An abstract approach to finite Ramsey theory and a self-dual Ramsey theorem

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Outline of Topics

1. Self-dual Ramsey theorem
2. Algebraic notions
3. Abstract pigeonhole and main theorem
4. Localizing and propagating pigeonhole

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Plan: present the concrete self-dual Ramsey result, present the abstract approach, and roughly outline how it is applied to get the above results.

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3. the Graham–Rothschild theorem for partitions is proved directly without proving it first for parameter sets; however, the parameter set generalization can also be obtained from the abstract result;

4. a hierarchy of the Ramsey results according to the number of times the abstract Ramsey theorem is applied in their proofs: the classical Ramsey theorem requires one application, the Hales–Jewett theorem requires two, the Graham–Rothschild theorem three, and the self-dual Ramsey theorem four.

Self-dual Ramsey theorem

Definition

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Theorem (Ramsey)

Given K, L and $d > 0$ there exists M such that for each d-coloring of all increasing injections $[K] \to [M]$ there exists an increasing injection $j_0\colon [L] \to [M]$ such that

$$\{j_0 \circ i\colon i\colon [K] \to [L] \text{ an increasing injection}\}$$

is monochromatic.

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Theorem (Graham–Rothschild)

Given K, L and $d > 0$ there exists M such that for each d-coloring of all rigid surjections $[M] \to [K]$ there exists a rigid surjection $t_0\colon [M] \to [L]$ such that

$$\{s \circ t_0 \colon s\colon [L] \to [K] \text{ a rigid surjection}\}$$

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Definition

A pair (s, i) is a **connection between** L **and** K if $s\colon [L] \to [K]$, $i\colon [K] \to [L]$ and for each $x \in [K]$

$$s(i(x)) = x \;\text{ and }\; \forall y < i(x)\; s(y) \leq x.$$

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We write

$$(s, i)\colon [L] \leftrightarrow [K].$$

Given connections $(s,i)\colon [L] \leftrightarrow [K]$ and $(t,j)\colon [M] \leftrightarrow [L]$, define

$$(s,i)\cdot(t,j)\colon [M] \leftrightarrow [K]$$

as

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Theorem (S.)

For natural numbers K, L and $d > 0$ there exists M such that for each d-coloring of all connections between M and K there is $(t_0, j_0)\colon [M] \leftrightarrow [L]$ such that

$$\{(s, i) \cdot (t_0, j_0)\colon (s, i)\colon [L] \leftrightarrow [K]\}$$

is monochromatic.

Algebraic notions

Abstract Ramsey statement:

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multiplication/action

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Algebraic approach:

multiplication/action, lifting them to sets, truncation operator.

Multiplicative part

Definition

A **local actoid**

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such that for $a, b \in A$ and $z \in Z$ if $a \,.\, (b \,.\, z)$ and $(a \cdot b) \,.\, z$ are both defined, then

$$a \,.\, (b \,.\, z) = (a \cdot b) \,.\, z.$$

Definition

A local actoid (A, Z) is called an **actoid** if for all $a, b \in A$ and $z \in Z$,

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Note: for a, b, z as above, one has $a \,.\, (b \,.\, z) = (a \cdot b) \,.\, z$.

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Then $s \circ t\colon [N_0] \to [K]$ is a rigid surjection for some $N_0 \leq N$.

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Lifting multiplication and action to sets

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For $F \subseteq A$ and $S \subseteq Z$, $F \,.\, S$ **is defined** if $f \,.\, x$ is defined for all $f \in F$ and $x \in S$, and we let

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$\mathcal{F}_0 = \mathcal{S}_0$ consist of sets of the form

$$F_{L,K} = S_{L,K} = \{s \in A_0 = Z_0 \colon s \colon [L] \to [K]\},$$

for $L \geq K > 0$.

$F_{N,M} \bullet F_{L,K}$ defined if and only if $M = L$ and

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$(\mathcal{F}_0, \mathcal{S}_0)$ with these operations is an **actoid of sets over** (A_0, Z_0).

Note that $F_{N,M} \cdot F_{L,K}$ and $F_{N,M} \,.\, S_{L,K}$ are defined if only $M \geq L$.

Truncation added

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∂ is call a **truncation**.

It is a type of a restriction operator.

Notation: for a background (A, Z) with a truncation ∂ and for $S \subseteq Z$, let

$$\partial S = \{\partial x \colon x \in S\}$$

and, more generally, for $t \in \mathbb{N}$

$$\partial^t S = \{\partial^t x \colon x \in S\}.$$

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∂_0 is a **truncation forgetting the largest value**.

