Topological Ramsey Theory and the Rationals

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Outline

1. Ramsey-classification theory
2. Topological Ramsey spaces
3. The Halpern-Läuchli space of strong subtrees
4. Ramsey theory of the countable dense linear ordering
5. The Hindman-Milliken space $\mathrm{FIN}^{[\infty]}$
6. Ramsey theory of the countable dense-in-itself metric space
7. Conclusion

Ramsey-classification theory

Fix a positive integer k. For a sequence $\vec{\rho} \in \{<, =, >\}^{k \times k}$, define an **atomic canonical relation** $R_{\vec{\rho}} \subseteq \mathbb{N}^k$ by

$$R_{\vec{\rho}}(\vec{x}) \quad \text{iff} \quad (\forall (i,j) \in k \times k)\ x_i\ \vec{\rho}(i,j)\ x_j.$$

We call $R \subseteq \mathbb{N}^k$ **canonical k-ary relation** on $\mathbb{N}$ if R is equal to the **disjunction** of a set of atomic canonical k-ary relations on $\mathbb{N}$.

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For every positive integer k and every relation $R \subseteq \mathbb{N}^k$ there is an infinite $M \subseteq \mathbb{N}$ such that $R \cap M^k$ is canonical.

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Theorem (Erdős-Rado, 1950)

For every positive integer k and every equivalence relation E on $[\mathbb{N}]^k$ there is infinite $M \subseteq \mathbb{N}$ such that $E \restriction [M]^k = E_I \restriction [M]^k$ for some $I \subseteq \{0, 1, \ldots, k-1\}$.

Here, $\{m_0, \ldots, m_{k-1}\}\ E_I\ \{n_0, \ldots, n_{k-1}\} \Leftrightarrow (\forall i \in I)\ m_i = n_i$.

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For $k = 1$ we have two Erdős-Rado relations E_{const} and E_{idd}, the equivalence relations induced by the constant and the identity function, respectively.

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An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is **selective** if for every $f : \mathbb{N} \to \mathbb{N}$ there is $M \in \mathcal{U}$ such that $f \restriction M$ is either constant or one-to-one

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The following are equivalent for $\mathcal{U} \in \beta\mathbb{N}$:

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Corollary

If two irreducible maps $\varphi : \mathcal{B} \to [\mathbb{N}]^{<\omega}$ and $\psi : \mathcal{B} \to [\mathbb{N}]^{<\omega}$ represents the same equivalence relation on a restriction of $\mathcal{B} \restriction M$ on an infinite set $M \subseteq \mathbb{N}$ then there is infinite set $N \subseteq M$ such that φ and ψ are actually equal on $\mathcal{B} \restriction N$.

An application to Tukey theory

For directed sets D and E, put $D \leq_T E$ if there is a **cofinal map** $f : E \to D$, i.e., a map such that

$$(\forall X \subseteq E)[X \text{ cofinal in } E \Rightarrow f[X] \text{ cofinal in } D].$$

Let $D \equiv_T E$ whenever $D \leq_T E$ and $E \leq_T D$.

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1. $\mathcal{U} \leq_T \mathcal{V}$.
2. $\mathcal{U} \equiv_{RK} \mathcal{V}^\alpha$ *for some countable ordinal* α.

Here $\mathcal{V}^\alpha$ denotes the αth **Fubini power** of $\mathcal{V}$ defined recursively on α up to *RK*-equivalence in the natural way:

$$A \in \mathcal{V}^\alpha \text{ iff } \{i : \{j : 2^i(2j+1) \in A\} \in \mathcal{V}^{\alpha_i}\} \in \mathcal{V}$$

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Corollary

Selective ultrafilters realize **minimal** *cofinal types in $\beta\mathbb{N} \setminus \mathbb{N}$.*

Ultrafilters on barriers

For a barrier $\mathcal{B}$ and $n \in \mathbb{N}$, set

$$\mathcal{B}_{\{n\}} = \{s \setminus \{n\} : s \in \mathcal{B}, n = \min(s)\}.$$

Then $\mathcal{B}_{\{n\}}$ is a barrier on $\mathbb{N} \setminus \{0, 1...., n\}$ for all $n \in \mathbb{N}$.

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Define an ultrafilter $\mathcal{U}^{\mathcal{B}}$ on $\mathcal{B}$ as follows:

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If $\mathcal{B} = [\mathbb{N}]^k$ for some positive integer k then

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Step 4: Show that this means that $\mathcal{U} \equiv_{RK} \mathcal{V}^{\varphi[\mathcal{B}]}$.

