Functional Interpretation of Proofs in Ergodic Theory and Combinatorics

Ulrich Kohlenbach
Department of Mathematics
Technische Universität Darmstadt

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- **interpret** the formulas A in P : $A \mapsto A^{\mathcal{I}}$,

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Our approach is based on novel forms and extensions of:

K. Gödel's functional interpretation!

Example: the monotone convergence principle

Let (a_n) be a nonincreasing sequence in $[0, 1]$. Then, clearly, (a_n) is convergent and so a Cauchy sequence which we write as:

$$\mathbf{(1)}\ \forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall m \in \mathbb{N} \forall i, j \in [n; n+m]\ (|a_i - a_j| \leq 2^{-k}),$$

where $[n; n+m] := \{n, n+1, \ldots, n+m\}$.

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Consider the (partial) Herbrand normal form of this statement is

$$\textbf{(2)}\ \forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i,j \in [n; n+g(n)]\ (|a_i - a_j| \le 2^{-k}).$$

In contrast to (1), there is a **simple (primitive recursive) bound** $\Phi^*(g,k)$ on (2) (Kreisel's **no-counterexample interpretation** also referred to as **'metastability'** by T.Tao):

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Proposition (G. Kreisel 1951)

Let (a_n) be any nonincreasing sequence in $[0,1]$ then

$$\forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \leq \Phi^*(g,k) \forall i,j \in [n; n+g(n)] \, (|a_i - a_j| \leq 2^{-k}),$$

where

$$\Phi^*(g,k) := \tilde{g}^{(2^k)}(0) \text{ with } \tilde{g}(n) := n + g(n).$$

Moreover, there exists an $i < 2^k$ such that n can be taken as $\tilde{g}^{(i)}(0)$.

Corollary (T. Tao's finite convergence principle)

$$\forall \mathbf{k} \in \mathbb{N}, \mathbf{g} : \mathbb{N} \to \mathbb{N} \exists \mathbf{M} \in \mathbb{N} \forall \mathbf{1} \geq \mathbf{a_0} \geq \ldots \geq \mathbf{a_M} \geq \mathbf{0} \exists \mathbf{N} \in \mathbb{N}$$
$$\left(\mathbf{N} + \mathbf{g(N)} \leq \mathbf{M} \wedge \forall \mathbf{n}, \mathbf{m} \in [\mathbf{N}, \mathbf{N} + \mathbf{g(N)}](|\mathbf{a_n} - \mathbf{a_m}| \leq \mathbf{2^{-k}})\right).$$

One may take $M := \Phi^*(g, k)$ as above.

Corollary (T. Tao's finite convergence principle)

$$\forall k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N} \exists M \in \mathbb{N} \forall 1 \geq a_0 \geq \ldots \geq a_M \geq 0 \exists N \in \mathbb{N} \\ (N + g(N) \leq M \wedge \forall n, m \in [N, N + g(N)](|a_n - a_m| \leq 2^{-k})).$$

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- In the simple case at hand, a bound on the no-counterexample interpretation coincides with the (monotone) functional interpretation.
- For more complicated formulas the latter is **much more involved** (already for the infinitary pigeonhole principle; compare again Tao).
- Proper understanding of functional interpretation requires treatment of systems based on **intuitionistic logic** (Brouwer).

Gödel's functional interpretation in five minutes

Gödel's **functional interpretation** D combined with Krivine's **negative translation** N results in an interpretation $Sh = D \circ N$ (Streicher/K.07)

$$\mathbf{A} \mapsto \mathbf{A}^{\mathbf{Sh}} \quad \text{(Shoenfield variant)}$$

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- $\mathbf{A} \leftrightarrow \mathbf{A}^{\mathbf{Sh}}$ by classical logic and **quantifier-free choice** in all types

$$\mathbf{QF\text{-}AC}: \ \forall \underline{\mathbf{a}} \exists \underline{\mathbf{b}}\, \mathbf{F}_{\mathbf{qf}}(\underline{\mathbf{a}}, \underline{\mathbf{b}}) \rightarrow \exists \underline{\mathbf{B}} \forall \underline{\mathbf{a}}\, \mathbf{F}_{\mathbf{qf}}(\underline{\mathbf{a}}, \underline{\mathbf{B}}(\underline{\mathbf{a}})).$$

