Ramsey's theorem for pairs and program extraction

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Program extraction

$$\mathsf{RCA}_0^\omega + \mathsf{RT}^2_2 \vdash \forall x\, \exists y\, A(x,y)$$

extract a term t, such that

$$\forall x\, A(x, t(x)).$$

Here $A(x,y)$ is quantifier-free.

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Here $A(x,y)$ is quantifier-free.
RCA_0^ω is the finite type extension of RCA_0:

- Sorted into type 0 for $\mathbb{N}$, type 1 for $\mathbb{N}^\mathbb{N}$, type 2 for $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$, $\ldots$,
- contains basic arithmetic $(+, \cdot)$, λ-abstraction,
- quantifier-free axiom of choice for numbers, i.e. the statement that definable functions of natural numbers exist,
- and a recursor R_0, which provides primitive recursion (for numbers),
- Σ^0_1-induction.

Program extraction via reduction to arithmetical comprehension

$$\mathsf{RCA}_0^\omega + \mathsf{RT}_2^2 \vdash \forall x\, \exists y\, A(x,y)$$

implies

$$\mathsf{RCA}_0^\omega + \Pi_1^0\text{-}\mathsf{CA} \vdash \forall x\, \exists y\, A(x,y).$$

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- Functional interpretation extracts a term t primitive recursive in the bar recursor $B_{0,1}$, such that
$$\forall x\, A(x, t(x)).$$
- Howard's ordinal analysis of the bar recursor shows that t is provably total relative to Π^0_∞-induction.

Functional interpretation of RT^2_2

- Formalization of RT^2_2

$$\forall c\colon [\mathbb{N}]^2 \to 2\, \exists H = \{h_0, h_1, \dots\} \quad \forall x, y\ (x \neq y \to c(\{h_x, h_y\}) = c(\{h_0, h_1\}))$$

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- Functional interpretation:

$$\forall c \qquad \exists H^1\, \forall x^0, y^0\, (x \neq y \to c(\{h_x, h_y\}) = c(\{h_0, h_1\}))$$

functional interpretation of RT^2_2

$$\forall c\, \forall X^2, Y^2\, \exists H^1 \qquad (X(H) \neq Y(H) \to c(\{h_{X(H)}, h_{Y(H)}\}) = c(\{h_0, h_1\}))$$

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$$\to c(\{h_{X(H)}, h_{Y(H)}\}) = c(\{h_0, h_1\}))$$

- A functional $\mathcal{R}^3(c, X, Y)$ yielding such an H is called a *solution-functional of the functional interpretation of* RT^2_2.

$$\mathsf{RCA}_0^\omega + \mathsf{RT}_2^2 \vdash \forall x \, \exists y \, A(x,y)$$

Applying the functional interpretation directly yields a term t primitive recursive in a solution-functional $\mathcal{R}$ of the functional interpretation of RT_2^2, such that

$$\forall x \, A(x, t(x))$$

The term t is made of

- $+$, $\cdot$,
- the primitive recursor R_0, i.e.

$$R_0(0, y, f) = y, \qquad R_0(x+1, y, f) = f(R_0(x, y, f), x).$$

- λ-abstraction and
- the solution functional $\mathcal{R}$.

With coding R_0 is of type 2, $\mathcal{R}$ is of type 3.
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Lemma

The term t can be normalized, such that each occurrence of $\mathcal{R}$ is of the form

$$\mathcal{R}(t_0[g], t_1[g], t_2[g])$$

for terms t_i containing only $g\colon \mathbb{N} \to \mathbb{N}$ free. This g can be chosen such that it is the same for each application.

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- For closed terms t:
  $\mathsf{RCA}^\omega_0 + \Pi^0_1\text{-CA}(t) \nvdash I\Sigma^0_3$

Theorem

For all f there exists an f', such that

$$\mathsf{uWKL}_0^\omega \vdash \forall c\colon [\mathbb{N}]^2 \to 2 \Big(\Pi^0_1\text{-CA}(f'(c))$$
$$\to \exists H \,(H \textit{ infinite and homogeneous for } c \wedge \Pi^0_1\text{-CA}(f(c,H)))\Big)$$

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- This theorem implies low_2-ness of RT^2_2
  - For computable c it follows that $0'$ plus WKL computes H and H'.
  - By the low basis theorem (Jockusch, Soare) then $0''$ computes H''. Hence H is low_2.

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- The proof of this theorem is based on Cholak's, Jockusch's and Slaman's proof (by first jump control) of the low_2-ness of RT^2_2.
- Finitely nested uses of instances of RT^2_2 are implied by a suitable single instance of Π^0_1-CA having the same parameters.

Interpreting t

Lemma

There exists an f, such that
uWKL$_0^\omega$ *and the functional interpretation of* $\forall g, x$ Π_1^0-CA$(f(g,x))$
proves that

1. t *is total,*
2. $\forall x\, A(x, t(x))$.

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Proof.

Normalize t such that it contains only finitely many $\mathcal{R}$ applications of the form $\mathcal{R}(t_0[g], t_1[g], t_2[g])$.

1. Replace the applications of $\mathcal{R}$ with he functional interpretation of the last theorem.

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1. Replace the applications of $\mathcal{R}$ with he functional interpretation of the last theorem.
2. Replace each occurrence of $\mathcal{R}$ in the proof of $\forall x\, A(x, t(x))$ by this interpretation. □

Theorem (K., Kohlenbach)

$$\mathsf{RCA}_0^\omega + \mathsf{RT}_2^2 \vdash \forall x\, \exists y\, A(x,y)$$

then one can extract a term t *provably total in* $\mathsf{RCA}_0^\omega + I\Sigma_2^0$ *such that*

$$\mathsf{RCA}_0^\omega + I\Sigma_2^0 \vdash \forall x\, A(x,t(x))$$

Proof.