Topological Ramsey spaces

A **topological Ramsey space** is a set $\mathcal{R}$ of sequences $A = (a_k)$ of objects and a quasi-ordering $\leq$ which defines the corresponding topology of basic-open sets of the form

$$[n, B] = \{A \in \mathcal{R} : A \leq B \text{ and } a_k = b_k \text{ for all } n < k\}$$

with the property that all **Baire subsets** $\mathcal{X}$ of $\mathcal{R}$ are **Ramsey**, i.e.,

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Notation:
$r_n(A) = \langle a_k : k < n \rangle$ is the nth **approximation** to A.
$\mathcal{AR}$ is the collection of all approximations to sequences in $\mathcal{R}$.
$|a|$ is the length of an approximation $a \in \mathcal{AR}$.
$\mathcal{AR}_l = \{a \in \mathcal{AR} : |a| = l\}$.
$[a, B] = \{A \in \mathcal{R} : A \leq B \text{ and } r_{|a|}(A) = a\}$.

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A.1. Finitization

There is a quasi-ordering $\leq_{\mathrm{fin}}$ on $\mathcal{AR}$ such that

(1) $\{a \in \mathcal{AR} : a \leq_{\mathrm{fin}} b\}$ is finite for all $b \in \mathcal{AR}$,

(2) $A \leq B$ iff $(\forall n)(\exists m)\ r_n(A) \leq_{\mathrm{fin}} r_m(B)$,

(3) $\forall a, b \in \mathcal{AR}[a \sqsubseteq b \wedge b \leq_{\mathrm{fin}} c \rightarrow \exists d \sqsubseteq c \quad a \leq_{\mathrm{fin}} d]$.

A sufficient condition for $\mathcal{R}$ being a topological Ramsey space is that $\mathcal{R}$ is a closed subset of $\mathcal{AR}^\omega$. and that the triple $(\mathcal{R}, \leq, r)$ has the following properties:

A.1. Finitization

There is a quasi-ordering $\leq_{\mathrm{fin}}$ on $\mathcal{AR}$ such that

(1) $\{a \in \mathcal{AR} : a \leq_{\mathrm{fin}} b\}$ is finite for all $b \in \mathcal{AR}$,
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A.2. Amalgamation

(1) If $\mathrm{depth}_B(a) < \infty$ then $[a, A] \neq \emptyset$ for all $A \in [\mathrm{depth}_B(a), B]$.
(2) $A \leq B$ and $[a, A] \neq \emptyset$ imply that there is $A' \in [\mathrm{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

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A.3. Pigeon-Hole

If $\mathrm{depth}_B(a) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$, then there is $A \in [\mathrm{depth}_B(a), B]$ such that $r_{|a|+1}[a, A] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, A] \subseteq \mathcal{O}^c$.

The Ramsey-Galvin-Prikry-Ellentuck space

Let $\mathcal{R} = \mathbb{N}^{[\infty]} = \{A \subseteq \mathbb{N} : |A| = \aleph_0\}$, $\leq = \subseteq$, and $r_n(A) = \{\text{first } n \text{ members of } A\}$.

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For every positive integer k *and every equivalence relation* $\sim$ *on* $\mathbb{N}^{[k]}$ *there is* $I \subseteq k$ *and infinite* $M \subseteq \mathbb{N}$ *such that for* $a, b \in \mathbb{N}^{[k]}$, $a \sim b$ *iff* $(\forall i \in I) a_i = b_i$.

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For every equivalence relation $\sim$ on U there is a strong subtree T of U of height ω such that one of the following holds:

1. $(\forall s, t \in T)[s \sim t \Leftrightarrow s = t]$,
2. $(\forall s, t \in T)[s \sim t \Leftrightarrow s = s]$,
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Corollary

U contains a strong subtree S of height ω such that either

1. *S is uniformly branching of some degree d*
2. *nodes of S of the same height have the same degree which increase as the height increase*
3. *different nodes of S have different branching degrees.*

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Theorem (Milliken, 1980)

For every positive integer k and every equivalence relation $\sim$ on $\mathcal{S}_k(U_d)$ there is a strong subtree T of U_d and a pair

$$(N, L) \in \mathcal{P}(d^{<k}) \times \mathcal{P}(k) \text{ with } \max\{|s| : s \in N\} < \min(L)$$

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For every positive integer k and every finite coloring of the set $[\mathbb{Q}]^k$ of all k-element subsets of $\mathbb{Q}$ there is $P \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$ such that $[P]^k$ uses at most t_k colors.

Here, $t_k = \sum_{l=1}^{k-1} \binom{2k-2}{2l-1} t_l \cdot t_{k-l}$ is the standard sequence of odd tangent numbers: $t_1 = 1$, $t_2 = 2$, $t_3 = 16$, etc.