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- $\underline{\mathbf{x}}, \underline{\mathbf{y}}$ are tuples of **functionals of finite type** over the base types of the system at hand.

$$A^{Sh} \equiv \forall u \exists x\, A_{Sh}(u, x), \quad B^{Sh} \equiv \forall v \exists y\, B_{Sh}(v, y).$$

$$\mathbf{A}^{\mathsf{Sh}} \equiv \forall \mathbf{u} \exists \mathbf{x}\, \mathbf{A}_{\mathsf{Sh}}(\mathbf{u}, \mathbf{x}), \quad \mathbf{B}^{\mathsf{Sh}} \equiv \forall \mathbf{v} \exists \mathbf{y}\, \mathbf{B}_{\mathsf{Sh}}(\mathbf{v}, \mathbf{y}).$$

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(SH3) $(\mathbf{A} \vee \mathbf{B})^{\mathsf{Sh}} \equiv \forall \mathbf{u}, \mathbf{v} \exists \mathbf{x}, \mathbf{y}\, \big(\mathbf{A}_{\mathsf{Sh}}(\mathbf{u}, \mathbf{x}) \vee \mathbf{B}_{\mathsf{Sh}}(\mathbf{v}, \mathbf{y})\big)$

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(SH7) $(A \wedge B)^{Sh} \equiv$
$\forall n, u, v \exists x, y\, (n{=}0 \rightarrow A_{Sh}(u,x)) \wedge (n{\neq}0 \rightarrow B_{Sh}(v,y))$
$\leftrightarrow \forall u, v \exists x, y\, \big(A_{Sh}(u,x) \wedge B_{Sh}(v,y)\big).$

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$$\mathbf{p} \vdash \forall \mathbf{x}\, \exists \mathbf{y}\, \mathbf{A_{qf}}(\mathbf{x}, \mathbf{y})$$

an explicit effective functional Φ realizing A^{Sh}, i.e.

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In p in $\mathbf{ACA}_0^{\omega}$: $\Phi \in T_0+\mathrm{BR}_{0,1}$ which for types ≤ 2 coincides with T (K.1998).

Monotone functional interpretation (K.1996)

Monotone *Sh* extracts Φ^* such that

$$\exists \mathbf{Y} \ \big(\mathbf{\Phi}^* \gtrsim \mathbf{Y} \wedge \forall \mathbf{x}\, \mathbf{A}_{\mathbf{Sh}}(\mathbf{x}, \mathbf{Y}(\mathbf{x}))\big),$$

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where $\gtrsim$ is some suitable notion of being a 'bound' that applies to higher order function spaces (W.A. Howard)

$$\begin{cases} \mathbf{x}^* \gtrsim_{\mathbb{N}} \mathbf{x} :\equiv \mathbf{x}^* \geq \mathbf{x}, \\ \mathbf{x}^* \gtrsim_{\rho \to \tau} \mathbf{x} :\equiv \forall \mathbf{y}^*, \mathbf{y}(\mathbf{y}^* \gtrsim_{\rho} \mathbf{y} \to \mathbf{x}^*(\mathbf{y}^*) \gtrsim_{\tau} \mathbf{x}(\mathbf{y})). \end{cases}$$

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Monotone Sh of WKL is solved by constant-1 functional!

- Refinements of the above functional interpretation of ACA_0 in $T_0 + \mathrm{BR}_{0,1}$ can be used for bound extractions from proofs that use RT^2_2 and variants thereof (see A. Kreuzer's talk).

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- Functional interpretations have been used in Ergodic Ramsey Theory (e.g. by H. Towsner in connection with the Furstenberg-Zimmer tower in the Bergelson-Liebman proof of the polynomial VDW and Szemerédi theorems).

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- This talk surveys recent results in nonlinear ergodic theory.

An Example from Ergodic Theory

X **Hilbert space**, $f : X \to X$ **linear** and $\|\mathbf{f(x)}\| \leq \|\mathbf{x}\|$ for all $x \in X$.

$$\mathbf{A_n(x)} := \frac{\mathbf{1}}{\mathbf{n+1}}\mathbf{S_n(x)}, \text{ where } \mathbf{S_n(x)} := \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{f^i(x)} \quad (\mathbf{n} \geq \mathbf{0}).$$

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Theorem (von Neumann Mean Ergodic Theorem)

For every $x \in X$, the sequence $(A_n(x))_n$ converges.

