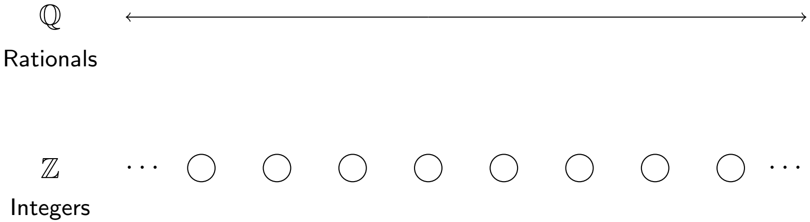
The Duplicator-Spoiler (Ehrenfeucht-Fraïssé) Game for an Ordinal Number of Turns

Gasarch, Pinkerton

Let's Play a Game!

Let's play $\mathcal{G}(\mathbb{Q};\mathbb{Z})$.



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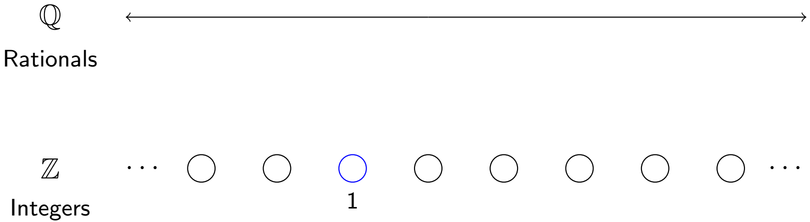

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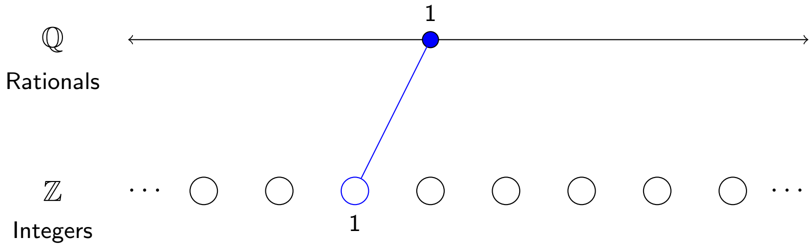

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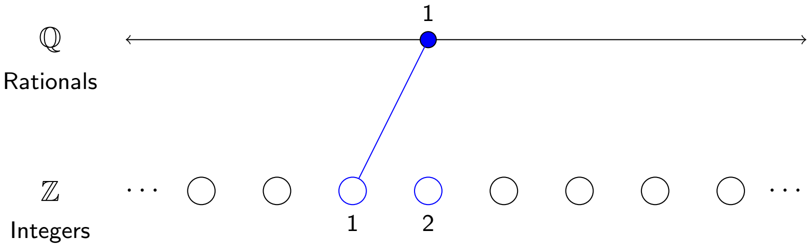

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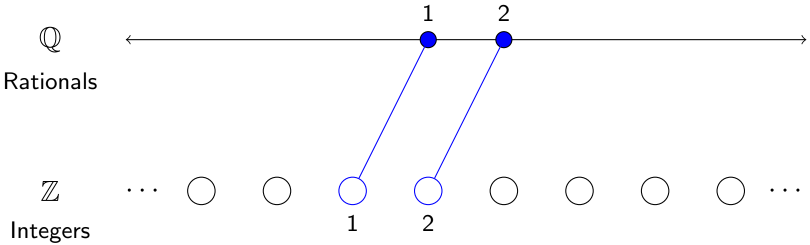

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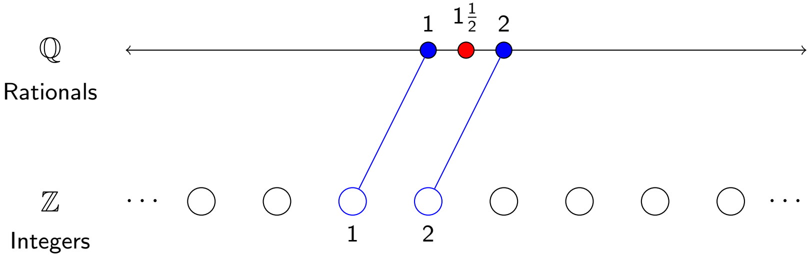