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Ramsey-classification problem for $[\mathbb{Q}]^k$

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Theorem (Vuksanovic, 2003)

For every positive integer k the class of all equivalence relations on the set $[\mathbb{Q}]^k$ of all k-element subsets of $\mathbb{Q}$ has a Ramsey basis of equivalence relations $E_{\mathcal{T}}$ determined by 'transitive sets'

$$\mathcal{T} \subseteq [2^{\leq 4k-2}] \times [2^{\leq 4k-2}].$$

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Remark

The exact size of an irredundant Ramsey basis of the class of equivalence relations on $[\mathbb{Q}]^3$ is not known.

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For every positive integer k *and every finite coloring of the set* $\mathbb{Q}^{[k]}$ *of rapidly increasing* k*-sequences of elements of* $\mathbb{Q}$ *there is* $P \subseteq \mathbb{Q}$ *order-isomorphic to* $\mathbb{Q}$ *such that* $P^{[k]}$ *is monochromatic.*

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Theorem

For every positive integer k the collection of equivalence relations on $\mathbb{Q}^{[k]}$ has a finite Ramsey basis of cardinality $h_k = 2\sum_{i=0}^{k} 5^i$.

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If $X = (x_n) \in \mathrm{FIN}^{[\infty]}$ then

$$\max(x_m) < \min(x_n) \text{ whenever } m < n$$

and for $X = (x_n), Y = (y_n) \in \mathrm{FIN}^{[\infty]}$, we set $X \leq Y$ whenever $X \subseteq [Y]$, where

$$[Y] = \{y_{n_0} \cup \cdots y_{n_k} : n_0 < \cdots < n_k\}.$$

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For every positive integer k *and every finite coloring of* $\mathrm{FIN}^{[k]}$ *there is* $Y = (y_n) \in \mathrm{FIN}^{[\infty]}$ *such that* $[Y]^{[k]}$ *is monochromatic.*

Theorem (Taylor, 1976)

For every equivalence relation E *on* FIN *there is* $Y = (y_n) \in \mathrm{FIN}^{[\infty]}$ *and*

$$\varphi \in \{\mathrm{const}, \mathrm{ident}, \mathsf{min}, \mathsf{max}, (\mathsf{min}, \mathsf{max})\}$$

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Theorem (Lefmann, 1996)

For every positive integer k *the class of equivalence relations on* $\mathrm{FIN}^{[k]}$ *has a Ramsey basis of cardinality* $s_k = \frac{1}{13 \cdot 2^{k+1}}[(13 + 3\sqrt{13})(7 + \sqrt{13})^k + (13 - 3\sqrt{13})(7 - \sqrt{13})^k]$.

Ramsey on countable topological spaces

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Theorem (Baumgartner, 1986)

Suppose X is a countable Hausdorff topological space. Then there is $c : [X]^2 \to \omega$ such that

$$c[P]^2 \supseteq \{0, 1, ..., 2n - 1\}$$

for all $n < \omega$ and all $P \subseteq X$ such that $P^{(n)} \neq \emptyset$

Here $P^{(0)} = P$, $P^{(n+1)} = (P^{(n)})'$, where for a subset A of X we let

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FIN as a topological copy of $\mathbb{Q}$

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Recall that FIN is the space of all finite nonempty subsets of $\mathbb{N}$ and we shall take it with the topology induced from $2^{\mathbb{N}}$ as our topological copy of the rationals $\mathbb{Q}$.

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Theorem

For every positive integers k and n there is integer $h(n,k)$ such that for every finite coloring of $[\mathbb{Q}]^k$ there is $P \subseteq \mathbb{Q}$ homeomorphic to ω^n such that $[P]^k$ uses at most $h(k,n)$ colors.

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Remark

The function seems expressible using the standard enumerating functions. For example, $h(n,2) = 2n$ for all n.

The oscillation mapping

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The **oscillation mapping** on $\mathbb{Q} = \{\emptyset\} \cup \mathrm{FIN} \subseteq 2^{\mathbb{N}}$:
Define $\mathrm{osc} : \mathbb{Q}^2 \to \omega$ by

$$\mathrm{osc}(s, t) = |(s \triangle t)/ \sim |,$$

where for $i, j \in s \triangle t$, we let

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Proposition

Suppose that $\partial^k(X) \neq \emptyset$ for some $X \subseteq \mathbb{Q}$ and some positive integer k. then

$$\mathrm{osc}[X^2] \supseteq \{2, 3, ..., 2k\}.$$

If, moreover, $X \cap \partial(X) \neq \emptyset$ then

$$1 \in \mathrm{osc}[X^2]$$

as well.

Corollary

The class of equivalence relations on the square of the countable dense-in-itself metric space has no finite (and, in fact, no countable) Ramsey basis.

