Chromatic Numbers and Unconditional Sequences in Banach Spaces

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(joint work with S. Todorcevic)

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The intend of this talk is to present a combinatorial approach to the existence of unconditional and subsymmetric basic sequences in a Banach space.

By a Banach space we mean a normed space with a complete norm which is infinite dimensional.

A normalized sequence $(x_\gamma)_{\gamma<\kappa}$ in a Banach space $(X, \|\cdot\|)$, indexed in some cardinal number κ is called a **basic** sequence when there is a constant $C \geq 1$ such that

$$\|\sum_{\gamma\in t} a_\gamma x_\gamma\| \leq C\|\sum_{\gamma\in s} a_\gamma x_\gamma\|$$

for every sequence of scalars $(a_\gamma)_{\gamma\in s}$ and every $t \sqsubseteq s$.

$(x_\gamma)_\gamma$ is called an **unconditional** basic sequence when there is a constant $C \geq 1$ such that $C \geq 1$ such that

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$(x_\gamma)_\gamma$ is called **subsymmetric** when there is a constant $C \geq 1$ such that

$$\| \sum_{k \in l} a_k x_{\gamma_k} \| \leq C \| \sum_{k<l} a_k x_{\xi_k} \|$$

for every sequence of scalars $(a_k)_{k<l}$ and every $\gamma_0 < \cdots < \gamma_{l-1}$, $\xi_0 < \cdots < \xi_{l-1}$.

$(x_\gamma)_\gamma$ is called **weakly-null** when the set

$$\{\gamma < \kappa \, : \, |x^*(x_\gamma)| \geq \varepsilon\}$$

is finite for every $x^* \in X^*$ and every $\varepsilon > 0$.

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Question 1

What is the minimal cardinal $\mathfrak{nc}$ such that every Banach space of density at least $\mathfrak{nc}$ has an unconditional basic sequence?

Question 2

What is the minimal cardinal $\mathfrak{ns}$ such that every Banach space of density at least $\mathfrak{ns}$ has an subsymmetric basic sequence?

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Question 3

What is the minimal cardinal $\mathfrak{nc}_0$ such that every normalized weakly-null sequence $(x_\gamma)_{\gamma<\mathfrak{nc}_0}$ has an unconditional basic sequence?

Question 4

What is the minimal cardinal $\mathfrak{ns}_{\mathrm{seq}}$ such that every normalized sequence $(x_\gamma)_{\gamma<\mathfrak{ns}_{\mathrm{seq}}}$ has a subsymmetric basic subsequence?

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Question 4

What is the minimal cardinal $\mathfrak{ns}_{\mathrm{seq}}$ such that every normalized sequence $(x_\gamma)_{\gamma<\mathfrak{ns}_{\mathrm{seq}}}$ has a subsymmetric basic subsequence?

Upper Bounds

$\mathfrak{ns}_{\rm seq}$ is smaller or equal than the first ω-Erdös cardinal. Consequently, $\mathfrak{nc}$, too. Ketonen 1974.

It is consistent with the existence of a measurable cardinal number that $\mathfrak{nc}_{\rm o} \leq \omega_\omega$. Dodos-LA-Todorcevic, 2011.

It is consistent with the existence of infinitely many strongly compact cardinals that $\mathfrak{nc} \leq \omega_\omega$. Dodos-LA-Todorcevic, 2011.

The proof relies on the following:

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Definition

By $\mathrm{Pol}_d(\kappa)$ we mean that for every coloring $c : [[\kappa]^d]^{<\omega} \to \omega$ there is a sequence of infinite sets $(x_n)_n$, $x_n \subseteq \kappa$ such that

$$c \restriction \prod_{k<n} [x_n]^d \text{ is constant for every } n.$$

Theorem

If $\mathrm{Pol}_1(\kappa)$, then $\mathfrak{nc}_0 \leq \kappa$.

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$\mathfrak{nc} > \omega$: Gowers-Maurey HI space, 1993.

Lower Bounds

$\mathfrak{nc} > 2^\omega$: Argyros-Tolias non-separable HI space, 2004.

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$\mathfrak{nc}_0 \geq \omega_\omega$.

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$\mathfrak{ns}_{\text{seq}} > \kappa$ *if and only if* κ *is not an* ω*-Erdös cardinal, i.e. if* $\kappa \not\to (\omega)_2^{<\omega}$.

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Theorem

$\mathfrak{ns}_{\mathrm{seq}} > \kappa$ iff κ is not ω-Erdös, i.e. $\kappa \not\to (\omega)^{\omega}_{2}$.