- Use functional interpretation to extract a term t primitive recursive in $\mathcal{R}$, such that $\forall x\, A(x,t(x))$.
- Normalize t. Replace the occurrence of $\mathcal{R}$ to obtain a proof in uWKL_0^ω + the functional interpretation of Π_1^0-$\mathsf{CA}(f(g,x))$.
- Solve the functional interpretation of Π_1^0-$\mathsf{CA}(f(g,x))$ with a *single use* of the bar recursor $B_{0,1}$.
- Kohlenbach's elimination of WKL and Howard's analysis of the bar recursor yield the result. □

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Proof.

- Use functional interpretation to extract a term t primitive recursive in $\mathcal{R}$ and the recursor R_1, such that $\forall x\, A(x,t(x))$.
- Normalize t. Replace the occurrence of $\mathcal{R}$ and R_1 to obtain a proof in uWKL_0^ω + the functional interpretation of Π^0_1-CA$(f(g,x))$.
- Solve the functional interpretation of Π^0_1-CA$(f(g,x))$ with a *single use* of the bar recursor $B_{0,1}$.
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Proof.

- Use functional interpretation to extract a term t primitive recursive in $\mathcal{R}$ and the recursor R_1 and the restricted bar recursor Φ_{WKL} for WKL, such that $\forall x\, A(x, t(x))$.
- Normalize t. Replace the occurrence of $\mathcal{R}$ and R_1 to obtain a proof in $\mathsf{uWKL}_0^\omega +$ the functional interpretation of Π_1^0-$\mathsf{CA}(f(g,x))$. Use Howard's analysis of the restricted bar recursor to interpret Φ_{WKL}.
- Solve the functional interpretation of Π_1^0-$\mathsf{CA}(f(g,x))$ with a *single use* of the bar recursor $B_{0,1}$.
- Kohlenbach's elimination of WKL and Howard's analysis of the bar recursor yield the result. □

Proofwise low

This is a general concept:
Call a principle P of the form

$$\forall X\,\exists Y\,P'(X,Y)$$

proofwise low over a system $\mathcal{T}$ if
for all f exists an f', such that

$$\mathcal{T} \vdash \forall X\ \left(\Pi^0_1\text{-CA}(f'(X)) \rightarrow \exists Y\ \left(P'(X,Y) \land \Pi^0_1\text{-CA}(f(X,Y))\right)\right).$$

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Theorem (K., Kohlenbach)

If P is proofwise low over WKL^ω_0 and P' is Π^0_1 then

$$\mathsf{WKL}^\omega_0 + I\Sigma^0_2 + P$$

is Π^0_3-conservative and admits term extraction over the system

$$\mathsf{RCA}^\omega_0 + I\Sigma^0_2.$$

Let $\mathsf{WKL}_0^{\omega *}$ be the system WKL_0^{ω} where $I\Sigma_1^0$ and R_0 is replaced by $I\Sigma_0^0$, 2^x and bounded primitive recursion.

Theorem (K.)

If P is proofwise low over $\mathsf{WKL}_0^{\omega *}$ and P' is Π_1^0 then

$$\mathsf{WKL}_0^{\omega} + B\Sigma_2^0 + P$$

is Π_3^0-conservative and admits extraction of primitive recursive terms over the system

$$\mathsf{RCA}_0^{\omega}.$$

Proof.

Use a refinement of Howard's ordinal analysis. □

The chain antichain principle

Definition

Let the *chain antichain principle* (CAC) be that statement the each partial order over $\mathbb{N}$ possesses an infinite chain or an infinite antichain.

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Lemma (Cholak, Jockusch, Slaman)

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \rightarrow \mathsf{CAC}$$

The chain antichain principle

Theorem (Chong, Slaman, Yang)

$$\mathsf{RCA}_0 + B\Sigma^0_2 + \mathsf{CAC}$$

is Π^1_1*-conservative over*

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The chain antichain principle

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Theorem (K.)

The chain antichain principle is proofwise low over $\mathsf{WKL}^{\omega*}_0$*.*
Hence

$$\mathsf{WKL}^{\omega}_0 + B\Sigma^0_2 + \mathsf{CAC}$$

is Π^0_3*-conservative over*

$$\mathsf{RCA}^{\omega}_0.$$

Moreover ***primitive recursive*** *terms can be extracted for* $\forall\exists$ *sentences.*

Connections to the Bolzano-Weierstraß principle

The chain antichain principle implies the following variant of the Bolzano Weierstraß principle:

Each bounded sequence in $\mathbb{R}$ contains a Cauchy-subsequence.

Compare to the reverse mathematics formulation of the Bolzano-Weierstraß principle:

Each bounded sequence in $\mathbb{R}$ contains a converging subsequence, i.e. a Cauchy-subsequence with Cauchy-rate 2^{-n}.

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Theorem (K.)

- RCA_0 *proves that the strong cohesive principle* ($\mathsf{COH} + B\Sigma^0_2$) *is equivalent to our variant of the Bolzano-Weierstraß principle.*
- ADS *is equivalent to the stronger statement that each sequence of real numbers contains a monotone subsequence.*

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- Application to RT^2_2 and CAC:
  - Extraction of terms of Ackermann type resp. primitive recursive terms.

Summary

- We introduced the notion of *proofwise low*.
  This is a refinement of low_2-ness.
- Program extraction and conservativity results for proofwise low principles.
- Application to RT^2_2 and CAC:
  - Extraction of terms of Ackermann type resp. primitive recursive terms.
  - New proof for the facts that RT^2_2 does not imply more than Ackermannian growth and
    that CAC does not imply Σ^0_2-induction.

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