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However, the following fact shows that the oscillation mapping is in some sense **canonical**.

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For every $f : [\mathbb{Q}]^2 \to \omega$ *and every positive integer* n *there is* $X \subseteq \mathbb{Q}$ *homeomorphic to* $\mathbb{Q}$ *such that for* $\{s,t\}, \{s',t'\} \in [X]^2$,

$$\mathrm{osc}(s,t) = \mathrm{osc}(s',t') \textit{ implies } f(s,t) = f(s',t')$$

provided that the numbers $f(s,t)$, $f(s',t')$, $\mathrm{osc}(s,t)$ *and* $\mathrm{osc}(s',t')$ *are all* $\leq n$.

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Remark

Since osc is not a continuous mapping on $\mathbb{Q}$ it is natural to examine the possibility of Ramsey-classification for **continuous** equivalence relations on powers of $\mathbb{Q}$.

Continuous colorings of $[\mathbb{Q}]^2$

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Theorem (T., 1994)

For every finite **continuous** *coloring of* $[\mathbb{Q}]^2$ *there is* $P \subseteq \mathbb{Q}$ *homeomorphic to* $\mathbb{Q}$ *such that* $[P]^2$ *is monochromatic.*

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Definition

An equivalence relation $\sim$ on $[\mathbb{Q}]^k$ is **continuous** if

$$s_n \sim t_n \text{ and } s_n \to s \text{ and } t_n \to t \text{ imply } s \sim t.$$

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An equivalence relation $\sim$ on $[\mathbb{Q}]^k$ is **discretely continuous** if the corresponding quotient mapping

$$q : [\mathbb{Q}]^k \to [\mathbb{Q}]^k / \sim$$

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Remark

Note that the equality relation $=$ on $[Q]^k$ is a continuous but not discretely continuous equivalence relation.

Theorem

The class of continuous equivalence relations on $[\mathbb{Q}]^2$ *does not have finite Ramsey basis.*

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Example

For a positive integer n define

$$\varphi_n : [\mathbb{Q}]^2 \to [\mathbb{Q}]^{\leq 2}$$

by letting

1. $\varphi_n(s,t) = \{s,t\}$ whenever $\mathrm{osc}(s,t) \leq n$, and
2. $\varphi_n(s,t) = \{s \restriction m, t \restriction m\}$ whenever $\mathrm{osc}(s,t) > n$ and where m is the minimum of the $n+1$'st class of $s \triangle t / \sim$.

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Then for $n \neq n'$, we have that

$$E_{\varphi_n} \restriction [P]^2 \neq E_{\varphi_{n'}} \restriction [P]^2$$

for any $P \subseteq \mathbb{Q}$ homeomorphic to $\mathbb{Q}$.

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The 26 equivalence relations have the form $E_{\varphi_1 \circ \varphi}$, where

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For $s, t \in \mathbb{Q}$, let
$$\triangle(s,t) = \min(s \triangle t).$$
Define $\mathrm{osc}^* : [\mathbb{Q}]^3 \to \omega$, by letting
$$\mathrm{osc}^*(s,t,u) = \mathrm{osc}(\{s \restriction m, t \restriction m, u \restriction m\}),$$
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Proposition

The mapping $\mathrm{osc}^* : [\mathbb{Q}]^3 \to \omega$ is **continuous** *and*

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Corollary

The class of discretely continuous equivalence relations on $[\mathbb{Q}]^3$ has no basis of cardinality smaller than the continuum

Four splitting patterns in $[\mathbb{Q}]^3$

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If $p(s, t, u) = 1$ or $p(s, t, u) = 3$, then $\mathrm{osc}^*(s, t, u) = 1$.

Lemma

Suppose $\partial^k(X) \neq \emptyset$ *for some* $k \geq 2$*. Then for every* $2 \leq j \leq 2k-2$ *there exists* $\{s, t, u\} \in [X]^3$ *such that* $p(s, t, u) = 2$ *and* $\mathrm{osc}^*(s, t, u) = j$.

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Theorem

There is a continuous mapping $c : [\mathbb{Q}]^3 \to \omega$ *such that*

$$c[X]^3 \supseteq \{0, 1, ..., 4(k-1)\}$$

for every positive integer k *and* $X \subseteq \mathbb{Q}$ *such that* $\partial^k(X) \neq \emptyset$.

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Theorem

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provided that the numbers $f(s, t, u)$, $f(s', t', u')$, $\mathrm{osc}^*(s, t, u)$ and $\mathrm{osc}^*(s', t', u')$ are all $\leq n$.

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Having available Ramsey space could serve as powerful tool when classifying a given class of structures.

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Work on a classification problem for a class of mathematical structure could lead to discovery of new Ramsey spaces.