Binary subtrees with few labeled paths

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May 24, 2011

This is joint work with **Rod Downey, Noam Greenberg**, and **Kevin Milans** in:

"Binary subtrees with few labeled paths," to appear in *Combinatorica*

The original motivation was to solve a problem in computability theory. This led to work in Ramseyan combinatorics, which I will describe first in this talk. Then I will describe the application to computability theory.

Terminology

Definition

Let T be a finite rooted tree.

1. The *depth* of a vertex of T is its distance from the root.
2. T is *complete* if all of its leaves have the same depth, and this common depth is called the *depth* of the tree.
3. T is *binary* if each vertex which is not a leaf has exactly 2 children.
4. T is *ternary* if each vertex which is not a leaf has exactly 3 children.

All trees we consider are both rooted and complete.

A useful preliminary result

Lemma

Monochromatic Subtree Lemma. *(Goldblatt) Suppose that T is a* **ternary** *tree of depth n, and each leaf of T is colored red or blue. Then T has a* **binary** *subtree S of depth n with all leaves of the same color.*

Labeled edges

From now on, each tree is assumed to come equipped with a $\{0, 1\}$ labeling of its **edges**.

A *path* in a tree is a path from the root to a leaf. In a tree of depth n, each path contains n labeled edges. The *label* of the path is obtained by reading the labels of its edges, starting from the root. This label is an n-bit binary word.

If S is a tree of depth n, let $L(S)$ be the set of labels of paths of S, so $L(S) \subseteq \{0, 1\}^n$.

Object of the game

Given a ternary tree T, find a binary subtree S of the same depth with as few path labels as possible, i.e. $|L(S)|$ as small as possible.

Example

If T has depth 2, there exists a binary subtree S with $|L(S)| \leq 2$. This is best possible for depth 2.

Definition

Let T a ternary tree of depth n. If S is a binary subtree of T of the same depth n, we write $S \sqsubset T$.

Definition

Let T be a ternary tree. Then the *weight* of T (denoted $w(T)$) is the *least* value of $|L(S)|$, over all $S \sqsubset T$.

Then we look at the worst case for each depth:

Definition

Let n be a positive integer. Then $f(n)$ is the *greatest* weight of a ternary tree of depth n.

Thus, $f(n)$ is the least number b such that every ternary tree of depth n has a binary subtree S of depth n with at most b path labels.

Examples

$$f(1) = 1$$

$$f(2) = 2$$

$$f(3) = 3$$

$$f(4) = 4$$

Don't worry – it gets more interesting!

The following gives recursive upper and lower bounds on f.

Proposition

Let m and n be positive integers.

- $f(n+1) \leq 2f(n)$.
- $f(m+n) \geq f(m) \cdot f(n)$

Corollary

$6 \leq f(5) \leq 8$.

Theorem

$f(5) = 8$.

Corollary

For all $n \geq 4$,

$$2^{\lfloor n/2 \rfloor} \leq f(n) \leq 2^{n-2}$$

Proposition

$\lim_n (f(n))^{1/n}$ *exists and is the supremum of the values of* $f(n)^{1/n}$ *for* $n \in \mathbb{N}$. *This limit is* ≤ 2.

Definition

$\lim_n (f(n)^{1/n}$ is denoted L.

Since $f(5) = 8$, we get the following lower bound on L.

Corollary

$L \geq \sqrt[5]{8} \geq 1.515$.

The above lower bound can be improved, as we shall see.

Question

What is the value of L? Is $L = 2$?

This can be compared with the situation for Ramsey numbers $R(n, n)$. It is not known whether $\lim_n R(n, n)^{1/n}$ exists, but it is known that

$$\sqrt{2} \leq \liminf_n \; R(n, n)^{1/n} \leq \limsup_n \; R(n, n)^{1/n} \leq 4$$

A lower bound

Theorem

For all $n \in \mathbb{N}$,

$$f(n) \geq 2^{\frac{n-2}{\log_2 3}} \geq (0.269) \cdot (1.548)^n$$

Corollary

$L \geq 2^{\frac{1}{\log_2 3}} \geq 1.548$

(This slightly improves the previous result $L \geq \sqrt[5]{8} \geq 1.515$.)

An upper bound on f

Theorem

There is a constant γ such that, for all $n \in \mathbb{N}$,

$$f(n) \leq \gamma 2^{n-0.6\sqrt{n}}$$

The proof uses probabilistic methods, the monochromatic subtree lemma from the beginning of this talk, and the pigeonhole principle.

Corollary

$\lim_n \frac{f(n)}{2^n} = 0$.