Proof: Suppose first that κ is ω-Erdös, and let $(x_\gamma)_{\gamma<\kappa}$ be a normalized sequence in a Banach space.
For each n-set $s \in [\kappa]^n$, color s by the type of the sequence $(x_\gamma)_{\gamma\in s}$. Since there are only 2^ω many types, by the Erdös-property of κ, there is some infinite set $X \subseteq \kappa$ such that if $s, t \subseteq X$ have the same cardinality, then they have the same type.
This means that $(x_\gamma)_{\gamma\in X}$ is subsymmetric.

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Now suppose that $\kappa \not\to (\omega)_2^{<\omega}$. Fix a bad coloring $f : [\kappa]^{<\omega} \to 2$ without infinite f-homogeneous sets.

Let $\mathcal{B}$ be the family of all f-homogeneous subsets of κ. By the hypothesis on κ, $\mathcal{B}$ consists on finite subsets of κ. Moreover, $\mathcal{B}$ is a compact family, and hereditary under inclusion.

Define the following Shreier norm on $c_{00}(\kappa)$: For each $x \in c_{00}(\kappa)$, let

$$\|x\|_{\mathcal{B}} := \max\{\langle x, \chi_s\rangle \ : \ s \in \mathcal{B}\}.$$

Let $X_{\mathcal{B}}$ be the completion of $(c_{00}(\kappa), \|\cdot\|_{\mathcal{B}})$. Since $\mathcal{B}$ is hereditary under inclusion, the unit sequence $(u_\gamma)_{\gamma<\kappa}$ is a unconditional basis of $X_{\mathcal{B}}$.

Claim

$(x_\gamma)_\gamma$ does not have subsymmetric subsequences.

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Let $A \subseteq \kappa$ be an infinite set, and fix $\varepsilon > 0$; Since the family $\mathcal{B} \upharpoonright A = \mathcal{B} \cap \mathcal{P}(A)$ is compact, it follows from Ptak's Lemma that there is some finite sequence of scalars $(a_\gamma)_{\gamma \in s}$, $s \subseteq A$, such that $\sum_{\gamma \in s} a_\gamma = 1$, $a_\gamma > 0$, and such that

$$\sum_{\gamma \in t} a_\gamma < \varepsilon \text{ for every } t \in \mathcal{B} \upharpoonright A. \quad (1)$$

Now, $\mathcal{B} \cap [A]^{\#(s)} \neq \emptyset$:

If v is a finite set with $|v| \geq R(k, 2k-1, 2^k)$, then v contains an f-homogeneous subset of cardinality k.

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So, if $v \in \mathcal{B} \restriction A$ is such that $\#v = \#s$, then

$$\| \sum_{\gamma \in s} a_\gamma u_\gamma \|_{\mathcal{B}} < \varepsilon \text{ and } \| \sum_{\gamma \in v} a_{\theta_{s,v}(\gamma)} u_\gamma \|_{\mathcal{B}} = 1 \tag{2}$$

so $(x_\gamma)_{\gamma \in A}$ is not $1/\varepsilon$-subsymmetric. Since $\varepsilon > 0$ is arbitrary, it follows that $(x_\gamma)_{\gamma \in A}$ is not subsymmetric.

Let κ be an infinite cardinal number.

Proposition

There is a graph $\mathcal{G}_{\mathfrak{nc}}(\kappa)$ whose vertexes are finite sequences of finite sets of κ such that if the chromatic number $\chi(\mathcal{G}_{\mathfrak{nc}}(\kappa))$ of $\mathcal{G}_{\mathfrak{nc}}(\kappa)$ is countable, then there is a reflexive Banach space of density κ without unconditional basic sequences.

Proposition

*There is a graph $\mathcal{G}_{\mathfrak{nc}_0}(\kappa)$ whose set of vertexes is a **dense** family of finite sequences of finite sets of κ such that if the chromatic number $\chi(\mathcal{G}_{\mathfrak{nc}_0}(\kappa))$ is countable, then there is a normalized weakly null sequence $(x_\alpha)_{\alpha<\kappa}$ without unconditional sequences.*

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Theorem

The graph $\mathcal{G}_{\mathfrak{nc}_0}(\omega_n)$ is countably chromatic for every integer n.