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(A_0, Z_0) with ∂_0 is a **background**.

Abstract pigeonhole and main theorem

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Recall the **abstract Ramsey statement**:

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Recall the **abstract Ramsey statement**:

find $F \in \mathcal{F}$ for which $F \bullet S$ is defined;
color $F \bullet S$;
find $f \in F$ with $f \,.\, S$ monochromatic.

We consider the **equivalence relation** $\sim$ on S given by

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We are looking for a **principle of the form**:

there is $F \in \mathcal{F}$ such that for each coloring of $F \bullet S$ there is $f \in F$ with multiplication by f stabilizing the coloring on equivalence classes of $\sim$.

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(ph): multiplication by f fixes color on equivalence classes of the equivalence relation on $\partial^t S$ given by $\partial x_1 = \partial x_2$.

Definition

A family $\mathcal{I}$ of subsets of Z for a background (A, Z) is called **vanishing** if for every $S \in \mathcal{I}$ there is $t \in \mathbb{N}$ such that $\partial^t S$ consists of at most one element.

Definition

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Theorem (S.)

Let $(\mathcal{F}, \mathcal{S})$ be a pigeonhole actoid. Assume $\mathcal{S}$ is vanishing. Then for every $d > 0$ and $S \in \mathcal{S}$ there exists $F \in \mathcal{F}$ such that $F \bullet S$ is defined and for each d-coloring of $F \bullet S$ there exists $f \in F$ for which $f.S$ is monochromatic.

Localizing and propagating pigeonhole

Localizing property (ph)

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$$x \leq_{A,Z} y \Rightarrow$$
$$\forall a \in A \text{ (if } a.y \text{ is defined, then } a.x \text{ is defined and } a.x \leq_{A,Z} a.y)$$

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In (ph), we color $F.(\partial^t S)$ and are asked to find $f \in F$ making the coloring constant on **each** equivalence class of the equivalence relation on $\partial^t S$ given by $\partial x_1 = \partial x_2$.

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Notation: $(\partial^t S)_y$

Localization: require making the coloring constant by multiplication by $f \in F$ on a **fixed** equivalence class $(\partial^t S)_y$ for some $y \in \partial^{t+1} S$; prove it implies full (ph).

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Price: need to keep a prescribed behavior of f on a part of the space of invariants $\partial^{t+1} S$ containing y.

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We consider the following **criterion**:

(ph^-) for $t \geq 0$, $d > 0$, $S \in \mathcal{S}$, and $y \in \partial\partial^t S$, there is $F \in \mathcal{F}$ and $a \in A$ such that $F \bullet S$ is defined, $a.y$ is defined, and for every d-coloring of $F.(\partial^t S)_y$ there is $f \in F$ such that $f.(\partial^t S)_y$ is monochromatic and f extends a on $\partial\partial^t S$.

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We consider the following **criterion**:

(ph^-) for $t \geq 0$, $d > 0$, $S \in \mathcal{S}$, and $y \in \partial^{t+1} S$, there is $F \in \mathcal{F}$ and $a \in A$ such that $F \bullet S$ is defined, $a.y$ is defined, and for every d-coloring of $F.(\partial^t S)_y$ there is $f \in F$ such that $f.(\partial^t S)_y$ is monochromatic and f extends a on $\partial^{t+1} S$.

Theorem (S.)

Let $(\mathcal{F}, \mathcal{S})$ be an actoid of sets over a background (A, Z). Assume that $\mathcal{S}$ consists of finite sets and that $\leq_{A,Z}$ is quasi-linear when restricted to S for each $S \in \mathcal{S}$. If $(\mathcal{F}, \mathcal{S})$ fulfills criterion (ph$^-$), then it is a pigeonhole actoid.

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Example.(ctd)
The actiod of sets $(\mathcal{F}_0, \mathcal{S}_0)$ over (A_0, Z_0) fulfills the assumptions of the above theorem, in particular, it has (ph^-). So it is a pigeonhole actoid of sets. The main theorem applied to it gives the Graham–Rothschild theorem for partitions.

Propagating property (ph)

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- the first result involves the notion of finite product of actoids of sets;
  result: finite products of actoids of sets are pigeonhole assuming the factors are pigeonhole;
  proof: uses the main theorem;
- the second result involves the notion of interpretability of sets from an actoid of sets in other actoids of sets;
  result: if each set from an actoid of sets is interpretable in some pigeonhole actoid of sets, then the actoid of sets is pigeonhole.

One obtains all the theorems mentioned in the introduction by repeated applications of the main theorem with the aid of the localization and propagation results.