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in general **no computable rate of convergence**.

But: **Prim.rec. bound on metastable version!**

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

Based on logical metatheorem to be discussed below:

Theorem (Leuştean/K., Ergodic Theor. Dynam. Syst. 2009)

X uniformly convex Banach space, η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.

Then for all $x \in X$ with $\|x\| \le b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \le \Phi(\varepsilon, g, b, \eta)\, \forall i, j \in [n; n + g(n)] \left(\|A_i(x) - A_j(x)\| < \varepsilon\right),$$

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$$\exists \mathbf{n} \leq \mathbf{\Phi}(\varepsilon, \mathbf{g}, \mathbf{b}, \eta)\, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}; \mathbf{n} + \mathbf{g}(\mathbf{n})] \left(\|\mathbf{A_i}(\mathbf{x}) - \mathbf{A_j}(\mathbf{x})\| < \varepsilon\right),$$

where

$$\mathbf{\Phi}(\varepsilon, \mathbf{g}, \mathbf{b}, \eta) := \mathbf{M} \cdot \tilde{\mathbf{h}}^{\mathbf{K}}(\mathbf{0}), \text{ with}$$

$$\mathbf{M} := \left\lceil \frac{\mathbf{16b}}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{\mathbf{16}} \eta\left(\frac{\varepsilon}{\mathbf{8b}}\right), \quad \mathbf{K} := \left\lceil \frac{\mathbf{b}}{\gamma} \right\rceil,$$

$$\mathbf{h}, \tilde{\mathbf{h}} : \mathbb{N} \to \mathbb{N}, \ \mathbf{h}(\mathbf{n}) := \mathbf{2}(\mathbf{Mn} + \mathbf{g}(\mathbf{Mn})), \quad \tilde{\mathbf{h}}(\mathbf{n}) := \max_{\mathbf{i} \leq \mathbf{n}} \mathbf{h}(\mathbf{i}).$$

Based on logical metatheorem to be discussed below:

Theorem (Leuştean/K., Ergodic Theor. Dynam. Syst. 2009)

X uniformly convex Banach space, η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.

Then for all $x \in X$ with $\|x\| \le b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \le \Phi(\varepsilon, g, b, \eta)\, \forall i, j \in [n; n + g(n)]\, \left(\|A_i(x) - A_j(x)\| < \varepsilon\right),$$

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$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^K(0), \text{ with}$$
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$$h, \tilde{h} : \mathbb{N} \to \mathbb{N},\ h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \le n} h(i).$$

Special Hilbert case: treated prior by Avigad/Gerhardy/Towsner (again based on logical metatheorem).

General logical metatheorems

Many abstract types of metric structures can be added as atoms: metric, hyperbolic, CAT(0), δ-hyperbolic, normed, uniformly convex, Hilbert, ... spaces or $\mathbb{R}$-trees X : add **new base type** X, all **finite types over** $\mathbb{N}, X$ and a new **constant** d_X representing d etc.

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Counterexamples (to extractability of uniform bounds): for the classes of strictly convex ($\rightarrow$ uniformly convex) or separable ($\rightarrow$ totally bounded) spaces!

A formal system for analysis

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

Functionals of type $\rho \to \tau$ map type-ρ objects to type-τ objects.

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$\mathcal{A}^{\omega}[X, d, \ldots]$ results by adding constants $d_X, \ldots$ with axioms expressing that $(X, d, \ldots)$ is a nonempty metric, hyperbolic ... space.

A novel form of majorization

y, x functionals of types ρ and $\widehat{\rho} := \rho[\mathbb{N}/X]$:

$$\mathbf{x}^{\mathbb{N}} \gtrsim_{\mathbb{N}} \mathbf{y}^{\mathbb{N}} :\equiv \mathbf{x} \geq \mathbf{y}$$
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For **linear** and **nonexpansive** f : $\mathrm{Id} \gtrsim f$.

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If $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle]$ proves

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then **monotone functional interpretation** extract a **computable functional** $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ s.t. for all g, k, b

$$\forall \mathbf{x} \in \mathbf{X}\, \forall \mathbf{f} : \mathbf{X} \to \mathbf{X}$$
$$\big(\mathbf{f} \text{ n.e.} \wedge \|\mathbf{x}\|, \|\mathbf{f(0)}\| \leq \mathbf{b} \to \exists \mathbf{v} \leq \mathbf{\Phi(g,k,b)}\, \mathbf{A}_\exists\big)$$

holds in **every (real) Hilbert space X.**

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Uniformly convex case: bound Φ depends on modulus of convexity η.

