, 19, 21, 22\}$ then $W_{\text{poly}}(x^2 + a, 4) = \infty$.

SQUARE DIFF FREE SETS AND $W_{\mathrm{poly}}(x^2, c)$ (I)

1. A set A is *Square Diff Free* if there is no $x, y \in A$ such that $x - y$ is a square.
2. $\mathrm{sdf}(n)$ is the largest sdf set of $[n]$.
3. If A is any subset of $[n]$ then there exists $\frac{n \log n}{|A|}$ translates of it that cover $[n]$ (Prob argument [CFL]).
4. If $c \geq O(\frac{n \log n}{\mathrm{sdf}(n)})$ then $W_{\mathrm{poly}}(x^2, c) \geq n$.
5. If $c \leq \Omega(\frac{n}{\mathrm{sdf}(n)})$ then $W_{\mathrm{poly}}(x^2, c) \leq n$.

SQUARE DIFF FREE SETS AND $W_{\text{poly}}(x^2, c)$ (II)

1. Ruzsa [Ru] showed

$$\mathrm{sdf}(n) \geq \Omega(n^{\log_{65} 7}) \geq \Omega(n^{0.733077\cdots}).$$

2. Beigel and Gasarch showed

$$\mathrm{sdf}(n) \geq \Omega(n^{0.5(1+\log_{205} 12)}) \geq \Omega(n^{0.7334\cdots}).$$

3. Pintz, Steiger, and Szemerédi [PSS,Wo] showed

$$\mathrm{sdf}(n) \leq \frac{n}{(\log n)^{O(\log \log \log \log n)}}.$$

SQUARE DIFF FREE SETS AND $W_{\mathrm{poly}}(x^2, c)$ (II)

Using Beigel-Gasarch [BG] and Pintz-Steiger-Szemerédi [PSS]

$$\Omega(c^{3.75}) \le W_{\mathrm{poly}}(x^2, c) \le 2^{c^{O(1/(\log\log\log\log c)^{\epsilon})}}.$$

PART III: OPEN QUESTIONS

1. Obtain smaller values for $W_{\mathrm{poly}}(ax^2 + bx, 3)$ (e.g., $O(m^6)$).
2. Obtain SANE values for $W_{\mathrm{poly}}(ax^3 + bx^2 + cx, 3)$.
3. Obtain SANE values for $W_{\mathrm{poly}}(ax^2 + bx, 4)$.
4. Obtain SANE values for $W_{\mathrm{poly}}(p_1(x), \ldots, p_k(x), 2)$.
5. Obtain SANE values for $W_{\mathrm{poly}}(p_1(x), \ldots, p_k(x), 3)$
6. Study more $W_{\mathrm{poly}}(f, c)$ where f is poly with constant term.
7. Start Study of $W_{\mathrm{poly}}(f, c)$ where f is non poly functions.

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