Ramsey properties of random discrete structures

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joint with

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Ramsey-type theorems

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$\forall$ *k-uniform hypergraph* F, $r \in \mathbb{N}$ $\exists\, n \in \mathbb{N}$: $\quad K_n^{(k)} \longrightarrow (F)_r^e$

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Main question in this talk

Can $K_n^{(k)}$ (resp. $[n]$ or $[n]^d$) be replaced by a random object?

Common framework

- sufficiently large complete structures have the Rasmey-property

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For a probability $p = p_n$ and a(finite) ground set $\Gamma = \Gamma_n$ let $\Gamma_{n,p}$ be the random subset, where each element from Γ_n is included independently with probability p.

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Examples:

- $[n]_p$, $[n]_p^d$, random (hyper)graph $G^{(k)}(n,p)$

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- $[n]_p$, $[n]_p^d$, random (hyper)graph $G^{(k)}(n,p)$
- Common framework: random subsets of $V_{n,p} \subseteq V_n$

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What is the smallest sequence $(p_n)_{n\in\mathbb{N}}$ of probabilities for which the property being (R,ξ)-Ramsey of $(H_n)_{n\in\mathbb{N}}$ can be transfered to $(H_n[V_{n,p_n}])_{n\in\mathbb{N}}$?

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- $p^{e(F')} n^{v(F')} \gg pn^k \quad \forall F' \subseteq F$ — Ramsey for k-uniform hypergraphs

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Corollary (random version of Gallai-Witt theorem)

$\forall F \subset \mathbb{N}^d$ *finite,* $\forall r \in \mathbb{N} > 0$, $\exists C > 0$ *such that if* $q_n > Cn^{-1/(|F|-1)}$ *then*

$$\lim_{n \to \infty} \mathbb{P}\big([n]_{q_n} \longrightarrow (F)_r\big) = 1\,.$$

New result (cont'd)

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  - graphs,
  - arithmetic progressions and more generally density regular A,
  - 0-statement for Rado's theorem,
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- sharp thresholds
  - Friedgut and Krivelevich (2000) vertex colorings for strictly balanced graphs
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- slightly stronger versions hold:
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  - 1-statement holds with probability $1 - 2^{-\Omega(p_n|V_n|)}$

Main technical result

Definition

Let $\mathbf{H} = (H_n)_{n\in\mathbb{N}}$ be a sequence of ℓ-uniform hypergraphs, let $\mathbf{p} = (p_n)_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ be a sequence of probabilities, and let $K \geq 1$. We say $\mathbf{H}$ is $(K, \mathbf{p})$-**bounded** if the following is true:

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For every $i \in [\ell - 1]$ there exists n_0 such that for every $n \geq n_0$ and $q \geq p_n$ we have

$$\mathbb{E}\left[\sum_{v\in V} \deg_i^2(v, V_{n,q})\right] \leq Kq^{2i}\frac{|E(H_n)|^2}{|V(H_n)|},$$

where for any subset $U \subseteq V_n$

$$\deg_i(v, U) = |\{e \in E\colon\ |e \cap (U \setminus \{v\})| \geq i \text{ and } v \in e\}|\ .$$

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Definition

Set $R(1,r)=R(\ell,1)=1$ and $R(\ell+1,r+1)=R(\ell,r+1)+(r+1)R(\ell+1),r)$.

Main lemma

Suppose $\mathbf{H}$ is $(K,\mathbf{p})$-bounded.

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$\forall i \in [k]$, $r \in \mathbb{N}$, $\xi > 0$ $\exists \zeta > 0$, $b > 0$, $C \geq 1$ and n_0 such that $\forall n \geq n_0$ and $\forall q \geq Cp_n$ the following holds.

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If $U\subseteq V_n$ with

$$H_n[U]\xrightarrow{\xi|E_n|}(e)^v_{R(i,r)}$$

then the binomial random subset U_q satisfies with probability at least

$$\mathbb{P}\Big(H_n[U_q]\xrightarrow{\zeta q^i|E_n|}(e_i)^v_r\Big)\geq 1-2^{-bq|V_n|}\,.$$

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- Find right generalization:
  - Ramsey-threshold
  - Extremal-threshold
  - Removal Lemma-threshold
  - KŁR-conjecture: probabilistic counting lemma for sparse regularity lemma