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‘We shall establish Theorem 1.6 by “finitary ergodic theory” techniques, reminiscent of those used in [Green-Tao]...’ ‘The main advantage of working in the finitary setting ... is that the underlying dynamical system becomes extremely explicit’...‘In proof theory, this finitisation is known as Gödel functional interpretation...which is also closely related to the Kreisel no-counterexample interpretation’
(T. Tao: Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theor. and Dynam. Syst. 28, 2008)

Theorem (J.-B. Baillon 1975): X Hilbert space, $C \subset X$ bounded closed and convex, $U : C \to C$ nonexpansive. Then for every $v_0 \in C$, the sequence of Cesàro means (u_n)

$$\mathbf{u_n := \frac{1}{n+1} \sum_{k=0}^{n} U^{(k)}(v_0)}$$

converges weakly to a fixed point of U.

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- **Strong convergence in general fails** (counterexample by Baillon).
- In important **special cases** (see below) **strong convergence** can be established.

Quantitative weak compactness

Theorem (K., APAL to appear)

Applying monotone functional interpretation to the proof of sequential weak compactness of $B_1(0)$ yields the extractability of a **closed term Ω^* in $T_0 + B_{0,1}$** s.t. (denoting strong majorizability rel. to 0_X by $\gtrsim$) the following is true in the model of all strongly majorizable functionals $\mathcal{M}^{\omega,X}$:

$$\begin{cases} \exists \tilde{\Omega} \lesssim \Omega^* \forall K \forall (x_n) \subset B_1(0) \exists v \in B_1(0) \exists \chi = \tilde{\Omega}(K,(x_n)) \forall w \in B_1(0) \\ \exists n \in [K(v,\chi), \chi(w, K(v,\chi))] \left(|\langle v - x_n, w)\rangle| <_{\mathbb{R}} 2^{-K(v,\chi)} \right), \end{cases}$$

where χ resp. K has type $X \times \mathbb{N} \to \mathbb{N}$ resp. $X \times (X \times \mathbb{N} \to \mathbb{N}) \to \mathbb{N}$.

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Also for general abstract b-bounded closed and convex $C \subset X$. Then Ω^* also depends on b (but not on C otherwise).

A quantitative 'metastable' version of Baillon's theorem

Theorem (K.2010)

Logical analysis of the proof of Baillon's theorem due to Brézis-Browder yields a primitive recursive functional φ such that for Ω^* as above, $\varepsilon > 0$, $g : \mathbb{N} \to \mathbb{N}$, we get $\varphi(\Omega^*, \varepsilon, b, g)$ as a bound on the metastable version of the weak Cauchy property of the Cesàro means (u_n), i.e.

$$\forall w \in B_1(0)\, \exists n \leq \varphi(\Omega^*, \varepsilon, b, g)\, \forall i, j \in [n; n+g(n)]\, \left(|\langle u_i - u_j, w\rangle| < \varepsilon\right).$$

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$$\forall \mathbf{w} \in \mathbf{B_1(0)}\, \exists \mathbf{n} \le \varphi(\boldsymbol{\Omega^*}, \varepsilon, \mathbf{b}, \mathbf{g})\, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}; \mathbf{n}+\mathbf{g}(\mathbf{n})]\ (|\langle \mathbf{u_i} - \mathbf{u_j}, \mathbf{w} \rangle| < \varepsilon).$$

Based on the detailed construction of Ω^* and results of W.A. Howard it follows that $\psi(\varepsilon, g) := \varphi(\Omega^*, \varepsilon, b, g)$ is definable in Gödel's T_4 (note that φ has type level 2).

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- Hence the quantitative version is a **finitization of Baillon's** theorem.

