Weak Consequences of Ramsey's Theorem for Pairs

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Motivated by the work of CJS, we want to analyze and calibrate the strength of various consequences of RT (pairs). All orders (chains) are countably infinite.

CAC (Dilworth): Every partial order has a chain or an antichain.

ADS (Erdös-Szekeres): Every linear order. has an ascending (ω) or descending (ω^*) suborder.

As for RT, decompose into stable and cohesive versions.

The stable linear orders are those of type ω, ω^* or $\omega + \omega^*$.

SADS: Every l.o. of type $\omega+\omega^*$ has a suborder type ω or ω^*.

CADS: Every l.o. has a suborder type ω, ω^*, or $\omega + \omega^*$.

Def.: a p.o. $\mathcal{P}$ is stable if

$$\forall i[(a.e.j)(i <_{\mathcal{P}} j) \vee (a.e.j)(i|_{\mathcal{P}}j)]$$

or $\forall i[(a.e.j)(i >_{\mathcal{P}} j) \vee (a.e.j)(i|_{\mathcal{P}}j)]$.

SCAC: Every stable p.o. has an chain or antichain.

CCAC: Every p.o. has a stable suborder.

We measure the strength of such principles either proof theoretically (reverse mathematics) or computational (Turing machines with oracles). The second, captures the intuition that problem A is easier than B if we can solve any instance of B by combining computable procedures with the ability to solve any instance of A that we can construct and then use the solutions as oracles.

If $\mathcal{C}$ is closed under Turing reducibility and join, $\mathcal{C}$ *computably satisfies* Ψ if Ψ is true in the standard model of arithmetic with set quantification over $\mathcal{C}$. Ψ *computably entails* Φ, $\Psi \models_c \Phi$, if (for closed $\mathcal{C}$), $\mathcal{C} \models_c \Psi \rightarrow \mathcal{C} \models_c \Phi$. Ψ and Φ are *computably equivalent*, $\Psi \equiv_c \Phi$, if $\Psi \models_c \Phi$ and $\Phi \models_c \Psi$.
Expresses the relations among mathematical theorems directly. Nonimplications are stronger than for r. m. as we consider only standard models.

Combinatorial Arguments

Prop 1: RT$\vdash$CAC; SRT$\vdash$SCAC.

Prop. 2: ADS$\vdash$COH.

Given $\langle R_i \rangle$ apply ADS to the lex order on $\langle R_i(x) | i \leq x \rangle$.

Prop 3: $\vdash$ CCAC $\leftrightarrow$ ADS.

$\rightarrow$ by def. $\leftarrow$: Let $\mathcal{L}$ be a linearization of $\mathcal{P}$. Apply ADS to get $S = \langle s_i \rangle$. If S is ascending, $(\forall i)(\forall j > i)(s_i <_P s_j \vee s_i \,|_P\, s_j)$. Apply COH to S and the sequence $R_i = \{s_j \mid s_i <_P s_j\}$. Any $\vec{R}$-cohesive subset of S is a stable suborder of $\mathcal{P}$. Similarly if S is descending.

Prop 4: SCAC $\vdash$ SADS.
Proof: Given $\leq_L$ of type $\omega + \omega^*$, color (m, n) blue if $m \leq_L n \wedge m \leq n$; otherwise, red. This is stable. A blue homogeneous set has order type ω. A red one has order type ω^*.

Prop: COH $\vdash$ CADS.
Proof: Given $\leq_L$, let $R_n = \{m \mid m \leq_L n\}$. If S is $\vec{R}$-cohesive then every element of $(S, \leq_L)$ has either finitely many predecessors or finitely many successors.

Classical degree theoretic arguments (Lerman, Tennenbaum, Denisov): none of these principles are computably true. More elaborate ones (Manaster, Hermann, DHLS, HS): they are not entailed by WKL; most of these implications cannot be reversed even in the sense of $\not\models_c$.
Also show that the splittings are true ones, i.e. CAC (ADS) is not entailed by either SCAC or CCAC=COH (SADS, CADS).