A generalized version of the corollary is what is needed for our application to logic. It can be proved using the monochromatic subtree lemma and the pigeonhole principle (without using probabilistic methods).

Infinite trees

We now consider **infinite** rooted trees which have no leaves. As before, every edge is labeled 0 or 1. Define *ternary* and *binary* ,and $S \sqsubset T$ as before.

If T is an infinite tree, then every path through T is labeled by an infinite binary word. Let $L(T)$ be the set of labels of paths through T.

Proposition

There is an infinite ternary tree T such that, for every $S \sqsubset T$, $|L(S)| = 2^{\aleph_0}$.

Hence, it no longer makes sense to try to minimize $|L(S)|$.

Measure of path labels

Let μ be the usual coin-toss measure on 2^ω.

Theorem

(Measure 0 path label theorem) Let T be an infinite ternary tree. Then there exists $S \sqsubset T$ such that $\mu(L(S)) = 0$.

Remarks

- The proof is effective. Hence, if T is computable, then S may be chosen to be computable.
- Let U be an infinite set of natural numbers. If we label only edges with depth in U, the result still holds, indeed effectively.

Application to computability theory

Goal. Compare the complexity of diagonalization using specified values with the complexity of constructing a random set, e.g. a 1-random set.

The problems of diagonalization and constructing a 1-random set are framed as **mass problems**.

Mass Problems

A **mass problem** is a set of total functions from ω to ω

We identify sets $A \subseteq \omega$ with their characteristic functions.

$\mathcal{A}$,$\mathcal{B}, \ldots$ are variables for mass problems.

The "solutions" to a mass problem $\mathcal{A}$ are simply the elements of $\mathcal{A}$.

Diagonalization mass problems

$\text{DNR} = \{f : (\forall e)[f(e) \neq \varphi_e(e)]\}$.

Note that no function in DNR is computable. Sometimes we restrict the values used to diagonalize:

Definition

For $k \in \omega$,
$\text{DNR}_k = \{f : (\forall e)[f(e) < k \quad \& \quad f(e) \neq \varphi_e(e)]\} = \text{DNR} \cap k^\omega$

Strong reducibility of mass problems

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems. $\mathcal{A}$ is *strongly reducible* to $\mathcal{B}$ (denoted $\mathcal{A} \leq_s \mathcal{B}$) if there is a **fixed** oracle Turing machine M such that $M^f \in \mathcal{A}$ for all $f \in \mathcal{B}$.

This definition is due to Medvedev, and the reducibility is also known as Medvedev reducibility.

The idea is that given any "solution" to $\mathcal{B}$, one can, in a uniform way compute a "solution" to $\mathcal{A}$.

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems. Then $\mathcal{A} >_s \mathcal{B}$ means that $\mathcal{B} \leq_s \mathcal{A}$ and $\mathcal{A} \nleq_s \mathcal{B}$.

The following old result of mine follows from a version of the monochromatic subtree lemma.

Theorem

$DNR_2 >_s DNR_3 >_s DNR_4 >_s \ldots$

Let 1-RAND be the class of all (characteristic functions of) 1-random sets. The following result is well-known:

Theorem

- *$DNR_2 >_s$ 1-RAND*
- *For all k, $DNR_k \nleq_s$ 1-RAND*

Question

(Joe Miller) Is 1-RAND strongly reducible to DNR_3?

We obtain a negative answer.

Theorem

(DGJM) 1-RAND $\not\leq_s$ DNR_3. Thus, there is no Turing machine which, given any function in DNR_3 as an oracle, computes the characteristic function of a 1-random set.

Recall: This fails for DNR_2 in place of DNR_3.

Applying the measure 0 path theorem

To show: 1-RAND $\not\leq_s$ DNR$_3$

Let M be an oracle Turing machine. WLOG, M^f is total for all $f \in 3^\omega$. Let $T = 3^{<\omega}$, labeled so that, for all $f \in 3^\omega$, M^f is the path label of f. Extract a computable $S \sqsubset T$ such that $\mu(L(S)) = 0$. Let f be a DNR path through S. Argue that Φ_e^f is *not* 1-random because it is in $L(S)$, a Π_1^0 class of measure 0. So $f \in$DNR$_3$ and $M^f \notin$ 1-RAND.

Corollary

There is a Π^0_1 class $P \subseteq 2^\omega$ of positive measure which is not strongly reducible to DNR_3.

Proof.

The class 1-RAND has a Π^0_1 subclass P of positive measure, and then $P \not\leq_s \mathrm{DNR}_3$. $\square$

This corollary answers a question raised by Steve Simpson, and this question led to all work presented here.