Definition

Given $s, t \in [\kappa]^{<\omega}$ with the same cardinality, let $\theta_{s,t} : s \to t$ denote the unique order-preserving bijection between s and t. For such s, t, let

$$M(s,t) := \{\alpha \in s \cap t \,:\, \theta_{s,t}(\alpha) = \alpha\}.$$

Definition

Let κ be a cardinal number. We define the following *positional graphs*: The set of vertices is the family $[\kappa]^{<\omega}$ of finite subsets of κ. We define the following sets of edges:

$$\begin{aligned} E_{\mathrm{pos},\emptyset} &:= \{(s,t) \in [\kappa]^{<\omega} \ : \ M(s,t) = \emptyset\} \\ E_{\mathrm{pos},f} &:= \{(s,t) \in [\kappa]^{<\omega} \ : \ |M(s,t)| < f(|s \cap t|)\} \\ E_{\mathrm{pos}} &:= \{(s,t) \in [\kappa]^{<\omega} \ : \ M(s,t) \neq s \cap t\}, \end{aligned}$$

where $f : \omega \to \omega$ is given. It is clear that $E_{\mathrm{pos},\emptyset} \subseteq E_{\mathrm{pos},f} \subseteq E_{\mathrm{pos}}$. Let $\mathfrak{G}_{pos,\emptyset}(\kappa)$, $\mathfrak{G}_{pos,f}(\kappa)$ and $\mathfrak{G}_{pos}(\kappa)$ be the corresponding graphs.

Definition

$s, t \in [\kappa]^{<\omega}$ are n-disjoint, $n \in \mathbb{N}$, if $\#(s \cap t) \leq n$.

Definition

Given two finite block sequences $\bar{s} = (s_i)_{i<k}$ and $\bar{t} = (t_i)_{i<k}$, we say that $\bar{s}$ and $\bar{t}$ form a *n-Δ-system* if

(a) $\{i < k \,:\, s_i = t_i\}$ is an initial interval $I_=$ of k.

(b) $\{i < k \,:\, s_i$ and t_i are n-disjoints$\}$ is a final interval $I_\emptyset$ of $k - 1$.

(c) $|I_= \cup I_\emptyset| = k - 2$.

We define the graph $(\mathrm{Bs}_{<\omega}(\kappa), \mathfrak{H}_n)$ whose edges are pairs $(\bar{s}, \bar{t})$ of sequences which does not form a n-Δ-system.

Case ω_n

Definition

A mapping $f : [\kappa]^n \to \gamma$ is min-dependant when $f(s) = f(t)$ implies that $\min s = \min t$.

Theorem

For every integer n, there is a mapping $f_n : [\omega_n]^{n+1} \to \omega$ such that the family

$$\mathcal{F} := \{s \in [\omega_n]^{<\omega} \ : \ f \upharpoonright s \text{ is min-dependant}\}$$

is ω-dense in ω_n, i.e. for every infinite set $A \subseteq \omega_n$, and every integer k, $\mathcal{F} \cap [A]^k \neq \emptyset$.

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The functions f_n are compositions of Todorcevic's ϱ-functions on ω_n's:

Definition

A function $\varrho : [\kappa^+]^2 \to \kappa$ is called a ϱ-function if

(a) f is subbaditive, i.e. for every $\alpha < \beta < \gamma < \kappa^+$

  (a.1) $\varrho(\alpha, \beta) \leq \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$,

  (a.2) $\varrho(\alpha, \gamma) \leq \max\{\varrho(\alpha, \beta), \varrho(\alpha, \gamma)\}$.

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(c) $\varrho(\alpha, \beta) \neq \varrho(\beta, \gamma)$ for every $\alpha < \beta < \gamma$.

For each integer n we fix a ϱ function $\varrho^{(n)}$ on ω_n. Let $n \in \omega$. For each $i \leq n$ we define recursively $f_i^{(n)} : [\omega_n]^{i+1} \to \omega_{n-i}$ as follows:

(1) $f_0^{(n)} := \mathrm{Id}_{\omega_n}$;

(2) $f_i(\alpha_0, \alpha_1, \dots, \alpha_i) := \varrho^{(n-(i-1))}(f_{i-1}(\alpha_0, \dots, \alpha_{i-1}), f_{i-1}(\alpha_1, \dots, \alpha_i))$ for each $\alpha_0 < \cdots < \alpha_i$ in ω_n and each $0 < i \leq n$.

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Let $f_n := f_n^{(n)} : [\omega_n]^{n+1} \to \omega$.

Then Using Erdös-Rado canonization, one can prove that f_n above has the desired property.

C1

It is consistently true *CH* and $\mathfrak{nc} = \omega_2$.

C2

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C2

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