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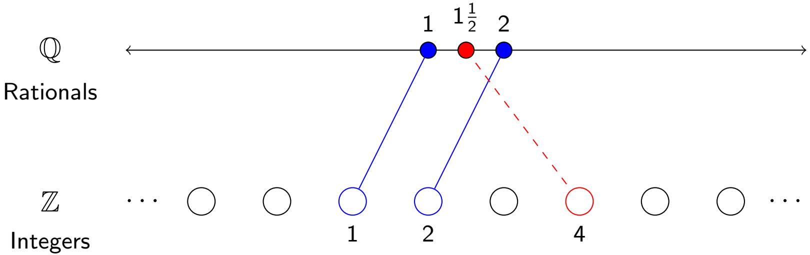


And Now with Logic

This sentence is true of $\mathbb{Q}$ but not of $\mathbb{Z}$.

$$\underset{1}{(\forall x)}\underset{2}{(\forall y)}[x < y \implies \underset{3}{(\exists z)}(x < z < y)]$$

Connection to Finite Model Theory

Theorem

Let S_1 and S_2 be sets. Let $\mathcal{R}$ be a relation on S_1 and S_2. The following are equivalent:

- *Duplicator wins the m-turn game $\mathcal{G}(S_1; S_2)$ with relation $\mathcal{R}$.*
- *For all first-order m-quantifier-depth sentences ϕ in the language of standard logical symbols and $\mathcal{R}$*

$$[S_1 \models \phi \iff S_2 \models \phi].$$

Applications of Duplicator-Spoiler Games

- Proving limits of expressibility
- Separating complexity classes

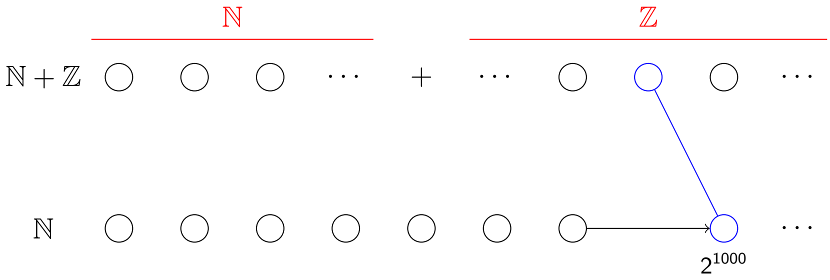
Finite Linear Orderings

Theorem

The spoiler wins the game $\mathcal{G}(\mathcal{F}_m; \mathcal{F}_n)$ $(m > n)$ *in* $\lfloor \log_2(n+1) \rfloor + 1$ *turns.*

The Problem...

Can the spoiler win $\mathcal{G}(\mathbb{N}+\mathbb{Z};\mathbb{N})$? Suppose there are 1000 turns.



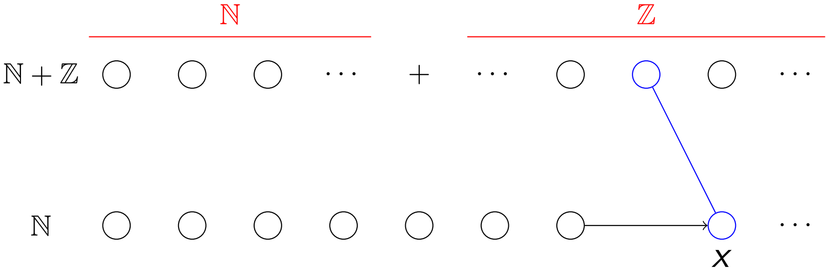
Ordinal Numbers of Turns

Definition

Let β and γ be ordinals. If a game has γ turns then after one turn, the spoiler chooses a $\beta < \gamma$ and there are β turns remaining.

$\mathcal{G}(\mathbb{N} + \mathbb{Z}; \mathbb{N})$

Can the spoiler win $\mathcal{G}(\mathbb{N} + \mathbb{Z}; \mathbb{N})$?



This game takes ω turns.