A theorem of R. Wittmann

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Halpern iterations: $U : C \to C$ nonexpansive, $u_0 \in C$, $\alpha_n \in [0,1]$

$$\mathbf{u_{n+1} := \alpha_{n+1}\, u_0 + (1 - \alpha_{n+1})\, U(u_n).}$$

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Using weak compactness, Wittmann proved in 1992:

Theorem (R. Wittmann 1992): $C \subseteq X$ closed and convex, $u_0 \in C$ and $Fix(U) \neq \emptyset$. Under suitable conditions on (α_n) (satisfied e.g. for $\alpha_n := \frac{1}{n+1}$) (u_n) converges strongly towards the fixed point of U that is closest to u_0.

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Remark: Wittmann's theorem is a **nonlinear generalization of the Mean ergodic theorem**: for $\alpha_n := 1/(n+1)$, $C := X$ and **linear** U, the Halpern iteration coincides with the Cesàro means. Hence the Mean ergodic theorem follows as a special case.

General features of the logical analysis of the proof due Wittmann

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- The possibility of elimination of weak compactness rests on the structure of the proof. **In more general cases, a genuine treatment of weak compactness seems necessary**.

In the following let $\mathbf{d \geq diam(C) := sup\{\|x - y\| : x, y \in C\}}$ or $\mathbf{d \geq 4\|u_0 - p\|}$ for some $p \in Fix(U)$.

A quantitative metastable version of Wittmann's theorem

Theorem (K., Advances in Mathematics 2011)

Let $\alpha_n := 1/(n+1)$ and (u_n) as above. Then for $\varepsilon \in (0,1)$

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where

$$\boldsymbol{\Phi}(\varepsilon, \mathbf{g}, \mathbf{d}) := \rho(\varepsilon^2/4\mathbf{d}^2, \chi_{\mathbf{d},\varepsilon}(\mathbf{N}_{\varepsilon,\mathbf{g},\mathbf{d}})) \text{ with}$$

$$\mathbf{N}_{\varepsilon,\mathbf{g},\mathbf{d}} := 16\mathbf{d} \cdot \left(\max\left\{(\boldsymbol{\Delta}^*_{\varepsilon,\mathbf{g}})^{(\mathbf{i})}(1) \;:\; \mathbf{i} \leq \mathbf{n}_{\varepsilon,\mathbf{d}}\right\}\right)^2, \ \mathbf{n}_{\varepsilon,\mathbf{d}} := \left\lceil \frac{\mathbf{d}^2}{\varepsilon_\mathbf{d}} \right\rceil,$$

$$\varepsilon_\mathbf{d} := \frac{\varepsilon^4}{8192\mathbf{d}^2}, \ \boldsymbol{\Delta}^*_{\varepsilon,\mathbf{g}}(\mathbf{n}) := \lceil 1/\boldsymbol{\Omega}_\mathbf{d}(\varepsilon/2, \tilde{\mathbf{g}}^\mathbf{M}, \chi_{\mathbf{d},\varepsilon}(16\mathbf{d} \cdot \mathbf{n}^2))\rceil,$$

with $\boldsymbol{\Omega}_\mathbf{d}(\varepsilon, \mathbf{g}, \mathbf{j}) := \delta_{\varepsilon, \tilde{\mathbf{g}}(\rho(\varepsilon^2/2\mathbf{d}^2, \mathbf{j}))}$, where $\delta_{\varepsilon,\mathbf{m}} := \frac{\varepsilon^2}{16\mathbf{dm}}$,

$$\rho(\varepsilon, \mathbf{n}) := \left\lceil \frac{\mathbf{n}+1}{\varepsilon} \right\rceil, \ \chi_{\mathbf{d},\varepsilon}(\mathbf{n}) := \max\left\{\chi_\mathbf{d}(\mathbf{n}), \left\lceil \frac{32\mathbf{d}^2}{\varepsilon^2} \right\rceil\right\},$$

$$\chi_\mathbf{d}(\mathbf{n}) := 4\mathbf{dn}(4\mathbf{dn} + 2), \tilde{\mathbf{g}}(\mathbf{n}) := \max\{\mathbf{n}, \mathbf{g}(\mathbf{n})\}, \ \mathbf{g}^+(\mathbf{n}) := \mathbf{n} + \mathbf{g}(\mathbf{n}).$$

Another strong nonlinear ergodic theorem

Already in 1976 Baillon proved strong convergence of Cesàro means if $\forall v \in C\,(-v \in C)$ and U is nonexpansive and odd, i.e. $U(-v) = -U(v)$.

