Embeddability as a “universal” quasi-order

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Bertinoro, May 27 2011

The framework

Framework: Establish connections between (basic) Model Theory for infinitary languages and Descriptive Set Theory (DST).

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Given an $\mathcal{L}_{\kappa\lambda}$-sentence φ, we set

$$\mathrm{Mod}^{\mu}_{\varphi} = \{x \in \mathrm{Mod}^{\mu}_{\mathcal{L}} \mid x \models \varphi\}.$$

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Every two uncountable Borel subsets of Polish spaces are Borel isomorphic: hence w.l.o.g. we may assume $\mathrm{dom}(R) = 2^\omega$.

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We abbreviate this statement with: $\sqsubseteq$ on $\mathrm{Mod}^{\omega}_{\mathcal{L}}$ is invariantly universal.

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3. A generalization of the F-MR theorem could allow to better understand the embeddability relation on $\mathrm{Mod}^\kappa_\mathcal{L}$, e.g. we would have that $(\mathcal{P}(\kappa), \subseteq^*)$ "Borel embeds" into $\sqsubseteq$ on (generalized) trees. In particular, we would have a generalization of Baumgartner result.

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4. show that $\mathrm{Sat}(f(2^\omega))$ is Borel (which also implies that g is Borel); [Sketch of the proof: each $f(x)$ is rigid by construction, hence the map $h \colon S_\infty \times 2^\omega \to \mathrm{Mod}^\omega_\mathcal{L} \colon (p, x) \mapsto j_\mathcal{L}(p, f(x))$ is injective. Since h is Borel and $\mathrm{range}(h) = \mathrm{Sat}(f(2^\omega))$, this last set is Borel.]
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In general, a κ^+-Borel subsets of X of size $> \kappa$ need not be κ^+-Borel isomorphic to 2^κ, but: for every analytic q.o. R there is an analytic q.o. R' on 2^κ s.t. $R \sim^\kappa_B R'$.

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Remark: if $\kappa^{<\kappa} > \kappa$ both directions of Lopez-Escobar theorem can fail.

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4. even if $\mathrm{Sat}(f(2^\kappa))$ is proved to be κ^+-Borel, the "inverse" reduction g need not be κ^+-Borel.

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This can be done in a completely different way w.r.t. the countable case.

The main idea

1. Find a suitable $\mathcal{L}_{\kappa^+\kappa}$-sentence ψ s.t. $f(2^\kappa) \subseteq \mathrm{Mod}^\kappa_\psi$ (ψ essentially "describes" the common part of the structures $f(x)$ for $x \in 2^\kappa$);

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Lemma

Assume κ is weakly compact. Then there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ s.t. $\mathrm{Sat}(f(2^\kappa)) = \mathrm{Mod}^\kappa_\varphi$.

Proof.

2^κ is κ-compact: since $h \circ f$ is continuous, $(h \circ f)(2^\kappa)$ is κ-compact and hence closed in 2^κ (because 2^κ is Hausdorff and κ is regular). Let $U = 2^\kappa \setminus (h \circ f)(2^\kappa)$: then $h^{-1}(U) = \mathrm{Mod}^\kappa_{\varphi_U}$, and hence it is enough to let φ be $\psi \wedge \neg\varphi_U$. □

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Since $\kappa^{<\kappa} = \kappa$, it is enough to show that $g^{-1}(\mathbf{N}_s)$ is κ^+-Borel for every $s \in {}^{<\kappa}2$. Notice that for every $A \subseteq 2^\kappa$, $g^{-1}(A) = \mathrm{Sat}(f(A))$. Each $\mathbf{N}_s$ is also closed, hence κ-compact: using an argument similar to the one in the previous lemma, find an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ_s such that $\mathrm{Sat}(f(\mathbf{N}_s)) = \mathrm{Mod}^\kappa_{\varphi_s}$. Then use the generalized Lopez-Escobar theorem. □

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Therefore we have shown:

Theorem (M.)

Let κ be a weakly compact cardinal. For every analytic q.o. R on 2^κ there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ s.t. $R \sim^\kappa_B \sqsubseteq\restriction \mathrm{Mod}^\kappa_\varphi$ (i.e. $\sqsubseteq$ on $\mathrm{Mod}^\kappa_{\mathcal{L}}$ is invariantly universal).

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Some comments:

1. The main difficulty is that in the previous arguments we heavily used the fact that 2^κ is κ-compact, which is equivalent to κ being weakly compact: therefore we need to use different ideas!
2. The condition $\kappa^{<\kappa} = \kappa$ would be optimal: if $\kappa^{<\kappa} > \kappa$ then one could find counterexamples (e.g. if $2^{\kappa^+} > 2^\kappa$).

Thank you for your attention!