$\mathcal{G}(\mathbb{Z}^k + \mathbb{Z}^k; \mathbb{Z}^k)$

Definition

1. $\mathbb{Z}^2 = \mathbb{Z} * \mathbb{Z} = \ldots + \mathbb{Z} + \mathbb{Z} + \mathbb{Z} + \mathbb{Z} + \ldots$
2. $\mathbb{Z}^3 = \mathbb{Z}^2 * \mathbb{Z} = \ldots + \mathbb{Z}^2 + \mathbb{Z}^2 + \mathbb{Z}^2 + \ldots$

etc.

Theorem

$\mathcal{G}(\mathbb{Z}^k + \mathbb{Z}^k; \mathbb{Z}^k)$ *takes* $\omega * k + 1$ *turns.*

Higher Ordinal Games

Definition

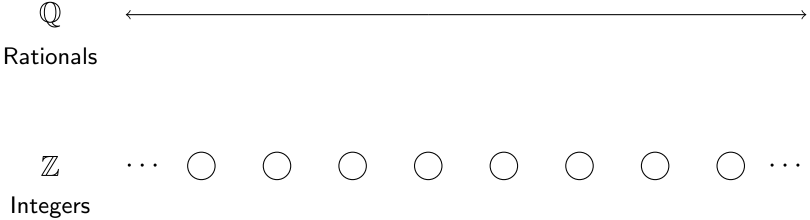
1. $\mathbb{Z}^0 = \mathcal{F}_1$
2. $\mathbb{Z}^\gamma * \mathbb{Z} = \mathbb{Z}^{\gamma+1}$
3. $\mathbb{Z}^\lambda = (\sum (\mathbb{Z}^\gamma * \omega | \gamma < \lambda))^* + \sum (\mathbb{Z}^\gamma * \omega | \gamma < \lambda)$

Theorem

For any ordinal λ, $\mathcal{G}(\mathbb{Z}^\lambda + \mathbb{Z}^\lambda; \mathbb{Z}^\lambda)$ takes $\omega * \lambda + 1$ turns.

The Addition Game

Let's play $\mathcal{G}(\mathbb{Q};\mathbb{Z})$ *with addition*.



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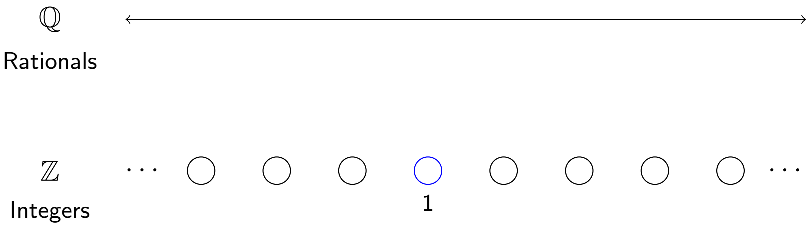

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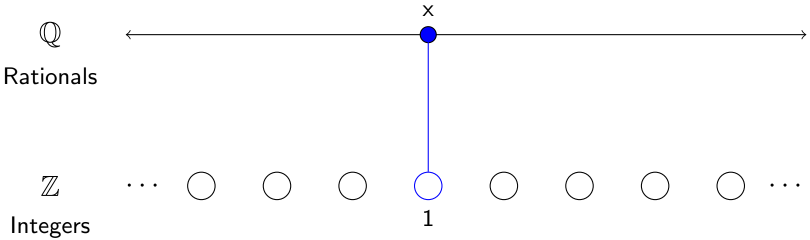

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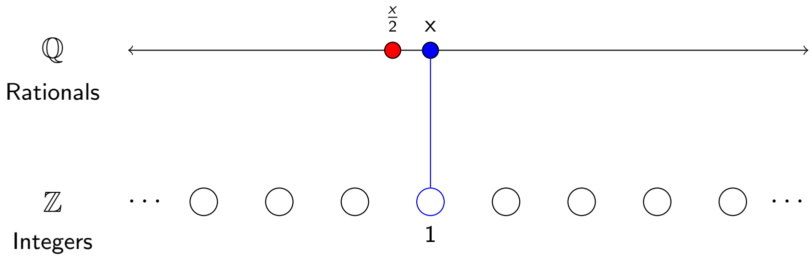
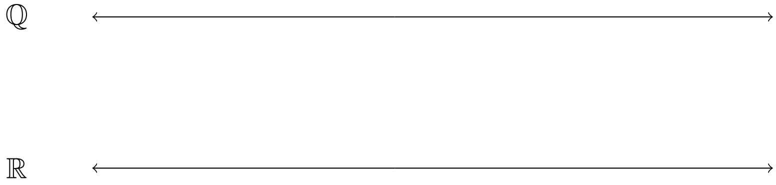