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$$\forall \mathbf{u}, \mathbf{v} \in \mathbf{C}\ (\|\mathbf{U}(\mathbf{u}) + \mathbf{U}(\mathbf{v})\| \leq \|\mathbf{u} + \mathbf{v}\|).$$

Then the sequence of Cesàro means (u_n) converges strongly.

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Then the sequence of Cesàro means (u_n) converges strongly.

- Holds for C closed under $v \mapsto -v$ and nonexpansive, odd U.

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Theorem [R. Wittmann 1990]: Let $C \subseteq X$ be an **arbitrary** subset and $U : C \to C$ s.t.

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Then the sequence of Cesàro means (u_n) converges strongly.

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- The condition above does **not even** imply that U is **continuous**. But U has a **trivial majorant** $U^* := Id$ $(v := u)$.

Another strong nonlinear ergodic theorem

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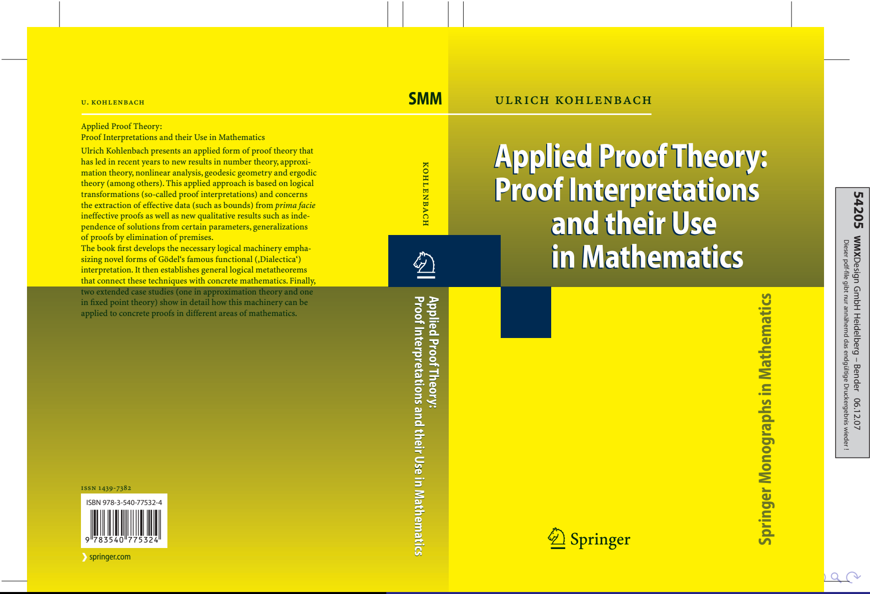
where

$$\Phi(k, b, g) := (N(2k+7, g) + P(2k+7, g)) \cdot b \cdot 2^{2k+8} + 1,$$
$$P(k, g) := P_0(k, F(k, g, N(k, g))),$$
$$F(k, g, n)(p) := p + n + \tilde{g}((n+p) \cdot b \cdot 2^{k+1}),$$
$$L(k, g)(n) := n + P_0(k, F(k, g, n)) + \tilde{g}((n + P_0(k, F(k, g, n))) \cdot b \cdot 2^{k+1}),$$
$$N(k, g) := (L(k, g))^{(b^2 2^{k+2})}(0),$$
$$P_0(k, f) := \tilde{f}^{(b^2 2^k)}(0),\ \tilde{f}(n) := n + f(n),\ f^M(n) := \max_{i \leq n+1} f(i).$$



U. KOHLENBACH

SMM

ULRICH KOHLENBACH

Applied Proof Theory: Proof Interpretations and their Use in Mathematics

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Ulrich Kohlenbach presents an applied form of proof theory that has led in recent years to new results in number theory, approximation theory, nonlinear analysis, geodesic geometry and ergodic theory (among others). This applied approach is based on logical transformations (so-called proof interpretations) and concerns the extraction of effective data (such as bounds) from *prima facie* ineffective proofs as well as new qualitative results such as independence of solutions from certain parameters, generalizations of proofs by elimination of premises.
The book first develops the necessary logical machinery emphasizing novel forms of Gödel's famous functional (,Dialectica') interpretation. It then establishes general logical metatheorems that connect these techniques with concrete mathematics. Finally, two extended case studies (one in approximation theory and one in fixed point theory) show in detail how this machinery can be applied to concrete proofs in different areas of mathematics.

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