Partitions properties of separable metric spaces

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Milman's theorem

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Theorem (Milman, 71)

Let $n > 0$, $\varepsilon > 0$.
Then there is $N \in \mathbb{N}$ such that whenever $\mathbb{S}^N = R \cup B$, we have $\mathbb{S}^n \hookrightarrow (R)_\varepsilon$ or $\mathbb{S}^n \hookrightarrow (B)_\varepsilon$. In symbols:

$$\forall n \in \mathbb{N} \;\; \exists N \in \mathbb{N} \;\; \forall \varepsilon > 0 \quad \mathbb{S}^N \xrightarrow{\varepsilon} (\mathbb{S}^n)_2.$$

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Remark: This is implied by:

Theorem (Matoušek-Rödl, 95)

Let $X \subset \mathbb{S}^\infty$ finite, affinely independent, with circumradius < 1.
Then there is a finite $Y \subset \mathbb{S}^\infty$, affinely independent, with circumradius < 1 such that

$$Y \longrightarrow (X)_2.$$

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No.

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Is there a direct, geometric or combinatorial, argument?

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The space above is the Urysohn space, denoted $\mathbb{U}$.
Up to isometry, there is a unique sphere of diameter 1 in $\mathbb{U}$. The corresponding metric space is the Urysohn sphere, denoted $\mathbb{S}$.

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Remark

- *For some finite approximate Ramsey type properties, $\mathbb{U}$ and ℓ_2 behave similarly. So do $\mathbb{S}$ and $\mathbb{S}^\infty$ (Gromov-Milman, 84 ; Pestov, 02).*
- *For exact finite Ramsey properties, the analogy is not clear yet (Kechris-Pestov-Todorcevic, Nešetřil, 05).*

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Corollary

Let $\varepsilon > 0$. Then:

$$\mathbb{S}_{\mathcal{C}([0,1])} \xrightarrow{\varepsilon} \left(\mathbb{S}_{\mathcal{C}([0,1])}\right)_2.$$

Note: in general, cannot require the copy to be linear.

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Proposition

- $\mathbb{S}$ *is the unique complete separable metric space with distances in* $[0,1]$ *into which any separable metric space with distances in* $[0,1]$ *embeds.*
- $\mathbb{S}$ *has a countable rational analogue: the space* $\mathbb{S}_{\mathbb{Q}}$*, unique countable ultrahomogeneous with distances in* $[0,1] \cap \mathbb{Q}$ *into which any countable metric space with distances in* $[0,1] \cap \mathbb{Q}$ *embeds.*
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Theorem (Delhommé-Laflamme-Pouzet-Sauer, 07)

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- *Up to isometry, there is a unique countable ultrahomogeneous metric space with distances in* $\{1, \dots, m\}$ *into which every countable metric space with distances in* $\{1, \dots, m\}$ *embeds. (*$\mathbb{U}_m$*, the Urysohn space with distances in* $\{1, \dots, m\}$*)*
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  - *S closed under addition, or initial segment of such a set.*
  - *S well founded with $s^+ > 2s$ for every $s \in S$.*

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Let $S \subset]0, +\infty[$ be bounded so that $\mathbb{U}_S$ exists. Is $\mathbb{U}_S$ indivisible?

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Theorem (Sauer, 10)

$\mathbb{U}_S$ *is indivisible for every finite* S.

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- Some spaces are not indivisible (resp. approximately indivisible). Still, some weaker partition relations can hold.
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- Idem for $\mathbb{U}_\mathbb{Q}$, $\mathbb{U}_\mathbb{N}$.

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2. $\mathbb{U} \xrightarrow{\varepsilon} (X, \mathbb{U})$ *for every finite* $X \subset \mathbb{U}$ *(even compact) and* $\varepsilon > 0$.

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Let X *be a unit distance pair of points. Does* $\mathbb{U}_{\mathbb{Q}} \longrightarrow (X, \mathbb{U}_{\mathbb{Q}})$*?*
Idem for $\mathbb{S}_{\mathbb{Q}}$ *with distance* $\delta < 1/2$*.*

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Theorem

$\mathbb{U}_{\mathbb{Q}} \stackrel{\varepsilon}{\longrightarrow} (X, \mathbb{U}_{\mathbb{Q}})$ *for every finite* $X \subset \mathbb{U}_{\mathbb{Q}}$ *and* $\varepsilon > 0$.
(for $\mathbb{S}_{\mathbb{Q}}$*, already know from LA-NVT-S that* $\mathbb{S}_{\mathbb{Q}} \stackrel{\varepsilon}{\longrightarrow} (\mathbb{S}_{\mathbb{Q}})_2$ *for every* $\varepsilon > 0$*)*

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Question

Is there a general finite Ramsey theorem for finite, affinely independent, metric subspaces of ℓ_2? (For points: cf Frankl-Rödl, 90)
Same question for $\mathbb{S}^\infty$, with same spaces, plus requirement that circumradius < 1. (For points, cf Matoušek-Rödl, 95)