Let's Play another Plus Game

$\mathcal{G}(\mathbb{Q};\mathbb{R})$ with addition takes $\omega + 1$ turns.



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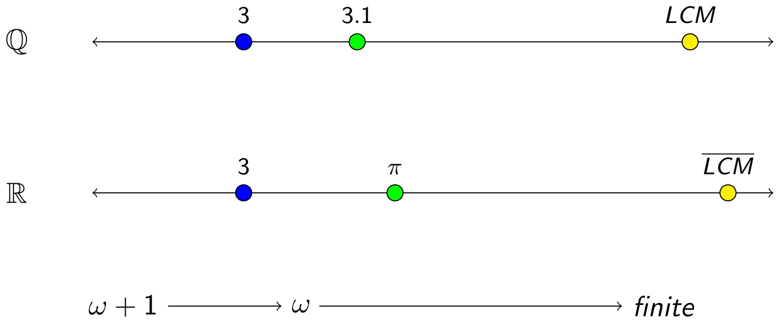
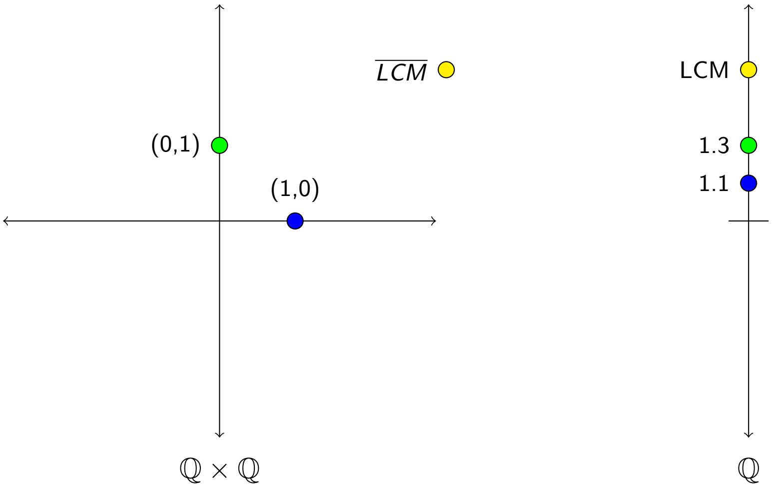


Vectors of Rationals

$\mathcal{G}(\mathbb{Q} \times \mathbb{Q}; \mathbb{Q})$

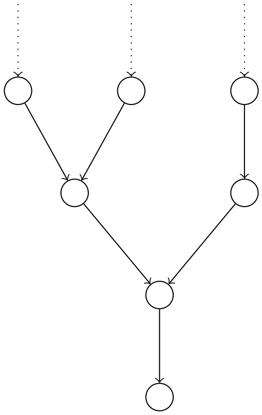


Theorem: Let $m, n \in \mathbb{N}$, $m > n$. $\mathcal{G}(\mathbb{Q}^m; \mathbb{Q}^n)$ takes $\omega + n$ turns.

Bounds on Operation Games

Theorem

Unary operation games take fewer than $\omega * 2$ turns.



Back to Logic

$\mathcal{G}(\mathbb{Z}+\mathbb{Z};\mathbb{Z})$ takes $\omega+1$ turns. $\omega+1$ corresponds to:

two moves of delay ↔ $(\forall x)(\forall y)$

then declaring the number of remaining turns. ↔ $(\exists S)(\forall z)[x < z < y \implies z \in S]$

Future Work

- Formalize matching higher-order system.
- Find natural operation game which take $\omega * 2$ turns.
- Explore generalized graphs.