Forcing arguments give more.

COH + WKL $\not\models_c$ SADS.

Iterate both Mathias and binary tree forcings.

ADS $\not\models_c$ WKL.

Iterate Mathias forcing and a special one adding solutions for ADS without a path through a specific instance of WKL. (Conditions are Δ_2.)

CAC $\not\models_c$ WKL.

Add a forcing with extension also Δ_2 to get SCAC without adding a solution to WKL. Iterate to get CAC.

A careful analysis of the forcings that avoids WKL shows we can get more.

Def: f is *diagonally nonrecursive* relative to A if $\forall e\,(f(e) \neq \Phi_e^A(e))$, where Φ_e is the e^{th} Turing functional.

(DNR**)** For every set A there is a function f that is diagonally noncomputable relative to A.

Immediate that WKL $\vdash$ DNR. Converse fails.

Thm: CAC $\not\models_c$ DNR

On the other hand, Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp & Slaman had already proven:

Thm: SRT $\vdash$ DNR

Cor: CAC $\not\models_c$ RT,

Indeed, CAC $\not\models_c$ SRT.

HS show $\vdash$ADS$\rightarrow$BΣ_2. Chong, Slaman and Yang have recently shown that CAC is conservative over BΣ_2. So all three principles have the same first order consequences.

An analysis of the proofs leads to conditions on the colorings associated with CAC and ADS that make them work. Transitivity is the key. Def: An *n-coloring* of $[X]^2$, $\langle C_i \mid i < n\rangle$, is *transitive* if each C_i is transitive: $C_i(x, y) \wedge C_i(y, z) \rightarrow C_i(x, z)$. It is *semi-transitive* if each C_i but one is transitive. If the C_i are not disjoint we call it an n-multicoloring. An infinite H is *homogeneous* for $\langle C_i\rangle$ if $(\exists i < n)(\forall x, y \in H)[x < y \rightarrow C_i(x, y)]$.

Let TrRT^2_n, STrRT^2_n, TrMRT^2_n and STrMRT^2_n be the assertions that colorings of the indicated type always have homogeneous sets.

$\forall n \geq 2, \vdash \mathrm{STrMRT}^2_n \rightarrow \mathrm{STrRT}^2_n$.

$\forall n \geq 2, \vdash \mathrm{STrMRT}^2_n \leftrightarrow \mathrm{STrMRT}^2_2$;

$\vdash \mathrm{STrRT}^2_3 \rightarrow \mathrm{CAC}; \vdash \mathrm{CAC} \rightarrow \mathrm{STrRT}^2_2$.

Only the last implication is not trivial. Still, we have the interesting equivalences:

$(\forall n \geq 2) \vdash \mathrm{CAC} \leftrightarrow$ every semi-transitive n-multicoloring has a homogeneous set.

For transitive partitions, the implication from multicolorings to colorings remains obvious but not the other direction. However,

$\vdash \mathrm{TrRT}^2_2 \to \mathrm{ADS} \to \mathrm{TrMRT}^2_2$.

so these three are equivalent.

Question: What are the relations among the TrRT^2_n and TrMRT^2_n as n varies over $\mathbb{N}$ and with STrRT^2_2.

Question: In particular, does ADS imply or entail CAC?

Remember they have the same first order consequences.

We have similar results and questions for the stable versions, SADS and SCAC.

More general open questions concern the possible generalizations of TrRT^2_n, TrMRT^2_n and STrRT^2_n to k-tuples for $k > 2$. Here, while we have some results, we do not even know what the "right" generalizations should be even for $k = 3$. There are many possibilities.

Some are related to canonical Ramsey theorems, some to regressive ones. Some correspond to the view that in the $n = 2$ case, we required that if the "first two" sides of a triangle have the same color so does the third. Other choices of the ordering of the sides are possible even for $n = 2$. Should we somehow consider the four sides of a pyramid for $k = 3$ or some other principle?

What generalizations make sense or might be useful?